REFLECTION AND TRANSMISSION OF SURFACE WATER WAVES BY UNDULATING BOTTOM TOPOGRAPHY

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Certificate

It is certified that the work contained in this thesis entitled **"Reflection and Transmission of Surface Water Waves by Undulating Bottom Topography"** by **Subash Chandra Martha,** a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

December, 2006

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Dedicated to

My Father **Satrughna Martha**

and

My Mother **Snehalata Martha**

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Abstract

The objective of this thesis is to investigate the scattering of a train of small amplitude harmonic surface water waves by small undulation of a sea-bed for both normal and oblique incidence.

In this study of scattering, mixed boundary value problems are set up for the determination of a velocity potential where the governing partial differential equation happens to be Laplace's equation in two dimensions for normal incidence and in three dimensions for oblique incidence within the fluid with a mixed boundary condition on the free surface and a condition on the bottom boundary. As the fluid domain extends to infinity, a far-field condition or an infinity condition arises to ensure uniqueness of the problem.

Applying a perturbation analysis, which involves a small parameter ε present in the representation of the small undulation of the sea-bed, directly to the boundary value problem the original problem is reduced to a simpler boundary value problem for the first order correction of the potential.

The mathematical tools utilised in the first part of the thesis in obtaining the solution of the simpler boundary value problems are (I) Green's function technique, and application of Green's integral theorem, (II) Fourier transform technique and application of residue theorem, and (III) Finite cosine transform. From the solutions of the velocity potential the reflection and transmission coefficients are evaluated approximately up to the first order of ε in terms of integrals involving the shape function $c(x)$ representing the bottom undulation.

Different special forms of the shape functions are considered to evaluate the integrals explicitly for these coefficients. Out of those shape functions, the particular case of a patch of sinusoidal ripples has considerable significance due to the ability of an undulating bed to reflect incident wave energy which is important in respect of both coastal protection, and of possible ripple growth if the bed is erodable. For this ripple patch it is observed that if the bed wave number is twice the surface wave number then there is a resonant Bragg-type interaction between the surface waves and the bed forms as observed earlier in the literature. At this resonance a large amount of reflection of the incident wave energy occurs which may be useful in the construction of an effective reflector of the incident wave energy for protecting coastal areas from the rough sea in the arctic regions. The same observation can be made even when the ripples do not have the same wavelength. The evaluation of the reflection and transmission coefficients for these different bed forms allow us to observe the behaviour of the scattered field associated with each undulation.

A direct method, namely eigenfunction expansion method, which includes both decaying and progressive wave mode terms, is also considered later in the thesis to solve the boundary

value problem. It is worthwhile to note that the analytical results obtained for the reflection and transmission coefficients by this method are quite different from the results obtained by previous methods employed in the thesis. This difference is due to the solution approach of this direct method containing an appropriate set of orthogonal eigenfunctions which depends upon a single parameter. However, while considering a patch of sinusoidal undulations on the bottom, when the results are computed and compared with the results obtained by other methods, excellent agreement is observed between the numerical estimates obtained by the present method and those by the known method.

In the concluding part of the thesis the problem of scattering of surface water waves by small undulation on a sea-bed of finite depth is investigated by assuming the sea-bed to be composed of porous material of specific type. Fourier transform technique is employed to obtain the complete solution of the mixed boundary value problem from which the reflection and transmission coefficients are determined which involve the shape function $c(x)$. It is observed that with negligible porosity, the results for these coefficients may be interpreted as the results obtained earlier in the thesis without porosity on the bed. These results are applied to the case of a patch of sinusoidal undulations on the bed to evaluate the coefficients numerically and the results are suitably presented graphically.

It is observed that the methods presented in the thesis in obtaining the first order potential, and hence the reflection and transmission coefficients, reduce the workload to a large extent. These methods lead to a computationally more tractable form of the solution for the scattered field.

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Nomenclature

σ :	frequency of the incoming water wave
g :	acceleration due to gravity
h :	uniform finite depth of water
t :	time
K :	$= \sigma^2/g$
Φ :	real velocity potential
ψ, ϕ :	complex velocity potential
ψ_0 :	complex incident velocity potential
(x, y, z) :	Cartesian coordinates
k_0	wave number of the wave
θ :	angle of the incidence of the wave train
μ :	x-component of k_0
ν :	z-component of k_0
a :	amplitude of the sinusoidal ripples
l :	wave number of the sinusoidal ripples
δ' :	phase angle

Chapter 1

Introduction

1.1 Preamble

"Fluid dynamics" is one of the most important part of the recent interdisciplinary activities concerning engineering and science, especially in applied mathematics. Fluid dynamics or hydrodynamics is the branch of science which is concerned with the study of the motion of fluid or that of bodies in contact with fluids.

Wave propagation and its applications are encountered in many areas of physical interest, including the fields of water waves, acoustics and electromagnetic waves. A wave is a disturbance such that when it propagates through a medium, energy is transmitted to distant points without any displacement of the particle of the medium. The energy from the sun is transmitted by waves through the ether. When some musical instrument is played upon in a room, sound waves spread through the room. If we throw a stone in a pond we observe waves in the pond which start from the point of striking of the stone and spread in all directions. Such water waves are also produced by pressure of wind upon the surface of water, by the relative motion of bodies like a ship moving in sea and by obstacles in the bed of the stream. These are some examples of wave motion.

Ever since studies on waves became central in science and engineering, water waves began serving as important models for investigation. Due to this water wave phenomena have attracted the attention of applied mathematicians and physicists since long back. The study of different kinds of water waves is of importance for various applications. For example, it is required for predicting the behaviour of floating structures (immersed totally or partially) such as ships, submarines, and tension leg plat forms, and for describing flows over bottom topography.

The wave motion in water may be classified into two categories: in one category the wave length is assumed to be greater than the depth of water and the study is called shallow water wave theory. The waves belonging to this category are tidal waves and long waves in shallow water. In the other category the wavelength is assumed to be much smaller than the depth of water so that the effect of the disturbance diminishes gradually as one goes downwards away from the free surface. Waves belonging to this category are named as surface water waves. In this case, if it is assumed that the amplitude of the wave is small compared to the wave length then the theory is called linearised theory of water waves. This theory has been derived as an approximation of the general theory on the basis of the assumption that the components of the velocities of water particles, the free surface elevation or depression and their derivatives are small quantities i.e. the motion is assumed to be very small. Though the ocean waves or surface waves are often nonlinear, the analysis in many physical problems is restricted to small amplitude waves mainly because consideration of linear theory is sufficient in offshore engineering and other related studies in handling most of the problems. Due to this reason, many scientists, researchers and ocean technologists use linearised theory of water waves to model mathematically various physical problems related to wave phenomena arising in costal engineering. In the present thesis the problems are formulated and solved on the basis of linearised theory of water waves.

Water waves (the terms *surface waves* and *gravity waves* are also in use) are created normally by a gravitational force in the presence of the free surface along which pressure is constant. Water is assumed to occupy a certain domain bounded by one or more moving or fixed surfaces that separate water from some other medium. Actually we consider boundaries of two types: the above mentioned free surface separating water from the atmosphere, and a rigid surface including the bottom and surface of bodies floating in and/or beneath the free surface.

Many physical problems arising in applied mathematics or mathematical physics can be formulated as mixed boundary value problems of elliptic partial differential equations. In the study of scattering or radiation of surface harmonic water waves under the assumption of the linearised theory, the partial differential equation happens to be Laplace's equation in either two or three dimensions, the domain is unbounded, some of the boundary conditions are of mixed type and the behaviour of the function is not known completely at large distances, and the third or fourth order derivative may occur in some of the boundary conditions in certain physical problems etc.

A long-standing but persistent problem in the area of wave theory is the determination of the effect of the bed topography and the obstacle(s) on a given wave field. An example of a practical problem faced by coastal engineers is to predict the amplitude of waves in harbours where both man-made break waters and the shape of the sea-bed affect wave behaviour. Such problems involve scattering, diffraction and refraction of waves and are mathematically formidable for linearised theory, even with relatively simple beds and/or obstacle geometries.

When a train of surface waves travelling from a large distance is incident on an obstacle submerged or partially immersed in water, some parts of the wave is reflected back by the obstacle and some part is transmitted over or below it. The waves which is reflected back is known as "reflected wave" and the wave which is transmitted is known as "transmitted wave". The reflected waves and transmitted waves are called outgoing waves as they go away from the obstacle after striking. This process is known as scattering. In this scattering process the incoming or the incident wave modifies and produces scattered waves which are nothing but outgoing waves such that a combination of incident wave and outgoing wave results from which the outgoing wave can be determined. The process of determining an outgoing wave for a given incoming wave is known as the mathematical problem of scattering. In the scattering theory, the reflected wave is accompanied by a constant, known as the "reflection coefficient" and the transmitted wave is accompanied by another constant, known as the "transmission coefficient". These two physical quantities play a vital role in the mathematical study of water wave scattering problem since they provide a measure for the amount of reflected and transmitted waves. This information is useful in the construction of offshore structures or in the problem of generation of surface waves by obstacles known as the wave-maker problem. Due to the complicacy, sometimes many researchers emphasise on the determination of these quantities directly instead of going into details of the solution.

Scattering problems involving barriers of different configurations have practical importance for various engineering applications. Modelling breakwaters constructed to protect the offshore areas from hazards of the rough sea is one of them. Also this class of problems is useful for designing ships, submarines, offshore structures etc. The other class of problem which studies the problem of free surface flow over a geometrical disturbance at the bottom of an ocean are important for their possible applications in the areas of coastal and marine engineering. The problem of reflection of surface waves by patches of large bottom undulations has received an increasing amount of attention recently as its mechanism is important in the development of shore-parallel bars. The work presented in this thesis is solely concerned with the effect of bed topography on an incident wave train.

With this preamble we now give a brief history of various problems investigated by researchers over the last few decades.

1.2 Brief history and motivation

The problems of free surface flow over an obstacle or a pre-existing (fixed) pattern of undulation on an otherwise flat bed are important for their possible applications in the areas of coastal and marine engineering, and as such these are being studied by scientists and engineers for a long time. The scattering of linear water waves by bed topography is governed by Laplace's equation together with appropriate boundary and radiation conditions. Such boundary value problems are, in general, somewhat difficult to solve, and explicit analytic expressions for the velocity potential are rare and occur only for certain special configurations usually involving vertical or horizontal strips as boundaries. Although the analytical solutions are rare for such boundary value problems, but by using various approximate mathematical techniques the quantities of physical interests, namely the reflection and transmission coefficients, can be estimated numerically. For example, Mei and Black [63] used a variational formulation of the problem of wave scattering by a bottom-standing rectangular thick vertical barrier to obtain numerical estimates for the reflection coefficient. Kanoria *et.al* [36] employed Galerkin approximations involving ultraspherical Gegenbauer polynomials for solving the integral equations arising in the integral equation formulation for the same problem and obtained very accurate numerical estimates for the reflection coefficient.

Due to the great mathematical complexity of linearised free surface flow in water of variable quiescent depth, a number of approximations to the boundary value problem has been proposed. In one class of approximation the vertical coordinate is removed by performing integration over the depth, thus reducing the dimension of the problem by one. Berkhoff [5, 6] developed a vertical integrated refraction-diffraction equation, known as *mild-slope equation*, which was the result of such a procedure. It was based on the assumption of a mild-slope bottom. Smith and Sprinks [85] gave another derivation of the *mild-slope equation*, similar to that of Berkhoff but more succinct, and the mild-slope approximation has since been used to produce other equations which model the effect of the bed topography on wave propagation. Booij [7] examined the accuracy of the mild-slope equation as a function of the bottom slope and showed that acceptable results could be achieved even though the bottom slope was not moderate. Kirby [39] derived the mild-slope equation for the linear surface wave-current interaction over slowly varying topography. Copeland [15] solved the mild-slope equation in the form of a pair of first-order equations (which consisted of a hyperbolic system) using a simple finite difference scheme. This offered the advantages of reduced computing time compared with the solution of a boundary value problem and incorporated useful features.

By recasting the mild-slope equation as a system of first-order differential equations, which were similar to the system of equations governing nearly horizontal flow in shallow water, a highly efficient algorithm for the later was used iteratively by Madsen and Larsen [50] to find the stationary solution. O'Hare and Davies [74] considered the propagation of monochromatic surface waves over a region of arbitrary, one-dimensional bottom topography. The smoothly varying bed topography was divided into a series of horizontal shelves and the surface wave field was calculated by successive approximation of Miles [66] scattering matrix at each of the intervening steps. Chamberlain [10] examined the scattering of a train of small amplitude harmonic waves on water by one-dimensional topography, using the mild-slope equation. The associated boundary value problem was converted into a pair of integral equations whose solutions were approximated by variational techniques. Chamberlain [11] showed that the reflection and transmission coefficients arising from the scattering of linear water waves by a one-dimensional topography were known to possess certain symmetry properties. He showed that the same relations hold in the mild-slope approximation to the full linear theory. These relations were used in the development of a decomposition method where solutions for relatively simple depth profiles might be combined to give solutions for complicated ones. The ideas and techniques used in the study of plane wave approximation over slowly varying depth were applied by Kellin and Johnson [38] to the axi-symmetric flow.

It was observed by a number of authors that the *mild-slope equation* could fail to produce adequate approximation for certain type of topography, such as ripple beds. These consist of a finite patch of periodic undulations or small amplitude sinusoidal ripples set in an otherwise horizontal bed. It is perhaps more accurate to refer to the bed perturbations as bars rather than ripples, but we follow what has become the standard terminology. To overcome the deficiency in the mild-slope equation, and Kirby [40] applied vertical integration process and developed a time-dependent extension of the reduced wave equation of Berkhoff for the case of waves propagating over the bed consisting of ripples superimposed on an otherwise slowly varying mean depth which satisfied the mild-slope assumption. The equation resulting out of Kirby's extension is known as *extended mild-slope equation*. Then Chamberlain and Porter [12] formalised the vertical averaging procedure by invoking a variational principle and returning to the relatively simple type of approximation, used by Berkhoff [5, 6] and Kirby [40], presented a new form (modified version) called *modified mild-slope equation* which contained the original *mild-slope equation* and its extended version as special cases. It was found that the *modified mild-slope equation* was capable of describing the scattering properties of singly and doubly periodic ripple beds, for which the mild-slope equation failed. Despite the marked improvement over the mild-slope equation which it produces for ripple bed scattering, Chamberlain and Porter [12] found that the modified mild-slope equation performed no better than its predecessor when used in Booij's problem [7]. Porter and Staziker [78] explained this result and, in the process, showed how approximations based on the original mild-slope equation could be significantly improved. The solutions which have been derived previously seem to be physically responsible in the sense they imply continuity of the approximation to the free surface and its slope. However such solutions were not consistent with the requirement of conservation of mass at locations where the bed profile had a discontinuous slope. Using the variational principle, Porter and Staziker [78] showed that the solutions of the mild-slope equation and its modified version must satisfy a jump condition where the bed slope was not continuous in order to ensure continuity of mass flow. Porter [76] re-examined the mild-slope equation and the modified mild-slope equation and resolved the discrepancies between their solutions. This was achieved by converting the above modified mild-slope equation in such a way that the reduced form of it, a new version of the mild-slope equation, contained all of the essential characteristics of the full modified mild-slope equation. A more complete review of linear water wave scattering by bed topography can be found in Porter and Chamberlain [77].

For several bed topographies consisting of horizontal and vertical sections, numerical solutions of integral equations arising from boundary value problems have been obtained. Staziker *et al.* [87] considered the problem of two-dimensional wave scattering by a local bed elevation of any shape on an otherwise horizontal bed (such bottom profiles are referred to as "humps") using linearised water wave theory. Green's function theory was used to convert the associated boundary value problem into a first kind integral equation and a variational approach was then used to obtain the approximations to the amplitude of the scattered wave. Miles and Chamberlain [71] developed a systematic hierarchy of partial differential equations of linear gravity waves in water of variable depth through the expansion of the average Lagrangian in powers of depth slope. For the propagation of small amplitude water waves over variable bathymetry regions, Athanassoulis and Belibassakis [1] derived a consistent coupled-mode theory from a variational formulation of the complete linear problem by representing the vertical distribution of the wave potential as a uniformly convergent series of local vertical modes at each horizontal position. Chamberlain and Porter [14] used the *mild-slope equation* to examine scattering and trapping by axisymmetric topography with the main emphasis on wholly submerged bed forms. Porter and Porter [80] considered the scattering and trapping of water waves by three dimensional submerged topography, infinite and periodic on one horizontal co-ordinate and of finite extent in the other, under the assumption of linearised theory. The mild-slope approximation was used to reduce the governing boundary value problem to one involving a form of the Helmholtz equation in which the coefficients depended on the topography.

Also many mixed boundary value problems corresponding to the problems of surface water waves travelling in regions of varying depth have been presented under various geometrical assumptions. Kirby [41] derived coupled equations governing the forward- and back-scattered components of a linear wave propagating in a region of varying depth from a second order wave equation for linear wave motion. Evans [25] described two mechanisms for the generation of standing edge waves over a sloping beach using classical linear water wave theory. Devillard *et al.* [22] presented a theoretical study of the localisation phenomenon of surface gravity waves by a rough bottom in a one-dimensional channel. They developed a renormalised-transfer-matrix approach to this problem. A detailed description of its experimental set-up and the corresponding results could be found in Belzons *et al.* [4]. Tsay *et al.* [90] developed a finite element model to calculate the wave refraction, diffraction, reflection and dissipation. The governing equation was a two-dimensional depth-integrated linear wave equation which considered the effect of the topographical variations and energy dissipation. Johnson [35] presented a straight forward method that yielded explicit transmission amplitudes for Kelvin wave scattering by topography whose isobaths were parallel sufficiently far from the vertical, but not necessarily planner, wall supporting the incident wave. Miles [69] derived the Eckart's [23] second-order, self adjoint partial differential equation for the free surface displacement of monochromatic gravity waves in water of variable depth from variational formulation by approximating the vertical variation of the velocity potential in the average Lagrangian by that for deep water waves. It was compared with the mild-slope equation, which also was second-order and self adjoint, and might be obtained in a similar manner for uniform finite depth. Chamberlain and Porter [13] described a method for determining those approximations to wave scattering by bed topography which were based on second order ordinary differential equations. The development of a decomposition method allowed the scattering matrix for an extended section of varying topography to be assembled in a piecemeal fashion.

The propagation of surface water waves in a channel with a rough bottom is, in general, a very difficult problem to solve analytically. A number of authors have studied water wave propagation over an irregular bottom topography. In the classical work of Lamb [44], the free surface elevation for the two-dimensional problem of steady flow over the bottom irregularities was obtained assuming irrotational motion. The problem of determining the scattering of long crested gravity wave over a bottom of arbitrary shape has received considerable attention in the literature. For a survey up to 1960, the work of Wehausen and

Laitone [91] is referred.

Kreisel [43] considered the case of wave propagation over uneven bottom with obstacles and determined the reflection and transmission coefficients of the wave train from obstacles when the surface remained free using the conformal transformation. Mei and Méhauté, [65] gave a note on the equations of long waves over an uneven bottom with the basic assumption that the depth was small in comparison with a horizontal length. Carrier [9] studied the propagation of gravity waves over a basin in which the propagation distance was large compared with the scale of the bottom topography which, in turn, was large compared with depth. Evans [24] constructed a new source of potential in the linearised theory of water waves and used this source potential to reduce the problem of the reflection and transmission waves by a shelf of arbitrarily profile to an integral equation. Restricting the shelf profile, he applied Fredholm theory to the integral equation so that, in general, a solution of the equation and hence to the problem, existed and was unique except possibly for certain discrete values of the parameter of the problem corresponding to trapping modes over the shelf. The general problem of wave-energy reflection by sea-bed topography was examined by Long [46].

A two-dimensional, irrotational, linear theory was used by Fitz-Gerald [27] to investigate the reflection of an incident surface gravity waves travelling over a region of varying depth. He used a Fourier transform procedure to convert the boundary value problem satisfied by the velocity potential into a pair of integro-differential equations. The required solution waves was then given by a linear combination of the solutions of the integro-differential equations. Since the uniqueness of the reflection and transmission coefficients has been proved by Kreisel [43] there was no corresponding result for the uniqueness of the velocity field. It remains an open question whether there exist solutions, tending to zero at infinity, which describes modes trapped on some topographical feature on the bottom profile. Fitz-Gerald was able to prove the existence and uniqueness of the velocity potential in two limiting cases and presented the associated asymptotic results. Hamilton [29] provided differential equations for long-period gravity waves on fluid of rapidly varying depth and he was able to calculate the reflection coefficient for long-period waves incident on a step change in depth and a half-depth barrier. A more complete set of reference is given by Mei [61]. Nachbin and Papanicolaou [73] analysed the linear water wave equations for shallow channels with arbitrary rapidly varying bottoms. They used the full, linear potential theory to study the reflection and transmission problem for time-harmonic (monochromatic) and pulse-shaped disturbances.

O'Hare and Davies [75] made comparisons between the predictions of the two models for surface wave propagation over rapidly-varying topography, one based on the extended mild-slope equation derived by Kirby [40] and the other on the successive-application matrix model described by O'Hare and Davies [74]. The models were applied to two types of undulating topography, namely sinusoidal (Davies and Heathershaw [19]) and doubly-sinusoidal beds (Guazzelli *et al.* [30]) and comparisons were made with existing laboratory data. Using the two-dimensional linear water wave theory to the scattering of water waves by a varying bottom topography, Evans and Linton [26] adopted a new approach in which the problem was first transferred into a uniform strip resulting in a variable free surface boundary condition. This was then approximated by a finite number of sections on which the free surface boundary condition was assumed to be constant. They also developed a transition matrix theory which was used to relate the wave amplitudes at infinity.

The scattering problem in which an arbitrary profile joins two semi-infinite horizontal bed sections at different depths is more interesting, from the point of applications, than that of involving an elevation of bounded extent on a single horizontal bed. It is also a more difficult problem from the integral equation point of view, although the one to which all the methods such as the mild-slope equation, topography discretization and Evans and Linton [26] mapping method might be applied. Linking to the above idea Porter and Porter [79] considered the two-dimensional scattering of water waves over a finite region of arbitrarily varying topography linking two semi-infinite regions of constant depth using linear water wave theory. The reflected and transmitted wave amplitudes were related to the incoming wave amplitudes by a scattering matrix which was defined in terms of inner product involving the solution of the corresponding integral equation system. Keller [37] derived a new shallow-water theory valid for arbitrary bottom slope by a scaling method.

There have been many investigations of the reflection of short incident gravity waves by bottom standing obstacles on the sea-bed, such as engineering works, but rather few of reflection by naturally occurring obstacles, such as sand ripples, which can be assumed to be small in some sense for which some sort of perturbation technique can be employed to obtain the first order corrections to the reflection and transmission coefficients. A number of approximations have been devised to the scattering of water waves by a finite number of periodic ripples or sand bars on an otherwise horizontal, flat bed. Miles [68] approximated the reflection and transmission coefficients up to the first order in terms of integrals involving the shape function of the bottom deformation by using small perturbation theory when the wave train was obliquely incident.

Davies [17] considered the interaction of first order incident surface progressive waves with small sinusoidal undulations of infinite horizontal extent and then showed that this interaction gave rise to two new waves with wave numbers which were the sum and difference of those of the surface waves and undulations. Davies [18] considered a sinusoidal undulation of small amplitude and finite horizontal extent in the sea-bed and used a somewhat elaborate method to handle the water wave scattering problem for normal incidence. Heathershaw [32] provided a supporting experimental proof to the theoretical results of reflection of wave energy due to the resonant interactions between surface water waves and undulating bed topography. Davies and Heathershaw [19] studied the problem both experimentally and theoretically for a horizontal bottom with sinusoidal undulation when the incidence was normal. Heathershaw and Davies [33] examined the case of wave interaction with a pre-existing pattern of bedforms, such as shore parallel bars on beaches and the way in which this interaction may lead to further growth of the bed forms.

Mandal and Basu [53] generalized the problem of Miles [68] to include the effect of surface tension at the free surface. A perturbation analysis, somewhat similar to that employed by Mandal and Chakrabarti [52] in connection with water wave diffraction by nearly vertical barriers, was used directly to the governing partial differential equation and the boundary conditions describing the physical problem. This analysis reduced the original problem to a boundary value problem of first order whose solution was obtained by an application of Green's integral theorem and then using this solution, the reflection and transmission coefficients were obtained up to the first order in terms of the integral involving the shape function describing the bottom undulation. Mandal and Basu [54] considered the two-dimensional problem of steady flow over a small two-dimensional geometrical disturbance at the bottom of a wide channel. They obtained the free surface elevation up to first order, in terms of an integral involving the shape of the channel bottom. Martha and Bora [57, 58] also considered water wave diffraction by small undulations on the sea-bed assuming linear water wave theory. A perturbation analysis and Green's integral theorem were used to evaluate the reflection and transmission coefficients for a patch of sinusoidal ripples.

Miles [70] calculated the reflection of straight-crested gravity waves by a non-secular perturbation in depth relative to an otherwise flat bottom of depth. Explicit results were developed up to second order for the sinusoidal patch and reduced for Bragg resonance. But Porter and Porter [80] investigated the case of interaction of linearized surface gravity water waves with three dimensional periodic topography. They considered the scattering and trapping of water waves by three dimensional submerged topography, infinite and periodic in one horizontal coordinate and of finite extent in the other, under the assumptions of linearized theory. Further, Porter and Porter [81] investigated the interactions between surface water waves over the periodic bed forms in three different situations: one of these was scattering of given incident waves by a finite section of periodic topography; the other two, in contrast, concern the existence of unforced waves over the periodically varying beds of infinite and finite extent.

Davies [18] and Davies and Heathershaw [19] highlighted that if the bottom contained periodic undulations Bragg resonance occurred when the wave length of the incident surface wave is twice that of the bottom undulation. As a result the Bragg scattered wave became resonant and could be greatly amplified. Such resonant wave interactions with bottom ripples play an important role in the evolution of nearshore surface waves. However, their theory breaks down near the Bragg resonance condition. To overcome this drawback, Mei [62] developed wave evolution and reflection theory at and near Bragg resonance condition for shore parallel sinusoidal bars. Studies on Bragg scattering by periodically spaced sand bars was given by Hara and Mei [31]. Further, Bragg scattering of surface waves by sinusoidal sand bars on a sea-bed and double periodic sea-bed, respectively, were studied by Mei *et al.* [64] and Naciri and Mei [72]. Davies *et al.* [20] considered the interaction of incident surface waves with a patch of sinusoidal ripples on an otherwise flat bed and obtained the solution for non-resonant and resonant cases. Kirby [42] also considered the reflection of linear surface waves by sinusoidal bottom undulations where the incident wave was not necessarily close to the resonant frequency.

It has been suggested that Bragg reflection, the combined coherent wave reflection from a few low-lying shore-parallel bars, might be used to protect a beach against storm-wave attack. Bailard *et al.* [2] used numerical models to examine issues relating to the feasibility of this concept. Guazzeli *et al.* [30] experimentally investigated higher order Bragg resonant interactions between linear gravity waves and doubly sinusoidal beds. Then Liu and Yue [47] studied the generalised Bragg scattering of surface waves over a wavy sea bottom. In their study, a nonlinear wave-wave interaction theory was imposed to analyse different Bragg resonance conditions for different combinations of waves and bottom undulations. Apparently, nonlinear wave-wave interaction theory could explain and predict the Bragg-scattering very accurately.

Further, Ting *et al.* [89] developed a time dependent wave equation for waves propagating with a current over permeable ripple beds. A one-dimensional wave field was solved numerically based on the derived equation to study the effect of current on the Bragg resonance condition. His numerical results indicated that the maximum reflection coefficient increases as current velocity increases from negative to positive value. Furthermore, the velocity of the current affected the position of the maximum reflection coefficient.

In coastal areas, porous structures are widely used as breakwaters to protect harbours, inlets and beaches from wave action, and as dissipating sea walls to attenuate the wave energy in harbours. The most widely used model of wave-induced flow within a porous medium is the one developed by Sollitt and Cross [86]. According to their approach, dissipation of wave energy inside porous medium was taken into account through a linearised friction term. Thereafter, several authors carried series of theoretical and experimental investigations on the phenomenon of wave interaction with porous bodies. Madsen [48] derived a simple solution for reflection and transmission from a rectangular porous structure under normal incidence of long waves based on the linearised form of the governing equations and linearised form of the flow resistance formula. Madsen [49] presented a theoretical solution for the reflection of linear shallow-water waves from a vertical porous wave absorber on a horizontal bottom. Gu and Wang [28] investigated water wave interactions with porous seabeds of granular materials theoretically and experimentally. Dalrymple *et al.* [16] adopted the Sollitt and Cross approach to analyse the reflection and transmission of oblique incident waves from infinitely long porous structures. Yu and Chwang [92] investigated the wave motion through a two layer porous structure and derived theoretical solutions for the wave reflection and transmission from a two-layer porous breakwaters.

Almost all the above existing investigations on the wave reflection and/or transmission problem of porous structures are restricted by assuming a horizontal sea-bed. A few authors, such as Mallayachari and Sundar [51], took into account the effects of an uneven sea-bed. However, in practical coastal engineering, an undulating sea-bed is more likely to be encountered. Several numerical models have been developed to understand wave propagation due to the bottom change, some of which assume an impermeable or a finite porous bottom.

Several studies have been carried out on wave propagation over an impermeable bed of any configuration. One such configuration is slowly varying depth for which mild-slope equation has been derived by many authors. These investigators centred their interests only on the wave motion of free water region. They did not include the wave motion in porous media or wave interaction with porous structures. In another development, investigations have also been directed to model wave transformation over a submerged porous structure and porous media.

Based on the Sollitt and Cross approach and on the assumption of mild-slope bottom, an analytical derivation of the basic equation for breaking and non-breaking waves travelling over a general finite porous bed was given by Rojanakamthorn *et al.* [82, 83]. Mase and Takeba [59] derived a particular equation for waves propagating over rapidly varying porous rippled beds with small amplitude, superimposed on both a slowly varying mean water depth and a gently varying thickness of porous layer. Then Zhu and Chwang [94, 95] studied the wave reflection from a composite porous absorber lying on a solid foundation with seaward slope. They applied an extended mild-slope equation for surface waves (Porter and Staziker [78]) to the sloping region in front of the porous absorber. More recently, by applying the Sollitt and Cross approach and the Galerkin eigenfunction expansions, Zhu [93] presented an idealised model for the evaluation of water wave refraction-diffraction within a porous medium on an undulating sea-bed. For non-dissipative porous medium, his present model reduces to that of Massel [60] and Porter and Staziker [78].

Many submerged structures built with slopes may produce inaccuracies. So Silva *et al.* [84] derived the extension of the time-dependent and time-independent *mild-slope equations* based on the equation for waves travelling over a finite porous bed using Green's second identity. This new modified *mild-slope equation* was presented by extending the theory of Kirby [40] and Mase and Takeba [59], which was applicable to structures with sharp slopes. Their model also accurately predicted the results using the solution presented by Mase and Takeba [59].

In the present thesis, we study the mixed boundary value problems of two and three dimensional Laplace's equation arising out of the physical problems of scattering of linearised surface water waves by small undulation on the sea-bed, including porous ones, for the case of finite depth. We solve these physical problems by different mathematical techniques. Our emphasis is on the treatment of this class of surface water wave scattering problems and to find the analytical solutions for the reflection and transmission coefficients. All the methods presented here are observed to be most suitable to handle the class of scattering problems under consideration.

In the following section a brief description of the basic equations of linearised theory of water waves for the case of uniform finite depth are presented in consultation with the classic treatise of Lamb [44], Stoker [88], Lighthill [45], Crapper [8] and Dean and Dalrymple [21].

1.3 Basic equations in linearised theory of water waves

In this section, we derive the basic equations associated with the linearised theory of water waves that are relevant to this thesis. The various boundary conditions corresponding to the free surface (without surface tension effect) and uniform finite depth are derived in brief.

We consider the motion of a homogeneous, incompressible and inviscid fluid with irrotational flow with constant volume density ρ under the action of gravity only. A right-handed rectangular Cartesian coordinate system is chosen in which the positive y-axis is taken vertically downwards and the horizontal xz-plane is taken along the undisturbed free surface.

The fluid occupies the region $-\infty < x, z < \infty,\ 0 \leq y \leq h$. In case of water of uniform finite depth h, the bottom surface is $y = h$. Since the motion is irrotational there exists a velocity potential $\Phi(x, y, z, t)$ such that the fluid velocity $\overrightarrow{q} = (u, v, w)$ can be expressed as

$$\overrightarrow{q} = \nabla\Phi. \tag{1.1}$$

The equation of continuity for such case is

$$\nabla.\overrightarrow{q} = 0 \tag{1.2}$$

and the Euler's equation of motion is

$$\frac{\partial \overrightarrow{q}}{\partial t} + (\overrightarrow{q}.\nabla)\overrightarrow{q} = g - \frac{1}{\rho}\nabla p\,, \tag{1.3}$$

where g is the gravitational constant and p is the pressure of the fluid.
Using equation (1.1), the equation of continuity becomes

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \text{ in the fluid region} \tag{1.4}$$

which is Laplace's equation.
After integration, Euler's equation of motion reduces to (Appendix I)

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}[u^2 + v^2 + w^2] + \frac{p}{\rho} - gy = 0, \tag{1.5}$$

which is known as Bernoulli's equation. Since the pressure at the free surface must be equal to the atmospheric pressure which is constant and may be taken to be equal to zero, the equation (1.5) gives rise to

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}[u^2 + v^2 + w^2] - g\eta = 0 \quad \text{on} \quad y = \eta(x, z, t). \tag{1.6}$$

This is known as the *dynamic boundary condition* at the free surface. The free surface is given by equation

$$F(x, y, z, t) = y - \eta(x, z, t) = 0, \tag{1.7}$$

where $\eta(x, z, t)$ denotes the free surface depression below mean water level $y = 0$ (if y is vertically upward then $\eta(x, z, t)$ will be the vertical displacement of free surface above mean water level $y = 0$).
For small motion, the term $|\overrightarrow{q}|^2$ is neglected so as to get the linearised form of the Bernoulli's equation as

$$\frac{\partial\Phi}{\partial t}(x, y, z, t) = g\eta(x, z, t) \;\; \text{on } y = \eta(x, z, t). \tag{1.8}$$

Expanding $\frac{\partial \Phi}{\partial t}$ about $y = 0$ by Taylor's series and neglecting the terms of second and higher orders of smallness, this reduces to the linearised dynamical boundary condition at the free surface given by

$$\frac{\partial \Phi}{\partial t} = g\eta \quad \text{on } y = 0. \tag{1.9}$$

If a fixed or moving surface varies with time, as would the water surface, then the total derivative of the surface with respect to time would be zero on the surface. In other words, if we move with the surface, it doesn't change (Dean and Dalrymple [21]). Hence,

$$\begin{aligned} &\frac{DF}{Dt}(x, y, z, t) = 0 \quad \text{on } F(x, y, z, t) = 0 \\ \Leftrightarrow \; &\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0. \end{aligned} \tag{1.10}$$

Substituting equation (1.7) in (1.10), we get

$$\frac{\partial \Phi}{\partial y}(x, y, z, t) = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial \eta}{\partial z}\frac{\partial \Phi}{\partial z} \qquad \text{on } y = \eta(x, z, t). \tag{1.11}$$

In other words, the vertical velocity component v is equal to the rate of rise of the water surface at any point. Mathematically

$$\begin{aligned} v &= \frac{d\eta}{dt}(x, z, t) \\ \Rightarrow \frac{\partial \Phi}{\partial y} &= \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + w\frac{\partial \eta}{\partial z} \qquad \text{on} \quad y = \eta(x, z, t) \\ \Rightarrow \frac{\partial \Phi}{\partial y} &= \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial \eta}{\partial z}\frac{\partial \Phi}{\partial z} \quad \text{on} \quad y = \eta(x, z, t). \end{aligned} \tag{1.12}$$

This is known as the *kinematic boundary condition* at the free surface.

Under the assumption of the linearised theory, the velocity components and the free surface depression together with their partial derivatives are small quantities, so their squares, higher powers and the products can be neglected so that this equation becomes

$$\frac{\partial \Phi}{\partial y}(x, y, z, t) = \frac{\partial \eta}{\partial t} \quad \text{on } y = \eta(x, z, t). \tag{1.13}$$

Again, expanding $\frac{\partial \Phi}{\partial y}(x, y, z, t)$ about $y = 0$ and neglecting terms of second and higher orders of smallness, the linearised kinematic boundary condition at the free surface is obtained as

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \eta}{\partial t} \quad \text{on} \quad y = 0. \tag{1.14}$$

Eliminating η between equations (1.9) and (1.14), we obtain the linearised free surface condition as

$$\frac{\partial^2 \Phi}{\partial t^2} - g\frac{\partial \Phi}{\partial y} = 0 \quad \text{on} \quad y = 0. \tag{1.15}$$

Once Φ is obtained the free surface depression $\eta(x,z,t)$ can be obtained by using the relation

$$\eta(x,z,t) = \frac{1}{g}\frac{\partial \Phi}{\partial t}(x,0,z,t). \tag{1.16}$$

Across any fixed impermeable obstacle, through which the fluid cannot pass, we have the condition that

$$\frac{\partial \Phi}{\partial n} = 0 \quad \text{on } L, \tag{1.17}$$

where L is the boundary of the obstacle and n is the outward normal to L and $\partial/\partial n$ denotes the normal derivative at a point (x,y,z) on L. In case of water of uniform finite depth h below the mean free surface, we have the condition

$$\frac{\partial \Phi}{\partial y} = 0 \quad \text{on} \quad y = h. \tag{1.18}$$

The equations (1.4), (1.15) and (1.18) are the basic equations of the linearised theory of water waves.
If the motion is to be simple harmonic in time with angular frequency σ, then the velocity potential Φ can be expressed as

$$\Phi(x,y,z,t) = \mathbf{Re}[\phi(x,y,z)e^{-i\sigma t}], \tag{1.19}$$

where ϕ is the complex-valued potential.

Under this assumption equations (1.4), (1.15) and (1.18) become

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{in the fluid region}, \tag{1.20}$$

$$\frac{\partial \phi}{\partial y} + K\phi = 0 \quad \text{on } y = 0, \tag{1.21}$$

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = h, \tag{1.22}$$

where $K = \sigma^2/g$.
Equations (1.20), (1.21) and (1.22) are also the basic equations of the linearised theory of water waves for time harmonic irrotational motion in the fluid.

The formulation for the two-dimensional problem can, similarly, be written as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{in the fluid region}, \tag{1.23}$$

$$\frac{\partial \phi}{\partial y} + K\phi = 0 \quad \text{on } y = 0, \tag{1.24}$$

$$\frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = h. \tag{1.25}$$

Further, as far as Laplace's equation is concerned, it has two types of solutions: the wave part (the progressive wave solution) and the non wave part (local solution).

Solution for the potential:

For two-dimensional motion, the solution of the Laplace equation representing progressive waves is given by

$$\phi(x, y) = \cosh k_0(h - y)e^{\pm ik_0 x}, \tag{1.26}$$

where k_0 is the unique real and positive root of the transcendental equation

$$K = k \tanh kh. \tag{1.27}$$

The local solutions are given by

$$\phi(x, y) = \cos k_n(h - y)e^{-k_n|x|}, \tag{1.28}$$

where k_n are the positive real roots of the transcendental equation

$$K + \kappa \tan \kappa h = 0. \tag{1.29}$$

These k_n's $(n = 1, 2, 3, \ldots)$ are nothing but the purely imaginary roots of equation (1.27). It may be noted that equation (1.27) has only the roots $\pm k_0$ and $\pm ik_n (n = 1, 2, 3, \ldots)$ and there is no other root.

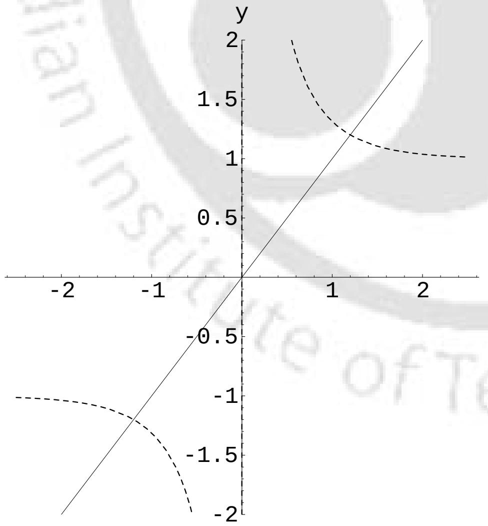

Figure 1.1: *Roots of* $k \tanh kh = K$ *when* $Kh = 1$

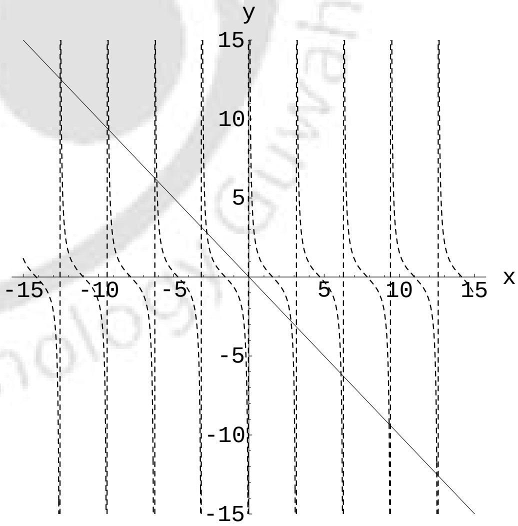

Figure 1.2: *Roots of* $K + \kappa \tan \kappa h = 0$ *when* $Kh = 1$

Value of Kh	Roots of $kh\ tanh(kh) - Kh = 0$
$Kh = 0.5$	0.7717, -0.7717
$Kh = 1.0$	1.1997, -1.1997
$Kh = 1.5$	1.6218, -1.6218
$Kh = 2.0$	2.0653, -2.0653
$Kh = 2.5$	2.5318, -2.5318
$Kh = 3.0$	3.0145, -3.0145
$Kh = 3.5$	3.5063, -3.5063
$Kh = 4.0$	4.0027, -4.0027
$Kh = 4.5$	4.5011, -4.5011
$Kh = 5.0$	5.0005, -5.0005

Table 1.1: *Roots of* $kh\ tanh(kh) - Kh = 0$ *for different values of* Kh

Value of Kh	Roots of $k_nh\ tan(k_nh) + Kh = 0$
$Kh = 0.4$	3.0095
	6.2190
	9.3822
	12.5345
	15.6825

Table 1.2: *Roots of* $k_nh\ tan(k_nh) + Kh = 0$ *for* $Kh = 0.4$

Value of Kh	Roots of $k_nh\ tan(k_nh) + Kh = 0$
$Kh = 1$	2.7984
	6.1213
	9.3179
	12.4865
	15.6441

Table 1.3: *Roots of* $k_nh\ tan(k_nh) + Kh = 0$ *for* $Kh = 1$

For three-dimensional motion with harmonic variation along the z-direction, the velocity potential $\Phi(x, y, z, t)$ can be represented as

$$\Phi(x, y, z, t) = \mathbf{Re}\left\{\phi(x, y)e^{i(\nu z - \sigma t)}\right\}, \tag{1.30}$$

where, now, $\phi(x, y)$ satisfies the Helmholtz equation with parameter ν, given by

$$(\nabla^2 - \nu^2)\phi = 0, \quad \text{in the fluid region.} \tag{1.31}$$

For water of uniform finite depth h, the progressive wave solutions are

$$\phi(x, y) = \cosh k_0(h - y)e^{\pm i(k_0^2 - \nu^2)^{1/2}x} \quad (\nu < k_0), \tag{1.32}$$

and the local solutions are

$$\phi(x, y) = \cos k_n(h - y)e^{-(k_n^2 + \nu^2)^{1/2}|x|}. \tag{1.33}$$

As far as the scattering and reflection of surface water waves are concerned, the progressive wave solution plays an important role. The sign in the coefficient of x depends upon whether the wave is an incoming or an outgoing one. It is known that the potential function ϕ behaves like a progressive wave at large distance (i.e. as $|x| \to \infty$) on the surface and are known as the far-field conditions or infinity conditions in the surface water wave theory.

1.4 Outline of the thesis

The work presented in the thesis is solely concerned with the effect of an undulating bed (including porous beds) on an incident wave train using linearised theory. We prescribe the incident wave train and the deviation in the still-water depth, and seek the additional waves, the scattered waves, caused by this deviation. A typical problem of this type requires the determination of a velocity potential satisfying Laplace's equation within the fluid, a mixed boundary condition on the free surface, and a given normal velocity on the rigid boundaries. If the fluid domain extends to infinity, a radiation condition is required to ensure uniqueness. This boundary value problem is well known and is formally presented in the subsequent chapters. We shall refer to the problem of finding a solution of the boundary value problem as the full linear problem. Analytical solutions of the full linear problem are rare for any deviation from the constant water depth case, that is, for any deviation from a flat sea-bed. Problems where analytical solutions exist are usually for a limited selection of straightforward geometries which include horizontal and/or vertical boundaries.

The mathematical tools utilised in this thesis are (i) Green's function technique, and application of Green's integral theorem, (ii) Generalised Fourier transform technique and application of residue theorem, (iii) Finite cosine transform, (iv) The eigenfunction expansion method.

The content of the thesis is presented in the form of eight chapters. Chapter 1 is devoted to the general introduction along with the historical background of the work done in the scattering theory and the derivation of the basic hydrodynamic equations in the linearised theory of water waves that is utilised in the thesis. In Chapter 2, we formulate the problem for the scattering of surface waves by small undulation of the sea-bed for normal and oblique incidence. Then by using a perturbation analysis which involves the small parameter ε, the governing boundary value problem arising out from the physical problem under consideration is reduced first to a simpler boundary value problem for the first order correction of the potential for both cases of normal and oblique incidence.

In Chapter 3, we solve the reduced boundary value problem for the first order correction of the potential for normal incidence as formulated in Chapter 2, obtaining the solutions by using three different techniques, namely Green's function and a suitable application of Green's integral theorem; Fourier transform; and finite cosine transform. From this solution the quantities of physical interest, namely the reflection and transmission coefficients, are evaluated up to the first order of ε in terms of integrals involving the shape function representing the bottom undulation.

In Chapter 4, we solve the reduced boundary value problem for first order correction of the potential for oblique incidence as formulated in Chapter 2. The solution of the boundary value problem for the first-order correction is obtained by using the same three techniques as applied in Chapter 3. Using this solution the reflection and transmission coefficients are evaluated up to the first order of ε in terms of integrals involving the shape function representing the bottom undulation.

In Chapter 5, different special forms of the undulations such as an exponentially damped undulation, a finite single hump and a patch of sinusoidal ripples are considered to evaluate the integrals explicitly and numerically for the reflection and transmission coefficients obtained in Chapters 3 and 4.

Chapter 6 is concerned with the same physical problem the solution of which is obtained by eigenfunction expansion method where both propagating and non-propagating modes of the wave are considered in the solution approach. A patch of sinusoidal ripples, a special form of the bottom undulation, is considered in order to evaluate the integrals for the reflection and transmission coefficients, and the results are compared with the ones obtained in earlier chapters.

Chapter 7 is devoted to the problem involving the scattering of surface water waves by small undulation on a porous sea-bed for both normal and oblique incidences. Using a perturbation analysis, the governing boundary value problem is reduced to a simpler boundary value problem which involves a small parameter ε present in the representation of the small undulation of the porous sea-bed. Fourier transform technique is applied to the simpler boundary value problem from which the reflection and transmission coefficients are determined. A patch of sinusoidal ripples is considered as a shape function, to evaluate explicitly the integrals for the reflection and transmission coefficients.

Finally, Chapter 8 is concerned with a brief summary of results highlighting the contribution made by this thesis. It also provides information for the scope for future investigations.

Chapter 2

Formulation for Scattering of Water Waves by Small Undulation of a Sea-bed

2.1 Introduction

In this chapter, the statement and formulation for the problem of scattering of surface waves by small undulation of the bottom of a laterally unbounded sea, using linear water wave theory, for both normal and oblique incidence are presented. We use a regular perturbation expansion in terms of a small undulation parameter ε to solve the governing boundary value problem associated with Laplace's equation and mixed type boundary conditions. The original problem then gets reduced to a simpler boundary value problem for the first order correction of the velocity potential.

The mathematical treatment of a phenomenon always requires the removal of "natural arbitrariness", the replacement of natural conditions by their idealised counterparts, and water wave mechanics is also no exception. While setting up the boundary value problem here, some of the assumptions and boundary conditions are based on idealised conditions.

2.2 Normal incidence

A right-handed rectangular Cartesian co-ordinate system is considered in which x-axis is measured along the undisturbed free surface of the sea and y-axis is measured positive vertically downwards from the undisturbed free surface.

The bottom of the sea with small undulation is described by $y = h + \varepsilon c(x)$ where $c(x)$ is a function with compact support and describes the bottom undulation, h denotes

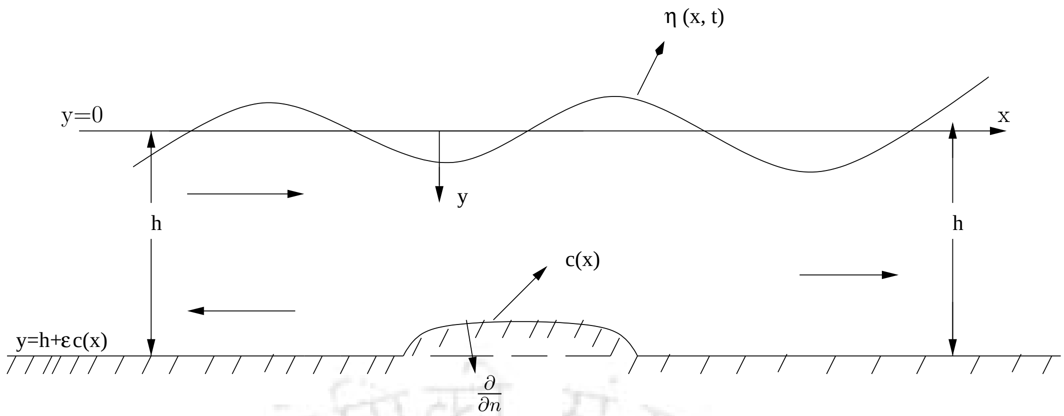

Figure 2.1: *The problem domain*

the uniform finite depth of sea far to either side of the undulation of the bottom so that $c(x) \to 0$ as $|x| \to \infty$ and the non-dimensional number $\varepsilon(\ll 1)$ a measure of smallness of the undulation. It is also assumed that the fluid is incompressible and inviscid, and the motion is irrotational. Assuming the linear theory, our interest is to solve for the complex-valued potential function $\phi(x, y)$ describing the small motion in water satisfying

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad 0 \le y \le h + \varepsilon c(x), \tag{2.1}$$

$$\frac{\partial\phi}{\partial y} + K\phi = 0, \quad y = 0, \tag{2.2}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad y = h + \varepsilon c(x), \tag{2.3}$$

where $K = \sigma^2/g, \sigma$ is the angular frequency of the incoming water wave train with time dependence $e^{-i\sigma t}$, g is the acceleration due to gravity, and $\partial/\partial n$ denotes the normal derivative at a point (x, y) on the bottom. The time dependent term is dropped throughout the analysis.

It is assumed that a progressive wave train represented by the velocity potential

$$\phi_0(x, y) = \cosh k_0(h - y)e^{ik_0 x}, \tag{2.4}$$

is incident upon the bottom undulation from negative infinity where k_0, the wave number of the incident wave, is the unique positive root of the equation (1.27).

It is, then, partially reflected by and partially transmitted over the undulation so that ϕ has far-field behaviour given by

$$\phi(x, y) \sim \begin{cases} \phi_0(x, y) + R\phi_0(-x, y), & x \to -\infty, \\ T\phi_0(x, y), & x \to +\infty, \end{cases} \tag{2.5}$$

where R and T are the usual reflection and transmission coefficients in water wave problems: the ratio of amplitudes of the reflected and transmitted waves respectively to that of the incident wave, and are to be determined. Hence,

$$\phi(x,y) \sim \begin{cases} (e^{ik_0x} + Re^{-ik_0x}) \cosh k_0(h-y), & x \to -\infty, \\ Te^{ik_0x} \cosh k_0(h-y), & x \to +\infty. \end{cases} \tag{2.6}$$

The bottom condition $\partial\phi/\partial n = 0$ on $y = h + \varepsilon c(x)$ can be approximated up to the first order of the small parameter ε as (Appendix J)

$$\frac{\partial \phi}{\partial y} - \varepsilon \frac{\partial}{\partial x}\left\{c(x)\frac{\partial \phi}{\partial x}\right\} = 0 \quad \text{on} \quad y = h. \tag{2.7}$$

The boundary condition (2.7) and the fact that a wave train propagating in a sea of uniform finite depth experiences no reflection, together suggest that ϕ, R and T introduced above can be expressed in terms of the small parameter ε as

$$\left.\begin{array}{l} \phi = \phi_0 + \varepsilon\phi_1 + O(\varepsilon^2) \\ R = \varepsilon R_1 + O(\varepsilon^2) \\ T = 1 + \varepsilon T_1 + O(\varepsilon^2) \end{array}\right\}. \tag{2.8}$$

Using (2.8) in equations (2.1), (2.2), (2.7) and (2.6) we find that $\phi_1(x,y)$ satisfies the following BVP described by

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0 \quad \text{in} \ -\infty < x < \infty, \ 0 \le y \le h, \tag{2.9}$$

$$\frac{\partial \phi_1}{\partial y} + K\phi_1 = 0 \quad \text{on} \quad y = 0, \tag{2.10}$$

$$\frac{\partial \phi_1}{\partial y} = ik_0 \frac{d}{dx}\{c(x)e^{ik_0x}\} \equiv p(x) \quad \text{on} \quad y = h, \tag{2.11}$$

$$\text{and} \quad \phi_1(x,y) \sim \begin{cases} R_1 e^{-ik_0x} \cosh k_0(h-y), & \text{as} \quad x \to -\infty, \\ T_1 e^{ik_0x} \cosh k_0(h-y), & \text{as} \quad x \to +\infty. \end{cases} \tag{2.12}$$

Consequently we are interested to solve the above BVP, given by equations (2.9)-(2.12), instead of solving the BVP represented by equations (2.1)-(2.3) and (2.6).

2.3 Oblique incidence

A right-handed rectangular Cartesian co-ordinate system is used in which xz-plane is the undisturbed free surface of the sea and y-axis is measured positive vertically downwards from the undisturbed free surface. The representation of the bottom of the sea with small undulation is the same as the one described in Section 2.2. Considering the same assumptions as described in Section 2.2 the complex-valued potential function $\psi(x,y,z)$ describing the small motion in water satisfies

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2} = 0 \quad \text{in} \quad 0\le y\le h+\varepsilon c(x), \tag{2.13}$$

$$\frac{\partial\psi}{\partial y}+K\psi = 0 \quad \text{on} \quad y=0, \tag{2.14}$$

$$\frac{\partial\psi}{\partial n} = 0 \quad \text{on} \quad y=h+\varepsilon c(x). \tag{2.15}$$

Here, $\partial/\partial n$ denotes the normal derivative at a point (x,y,z) on the bottom.

It can be assumed that a progressive wave train represented by the velocity potential

$$\psi_0(x,y,z)=\cosh k_0(h-y)e^{i(\mu x+\nu z)} \tag{2.16}$$

is obliquely incident upon the bottom undulation from negative infinity, where

$$\mu=k_0\cos\theta, \quad \nu=k_0\sin\theta, \tag{2.17}$$

where θ is the angle of incidence of the wave train ($\theta=0$ corresponds to normal incidence), μ and ν are, respectively, the x and z components of k_0, the wave number of the incident wave, which is the unique positive real root of equation (1.27).

It is, then, partially reflected by and partially transmitted over the undulation so that ψ has the asymptotic behaviour given by

$$\psi(x,y,z)\sim\begin{cases}\psi_0(x,y,z)+R\psi_0(-x,y,z) & \text{as} \quad x\to-\infty,\\ T\psi_0(x,y,z) & \text{as} \quad x\to+\infty,\end{cases}$$
$$\Rightarrow \psi(x,y,z)\sim\begin{cases}(e^{i\mu x}+Re^{-i\mu x})e^{i\nu z}\cosh k_0(h-y) & \text{as} \quad x\to-\infty,\\ Te^{i(\mu x+\nu z)}\cosh k_0(h-y) & \text{as} \quad x\to+\infty,\end{cases} \tag{2.18}$$

where R and T have the usual meanings.

Assuming ε to be very small and neglecting $O(\varepsilon^2)$ terms, the boundary condition $\partial\psi/\partial n=0$ on $y=h+\varepsilon c(x)$ can be expressed in an appropriate form as

$$\frac{\partial\psi}{\partial y}-\varepsilon\left[\frac{\partial}{\partial x}\left\{c(x)\frac{\partial\psi}{\partial x}\right\}+c(x)\frac{\partial^2\psi}{\partial z^2}\right]+O(\varepsilon^2)=0 \quad \text{on} \quad y=h. \tag{2.19}$$

Now, in view of the geometry of the problem, i.e, because of the uniformity in the $z-$direction, $\psi(x,y,z)$ can be written as

$$\psi(x,y,z)=\phi(x,y)e^{i\nu z}. \tag{2.20}$$

Then $\phi(x,y)$ satisfies the equations

$$(\nabla^2-\nu^2)\phi = 0 \quad \text{in} \quad -\infty < x < \infty,\ 0 \le y \le h, \tag{2.21}$$

$$\frac{\partial \phi}{\partial y} + K\phi = 0 \quad \text{on} \quad y = 0, \tag{2.22}$$

$$\frac{\partial \phi}{\partial y} - \varepsilon\left[\frac{\partial}{\partial x}\left\{c(x)\frac{\partial \phi}{\partial x}\right\} - \nu^2 c(x)\phi(x,y)\right] + O(\varepsilon^2) = 0 \quad \text{on} \quad y = h, \tag{2.23}$$

and

$$\phi(x,y) \sim \begin{cases} (e^{i\mu x} + Re^{-i\mu x})\cosh k_0(h-y) & \text{as} \quad x \to -\infty, \\ Te^{i\mu x}\cosh k_0(h-y) & \text{as} \quad x \to +\infty, \end{cases} \tag{2.24}$$

where ∇^2 is the two-dimensional Laplacian operator.

The form of the approximate boundary condition (2.23) and the fact that a wave train propagating in an sea of uniform finite depth experiences no reflection, together suggest that ϕ, R and T introduced above can be expressed in terms of the small parameter ε as

$$\left.\begin{aligned} \phi &= \cosh k_0(h-y)e^{i\mu x} + \varepsilon\phi_1 + O(\varepsilon^2) \\ R &= \varepsilon R_1 + O(\varepsilon^2) \\ T &= 1 + \varepsilon T_1 + O(\varepsilon^2) \end{aligned}\right\}. \tag{2.25}$$

Using equation (2.25) in equations (2.21)-(2.24) we find that $\phi_1(x,y)$ satisfies the following BVP described by

$$(\nabla^2-\nu^2)\phi_1 = 0 \qquad \text{in} \quad -\infty < x < \infty,\ 0 < y < h, \tag{2.26}$$

$$\frac{\partial \phi_1}{\partial y} + K\phi_1 = 0 \qquad \text{on} \quad y = 0, \tag{2.27}$$

$$\frac{\partial \phi_1}{\partial y} = i\mu\frac{d}{dx}\left\{c(x)e^{i\mu x}\right\} - \nu^2 c(x)e^{i\mu x} \equiv V(x) \text{ on } y = h, \tag{2.28}$$

$$\text{and} \quad \phi_1(x,y) \sim \begin{cases} R_1 e^{-i\mu x}\cosh k_0(h-y) & \text{as} \quad x \to -\infty, \\ T_1 e^{i\mu x}\cosh k_0(h-y) & \text{as} \quad x \to +\infty. \end{cases} \tag{2.29}$$

Consequently we are interested to solve the above BVP given by equations (2.26)-(2.29) instead of solving the BVP represented by equations (2.13)-(2.15) and (2.18) for the case of oblique incidence.

Chapter 3

Solution to the Scattering Problem for Normal Incidence

3.1 Introduction

In the last chapter we formulated the problem for scattering of surface waves by small undulation of the bottom of a laterally unbounded sea using linear water wave theory for both normal and oblique incidences. In this chapter, we solve the BVP for normal incidence by using three different techniques, namely, Green's function and a suitable application of Green's integral theorem; Fourier transform and finite cosine transform. From the solution of the BVP, the quantities of physical interest, namely the reflection and transmission coefficients, are evaluated up to the first order of ε in terms of integrals involving the shape function $c(x)$.

3.2 Solution by Green's function technique

In this section, the simpler BVP for the normal incidence of surface waves is solved by using Green's function. The solution for the first order correction to the velocity potential is obtained from which the reflection and transmission coefficients are evaluated.

3.2.1 Solution procedure

To solve the above boundary value problem (equations (2.9) -(2.12)) we require the Green's function $G(x, y; \xi, \eta)$ which satisfies the following equations:

$$\nabla^2 G = 0 \qquad \text{in} \quad -\infty < x < \infty,\ 0 \leq y \leq h, \tag{3.1}$$
$$\text{except at} \quad (\xi, \eta) \text{ where } 0 < \eta < h,$$
$$\frac{\partial G}{\partial y} + KG = 0 \quad \text{on} \quad y = 0, \tag{3.2}$$
$$\frac{\partial G}{\partial y} = 0 \qquad \text{on } y = h, \tag{3.3}$$
$$G \sim \ln r \qquad \text{as} \quad r = \{(x-\xi)^2 + (y-\eta)^2\}^{1/2} \to 0, \tag{3.4}$$
$$G \sim \text{multiple of } \cosh k_0(y-h) e^{ik_0|x-\xi|} \quad \text{as} \quad |x-\xi| \to \infty. \tag{3.5}$$

The condition (3.5) implies that G represents an outgoing wave as $|x - \xi| \to \infty$. The solution of the above BVP (equations (3.1)-(3.5)) is given by Mandal and Chakrabarti [55] as,

$$G(x, y; \xi, \eta) = \frac{-4\pi i \cosh k_0(h-y) \cosh k_0(h-\eta) e^{ik_0|x-\xi|}}{2k_0 h + \sinh 2k_0 h} - \sum_{n=1}^{\infty} \frac{4\pi \cos k_n(h-y) \cos k_n(h-\eta) e^{-k_n|x-\xi|}}{2k_n h + \sin 2k_n h}, \tag{3.6}$$

where k_n are real and positive roots of (1.29).

The behaviour of G when $|x| \to \infty$ is given by the first term of equation (3.6). Now applying the Green's integral theorem to $\phi_1(x, y)$ and $G(x, y; \xi, \eta)$, we get

$$\int_C \left(\phi_1 \frac{\partial G}{\partial n} - G \frac{\partial \phi_1}{\partial n} \right) ds = 0, \tag{3.7}$$

where C is a closed contour bounded externally by the lines $y = 0, y = h$ $(-X \leq x \leq X)$, $x = \pm X$ $(0 \leq y \leq h)$ and internally by a small circle of radius δ with centre at (ξ, η), as shown in Figure 3.1, and ultimately letting $X \to \infty$ and $\delta \to 0$.

Now the contribution to the integral in equation (3.7) from the small circle as its radius $\delta \to 0$ is $2\pi\phi_1(\xi, \eta)$ and the contribution from the line $y = 0, -X \leq x \leq X$ (i.e. the free surface) is

$$\int_{-X}^{X} \left(\phi_1 \frac{\partial G}{\partial y} - G \frac{\partial \phi_1}{\partial y} \right)_{y=0} dx = 0.$$

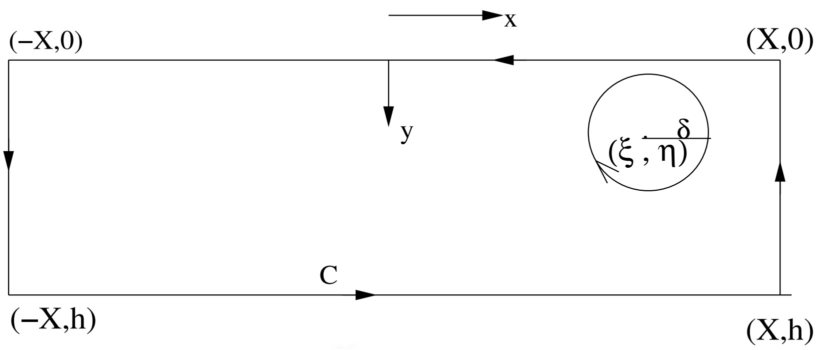

Figure 3.1: *Contour Integration*

Also there will be no contribution from the lines $x = \pm X, (0 \leq y \leq h)$ due to the conditions satisfied by ϕ_1 and G as $|x| \to \infty$. Finally, the contribution from the line $y = h, (-X \leq x \leq X)$ is

$$\int_{-X}^{X} \left(\phi_1 \frac{\partial G}{\partial y} - G \frac{\partial \phi_1}{\partial y} \right)_{y=h} dx$$
$$= -\int_{-\infty}^{\infty} G(x, h; \xi, \eta)\ p(x)\ dx \text{ as } X \to \infty.$$

Thus, collecting all the results, the integral equation (3.7) will give rise to the determination of ϕ_1 as:

$$\phi_1(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x, h; \xi, \eta)\ p(x)\ dx, \tag{3.8}$$

which solves the boundary value problem for $\phi_1(x, y)$.

3.2.2 Reflection and transmission coefficients

The reflection and transmission coefficients, R_1 and T_1, are now obtained by letting $\xi \to \mp\infty$ and using the far-field condition (2.12) (with (x, y) replaced by (ξ, η)) in equation (3.8). For this, from (3.6) we require the result

$$G(x, y; \xi, \eta) = \frac{-4\pi i \cosh k_0(h-y) \cosh k_0(h-\eta) e^{ik_0|x-\xi|}}{2k_0h + \sinh 2k_0h} \quad \text{as } |x - \xi| \to \infty. \tag{3.9}$$

To find R_1, we note from the far-field condition (2.12) and equation (3.9), respectively, that

$$\phi_1(\xi, \eta) = R_1 \cosh k_0(h-\eta) e^{-ik_0\xi} \quad \text{as} \quad \xi \to -\infty, \tag{3.10}$$

$$\text{and} \qquad G(x, h; \xi, \eta) = \frac{-4\pi i \cosh k_0(h-\eta) e^{ik_0(x-\xi)}}{2k_0h + \sinh 2k_0h} \quad \text{as} \quad \xi \to -\infty. \tag{3.11}$$

Substituting equations (3.10) and (3.11) in equation (3.8), we obtain

$$\begin{aligned} R_1 &= \frac{-2i}{2k_0h+\sinh 2k_0h}\int_{-\infty}^{\infty} e^{ik_0x}p(x)dx \\ &= \frac{-2ik_0^2}{2k_0h+\sinh 2k_0h}\int_{-\infty}^{\infty} e^{2ik_0x}c(x)dx. \end{aligned} \tag{3.12}$$

Again, to find T_1, we note from the far-field condition (2.12) and equation (3.9), respectively, that

$$\phi_1(\xi,\eta) = T_1 \cosh k_0(h-\eta)e^{ik_0\xi} \quad \text{as} \quad \xi \to +\infty\ , \tag{3.13}$$

and $$G(x,h;\xi,\eta) = \frac{-4\pi i \cosh k_0(h-\eta)e^{-ik_0(x-\xi)}}{2k_0h+\sinh 2k_0h} \quad \text{as} \quad \xi \to +\infty. \tag{3.14}$$

Using equations (3.13) and (3.14) in equation (3.8), we obtain

$$\begin{aligned} T_1 &= \frac{-2i}{2k_0h+\sinh 2k_0h}\int_{-\infty}^{\infty} e^{-ik_0x}p(x)dx \\ &= \frac{2ik_0^2}{2k_0h+\sinh 2k_0h}\int_{-\infty}^{\infty} c(x)dx. \end{aligned} \tag{3.15}$$

The equations (3.12) and (3.15) are equivalent to the equations of Miles [68] for the case of normal incidence and Mandal and Basu [53] for the case of normal incidence in the absence of surface tension.

The reflection and transmission coefficients can be evaluated from equations (3.12) and (3.15) once the shape function $c(x)$ is known.

3.3 Solution by Fourier transform technique

In the previous section, we solved the BVP by using Green's function and by a suitable application of Green's integral theorem. Here we consider the same problem and solve it by a different analytical method. The reduced boundary value problem is solved with the help of Fourier transform technique. While employing this technique it is found that the integrand contains certain singularities. Hence, the residue theorem is used while using contour integration to evaluate the first order correction of the potential. From the solution of the expression of the first order correction to the potential, the reflection and transmission coefficients are evaluated.

3.3.1 Solution procedure

To solve the boundary value problem (equations (2.9) -(2.12)) we now assume that ϕ_1 is such that the Fourier transform of ϕ_1 with respect to x, denoted by $\overline{\phi}_1$, exists and is given by

$$\overline{\phi}_1(\xi, y) = \int_{-\infty}^{\infty} \phi_1(x, y) e^{i\xi x} dx, \tag{3.16}$$

together with the inverse

$$\phi_1(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\phi}_1(\xi, y) e^{-i\xi x} d\xi. \tag{3.17}$$

We observe that such Fourier transform exists if we make an artificial assumption that K possesses a small imaginary part, as given by $i\mu'\sigma/g$, where $\mu' > 0$ is very small which will be taken to be zero (in eliminating sense) at the end of the analysis.

Even instead of K, if we assume k_0 to possess a small imaginary part then also we will arrive at the same conclusion. For, assuming $k_0 = k_0^{(1)} + ik_0^{(2)}$,

$$\begin{aligned}
K &= k_0 \tanh k_0 h \\
&= (k_0^{(1)} + ik_0^{(2)}) \tanh (k_0^{(1)} + ik_0^{(2)}) h \\
&= (k_0^{(1)} + ik_0^{(2)}) \left[\frac{\tanh (k_0^{(1)} h) + i \tan (k_0^{(2)} h)}{1 + i \tanh (k_0^{(1)} h) \tan (k_0^{(2)} h)} \right] \\
&\approx (k_0^{(1)} + ik_0^{(2)}) \left[\frac{\tanh (k_0^{(1)} h) + i (k_0^{(2)} h)}{1 + i (k_0^{(2)} h) \tanh (k_0^{(1)} h)} \right] \\
&\approx (k_0^{(1)} + ik_0^{(2)}) \left[\tanh (k_0^{(1)} h) + ik_0^{(2)} h \right] \left[1 - ik_0^{(2)} h \tanh (k_0^{(1)} h) \right] \\
&\approx (k_0^{(1)} + ik_0^{(2)}) \left[\tanh (k_0^{(1)} h) - i(k_0^{(2)} h) \tanh^2 (k_0^{(1)} h) + i(k_0^{(2)} h) + (k_0^{(2)} h)^2 \tanh (k_0^{(1)} h) \right] \\
&\approx (k_0^{(1)} + ik_0^{(2)}) \left[\tanh (k_0^{(1)} h) - i(k_0^{(2)} h) \{ \tanh^2 (k_0^{(1)} h) - 1 \} \right] \\
&\approx (k_0^{(1)} + ik_0^{(2)}) \left[\tanh (k_0^{(1)} h) + i(k_0^{(2)} h) sech^2 (k_0^{(1)} h) \right] \\
&\approx k_0^{(1)} \tanh (k_0^{(1)} h) + ik_0^{(1)} (k_0^{(2)} h) sech^2 (k_0^{(1)} h) + ik_0^{(2)} \tanh (k_0^{(1)} h) - h (k_0^{(2)})^2 sech^2 (k_0^{(1)} h) \\
&\approx k_0^{(1)} \tanh (k_0^{(1)} h) + ik_0^{(1)} (k_0^{(2)} h) sech^2 (k_0^{(1)} h) + ik_0^{(2)} \tanh (k_0^{(1)} h) \\
&\approx k_0^{(1)} \tanh (k_0^{(1)} h) + ik_0^{(2)} \left[\tanh (k_0^{(1)} h) + hk_0^{(1)} sech^2 (k_0^{(1)} h) \right] \\
&\approx \widehat{K_1} + i\widehat{K_2} \quad (\widehat{K_2} \sim \text{small}).
\end{aligned}$$

Now, taking the Fourier transform of the governing equation (2.9) and boundary conditions (2.10) and (2.11) with respect of the horizontal space variable x, we obtain

$$\frac{\partial^2\overline{\phi}_1}{\partial y^2} - \xi^2\overline{\phi}_1 = 0 \quad \text{in } -\infty < \xi < \infty,\ 0 \leq y \leq h, \tag{3.18}$$

$$\frac{\partial\overline{\phi}_1}{\partial y} + K\overline{\phi}_1 = 0 \quad \text{on } y = 0, \tag{3.19}$$

$$\frac{\partial\overline{\phi}_1}{\partial y} = \Lambda(\xi) \quad \text{on } y = h, \tag{3.20}$$

where

$$\Lambda(\xi) = \int_{-\infty}^{\infty} p(x)e^{i\xi x}dx, \tag{3.21}$$

with $p(x)$ is given by equation (2.11).

The solution of (3.18) is given by

$$\overline{\phi}_1(\xi, y) = C_1(\xi)\cosh\xi y + C_2(\xi)\sinh\xi y. \tag{3.22}$$

Applying the boundary condition (3.19) to (3.22), we obtain

$$C_2(\xi) = \frac{-KC_1(\xi)}{\xi}\ . \tag{3.23}$$

Applying the boundary condition (3.20) to (3.22), we obtain

$$C_1(\xi) = \frac{\Lambda(\xi)}{\xi\sinh\xi h - K\cosh\xi h}\ . \tag{3.24}$$

Substituting the values of $C_1(\xi)$ and $C_2(\xi)$ in equation (3.22) we obtain

$$\overline{\phi}_1(\xi, y) = \frac{\xi\cosh\xi y - K\sinh\xi y}{\xi[\xi\sinh\xi h - K\cosh\xi h]}\Lambda(\xi). \tag{3.25}$$

Taking inverse Fourier transform, the solution for the velocity potential can be written in the form

$$\phi_1(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\xi\cosh\xi y - K\sinh\xi y}{\xi[\xi\sinh\xi h - K\cosh\xi h]}\Lambda(\xi)e^{-i\xi x}d\xi. \tag{3.26}$$

We now obtain the final result from (3.26) by contour integration using the residue theorem.

We observe that the equation (3.26) also has certain singularities (lying on the ξ-axis) other than $\xi = 0$. Replacing K by $\widehat{K} = (\sigma^2 + i\mu'\sigma)/g$ in equation (3.26), the singularities of (3.26) are displaced off the ξ-axis to the upper and the lower half planes. Hence, we write

$$\phi_1(x, y) = \lim_{\mu'\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\xi, y)}{G_{\mu'}(\xi, h)}e^{-i\xi x}d\xi, \tag{3.27}$$

where

$$F(\xi, y) = [\xi \cosh \xi y - \widehat{K} \sinh \xi y]\Lambda(\xi), \tag{3.28}$$
$$G_{\mu'}(\xi, h) = \xi[\xi \sinh \xi h - \widehat{K} \cosh \xi h]. \tag{3.29}$$

If $\widehat{K} = \widehat{K}_1 + i\widehat{K}_2$, then $\widehat{K}_2 = \mu' K/\sigma$ which is very small and if $\zeta = \alpha + i\beta$ is a zero of the expression (3.29), then ζ can be determined up to first order of $\widehat{K}_2$ as

$$\zeta = \pm\alpha_n \pm i\beta_n, \quad \text{and} \quad \zeta = \pm(k_0 + \gamma) \pm i\beta_n', \tag{3.30}$$

where

$$\left.\begin{array}{l} \alpha_n = \widehat{K}_2\alpha_n^{(1)}, \quad \alpha_n^{(1)} = \dfrac{-\cos\beta_n h}{(\widehat{K}_1 h - 1)\sin(\pm\beta_n h) - (\pm\beta_n h)\cos\beta_n h}, \text{ for } \beta_n > 0 \\ \beta_n\text{'s are roots of } \beta\tan\beta h + K = 0, \\ \gamma = \widehat{K}_2\alpha_n'^{(1)}, \quad \alpha_n'^{(1)} = \dfrac{k_0^2 h - \widehat{K}_1[1 + (k_0 h)^2/2]}{\pm k_0\widehat{K}_1\widehat{K}_2 h - (\pm 2k_0\widehat{K}_2 h)}, \\ \beta_n' = \widehat{K}_2\beta_n'^{(1)}, \quad \beta_n'^{(1)} = \dfrac{2 + (k_0 h)^2}{\pm 4k_0 h \pm (k_0 h)^3 - (\pm 2k_0 h^2\widehat{K}_1)} \end{array}\right\}. \tag{3.31}$$

3.3.2 Results

Here the contour consists of the portion $-R$ to R on the real ξ-axis and a semicircle centred at the origin and having a large radius R. The semicircle must be taken in the upper half ζ-plane ($\zeta = \xi + i\eta$) in anticlockwise direction or in the lower half plane in clockwise direction, according as $x < 0$ or $x > 0$. In the limit as $R \to \infty$, the required range of integration is recovered, since the integration along the semicircle makes a zero contribution. Hence, by using the residue theorem,

$$\begin{aligned}\phi_1(x, y) = & \lim_{\mu'\to 0}(-i)\left[\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\zeta, y)\, e^{-i\zeta x}}{G_{\mu'}(\zeta, h)}\right\}\bigg|_{\zeta=\alpha_n+i\beta_n}\right. \\ & \left. +\text{Res}\left\{\frac{F(\zeta, y)\, e^{-i\zeta x}}{G_{\mu'}(\zeta, h)}\right\}\bigg|_{\zeta=(k_0+\gamma)+i\beta_n'}\right] \quad \text{for } x < 0,\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\phi_1(x, y) = & \lim_{\mu'\to 0} i\left[\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\zeta, y)\, e^{-i\zeta x}}{G_{\mu'}(\zeta, h)}\right\}\bigg|_{\zeta=-\alpha_n-i\beta_n}\right. \\ & \left. +\text{Res}\left\{\frac{F(\zeta, y)\, e^{-i\zeta x}}{G_{\mu'}(\zeta, h)}\right\}\bigg|_{\zeta=-(k_0+\gamma)-i\beta_n'}\right] \quad \text{for } x > 0,\end{aligned} \tag{3.33}$$

which imply that

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0}(-i)\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\zeta,y)\ e^{-i\zeta x}}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=\alpha_n+i\beta_n}\\ &+ \frac{-2ik_0^2}{2k_0h+\sinh 2k_0h}\left\{\int_{-\infty}^{\infty}c(x)e^{2ik_0x}dx\right\}\cosh k_0(h-y)e^{-ik_0x}\\ &\text{for } x<0, \end{aligned} \tag{3.34}$$

and

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0} i\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\zeta,y)\ e^{-i\zeta x}}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=-\alpha_n-i\beta_n}\\ &+ \frac{2ik_0^2}{2k_0h+\sinh 2k_0h}\left\{\int_{-\infty}^{\infty}c(x)dx\right\}\cosh k_0(h-y)e^{ik_0x}\\ &\text{for } x>0. \end{aligned} \tag{3.35}$$

The first term on right hand side of each of the equations (3.34) and (3.35) represents the non-propagating modes which decay rapidly away from the undulation and the second term represents a propagating mode from the region of the bed disturbance. Comparing the equations (3.34) and (3.35) with equation (2.12), the reflection and transmission coefficients can, respectively, be written as

$$R_1 = \frac{-2ik_0^2}{2k_0h+\sinh 2k_0h}\int_{-\infty}^{\infty}c(x)e^{2ik_0x}dx, \tag{3.36}$$

and

$$T_1 = \frac{2ik_0^2}{2k_0h+\sinh 2k_0h}\int_{-\infty}^{\infty}c(x)dx. \tag{3.37}$$

R_1 and T_1 obtained in equations (3.36) and (3.37) are same as equations (3.12) and (3.15) obtained in Section 3.2.

3.4 Solution by finite cosine transform technique

In the last two sections, we solved the BVP by utilising two different methods: one was Green's function and by a suitable application of Green's integral theorem, and the other Fourier transform technique. Here we consider the same boundary value problem and solve it by a different method. The reduced boundary value problem is solved by the help of finite cosine transform. By this method the expression for the first order correction to the potential is obtained. From the solution of the expression of the first order correction to the potential, the reflection and transmission coefficients are evaluated.

3.4.1 Solution procedure

Now, the reduced boundary value problem, described by equations (2.9) -(2.12), can be solved by introducing finite cosine transformation with respect to y, which is more appropriate than Fourier transformation which involves contour integration and related results regarding the path of integration which are cumbersome, as described in Davies and Heathershaw [19] and in the previous section.

Define the finite cosine transformation of $\phi_1(x, y)$ with respect to y as

$$F_\kappa(x) = \int_h^0 \phi_1(x, y) \cos \kappa(h - y) dy, \tag{3.38}$$

where κ is determined by equation (1.29).

The corresponding inverse transform (Miles [67]) is given by

$$\phi_1(x, y) = \sum_\kappa \frac{4\kappa}{2\kappa h + \sin 2\kappa h} F_\kappa(x) \cos \kappa(h - y). \tag{3.39}$$

where the summation is over the infinite, discrete set of positive real roots $\kappa = k_n$, $n = 1, 2, 3, \ldots$ of (1.29) and the imaginary root $\kappa = -ik_0$.

Applying finite cosine transform to equation (2.9) and then using the given conditions (2.10), (2.11) and (1.29), we obtain the following second order ordinary differential equations

$$F_n''(x) - k_n^2 F_n(x) = p(x), \tag{3.40a}$$

and

$$F_0''(x) - (-ik_0)^2 F_0(x) = p(x). \tag{3.40b}$$

The above ordinary differential equations (3.40a) and (3.40b) can be solved successfully in the forms

$$F_n(x) = -\frac{1}{2k_n} \int_{-\infty}^{\infty} e^{-k_n|x-\xi|}\, p(\xi) d\xi, \tag{3.41a}$$

and

$$F_0(x) = -\frac{1}{2\beta_0} \int_{-\infty}^{\infty} e^{-\beta_0|x-\xi|}\, p(\xi) d\xi, \tag{3.41b}$$

where

$$\beta_0 = -ik_0, \tag{3.42}$$

if and only if we make an artificial assumption that k_0 possesses a small positive imaginary part which will be taken to be zero (in eliminating sense) at the end of analysis.

Assuming $k_0 = k_0^{(1)} + ik_0^{(2)}$ and substituting (3.41) in (3.39), we get

$$\begin{aligned}\phi_1(x,y) &= \sum_{n=1}^{\infty} \frac{(-2)}{2k_nh + \sin 2k_nh} \cos k_n(h-y) \int_{-\infty}^{\infty} e^{-k_n|x-\xi|}\, p(\xi)d\xi \\ &+ \frac{(-2i)}{2k_0h + \sinh 2k_0h} \cosh k_0(h-y) \int_{-\infty}^{\infty} e^{i(k_0^{(1)}+ik_0^{(2)})|x-\xi|}\, p(\xi)d\xi. \end{aligned} \tag{3.43}$$

$$\begin{aligned}\Rightarrow \phi_1(x,y) &= \sum_{n=1}^{\infty} \frac{(-2)}{2k_nh + \sin 2k_nh} \cos k_n(h-y) \int_{-\infty}^{\infty} e^{-k_n|x-\xi|}\, p(\xi)d\xi \\ &+ \frac{(-2i)}{2k_0h + \sinh 2k_0h} \cosh k_0(h-y) \int_{-\infty}^{\infty} e^{\mp i(k_0^{(1)}+ik_0^{(2)})|x-\xi|}\, p(\xi)d\xi, \;\; x \to \mp\infty. \end{aligned} \tag{3.44}$$

3.4.2 Reflection and transmission coefficients

Now taking $k_0^{(2)} \to 0$ and comparing equation (3.44) with equation (2.12),

for $x \to -\infty$, the reflection coefficient can be written as

$$\begin{aligned} R_1 &= \frac{-2i}{2k_0h + \sinh 2k_0h} \int_{-\infty}^{\infty} e^{ik_0x}\, p(x)dx \\ &= \frac{-2ik_0^2}{2k_0h + \sinh 2k_0h} \int_{-\infty}^{\infty} c(x)\, e^{2ik_0x} dx, \end{aligned} \tag{3.45}$$

and for $x \to +\infty$, the transmission coefficient can be written as

$$\begin{aligned} T_1 &= \frac{-2i}{2k_0h + \sinh 2k_0h} \int_{-\infty}^{\infty} e^{-ik_0x}\, p(x)dx \\ &= \frac{2ik_0^2}{2k_0h + \sinh 2k_0h} \int_{-\infty}^{\infty} c(x)dx. \end{aligned} \tag{3.46}$$

The reflection and transmission coefficients obtained in equations (3.45) and (3.46) are same as equations (3.12) and (3.15) obtained by Green's function method.

3.5 Conclusion

In this chapter, the reduced boundary value problem for the first order correction of the potential for normal incidence is solved by using Green's function technique; Fourier transform technique and finite cosine transform technique. After obtaining the velocity potential, the reflection and transmission coefficients were obtained in terms of integrals involving the shape function $c(x)$. These coefficients can be found out numerically once the shape function $c(x)$ is known. In Chapter 5, we will consider different examples of the shape function to evaluate these coefficients numerically.

It is found that the results obtained for these coefficients, in terms of integrals involving the shape function $c(x)$, by the above three different methods are the same.

In the application of Fourier transform technique, by taking Fourier transform of ϕ_1 with respect to x to the BVP it is found that the integrand (in equations (3.26) and (3.27)) contains certain singularities and hence residue theorem is used while employing contour integration to evaluate the integrals appearing in the first order correction of the potential. This technique has a more general approach than that employed by Davies and Heathershaw [19] to the problem of scattering of water waves by sinusoidal undulations on an otherwise flat bed.

On the other hand, the application of finite cosine transform technique of ϕ_1 does not involve the contour integration and the related results regarding the path of integration. This method also does not require determining a Green's function as was the case for the first method. Also, the use of this technique in obtaining the first-order potential, and hence the reflection and transmission coefficients, reduces the workload to a large extent. Hence, this technique has edge over the other two techniques.

Chapter 4

Solution to the Scattering Problem for Oblique Incidence

4.1 Introduction

In the last chapter we solved the problem of scattering of surface waves by small undulation of the bottom of a laterally unbounded sea, using linear water wave theory, for normal incidence using three different techniques, namely, Green's function and by a suitable application of Green's integral theorem; Fourier transform technique; and finite cosine transform. In the present chapter, we consider the same problem for oblique incidence and we solve it by these same three analytical methods. By these methods, the solution of the first order correction to the potential involved in the reduced boundary value problem is determined from which the quantities of physical interest namely, the reflection and transmission coefficients, are evaluated up to the first order of the small parameter ε in terms of integrals involving the shape function $c(x)$.

4.2 Solution by Green's function technique

In this section, the simpler boundary value problem (equations (2.26)-(2.29)) for the oblique incidence of surface waves is solved by using Green's function and by a suitable application of Green's integral theorem and the solution for the first order correction to the potential is obtained from which the reflection and transmission coefficients are evaluated.

4.2.1 Solution procedure

To solve this boundary value problem described by equations (2.26)-(2.29) we require the Green's function $G(x, y; \xi, \eta)$ which satisfies the following equations:

$$(\nabla^2 - \nu^2)G = 0 \quad \text{in} \quad -\infty < x < \infty,\ 0 < y < h, \tag{4.1}$$
$$\text{except at} \quad (\xi, \eta) \text{ where } 0 < \eta < h,$$
$$\frac{\partial G}{\partial y} + KG = 0 \quad \text{on} \quad y = 0, \tag{4.2}$$
$$\frac{\partial G}{\partial y} = 0 \quad \text{on } y = h, \tag{4.3}$$
$$G \sim \ln r \quad \text{as} \quad r = \{(x-\xi)^2 + (y-\eta)^2\}^{1/2} \to 0, \tag{4.4}$$
$$G \sim \text{multiple of } \cosh k_0(h-y)e^{i\mu|x-\xi|} \quad \text{as} \quad |x-\xi| \to \infty. \tag{4.5}$$

The condition (4.5) implies that G represents an outgoing wave as $|x - \xi| \to \infty$. Based on the solution given by Heins [34], we obtain the solution for the BVP (equations (4.1) - (4.5)) as

$$\begin{aligned} G(x, y; \xi, \eta) &= \frac{-4\pi i \sec\theta \cosh k_0(h-y) \cosh k_0(h-\eta) e^{i\mu|x-\xi|}}{2k_0 h + \sinh 2k_0 h} \\ &- \sum_{n=1}^{\infty} \frac{4\pi k_n \cos k_n(h-y) \cos k_n(h-\eta) e^{-\sqrt{(k_n^2+\nu^2)}\ |x-\xi|}}{(2k_n h + \sin 2k_n h)\sqrt{k_n^2+\nu^2}}, \end{aligned} \tag{4.6}$$

where k_n are real and positive roots of the equation (1.29).

The behaviour of G when $|x| \to \infty$ is given by the first term of the equation (4.6). We now apply the Green's integral theorem to $\phi_1(x, y)$ and $G(x, y; \xi, \eta)$ in the form

$$\int_C \left(\phi_1 \frac{\partial G}{\partial n} - G \frac{\partial \phi_1}{\partial n} \right) ds = 0, \tag{4.7}$$

where C is a closed contour in the xy-plane consisting of the lines $y = 0, y = h$ $(-X \le x \le X)$, $x = \pm X$ $(0 \le y \le h)$ and a small circle of radius δ with centre at (ξ, η), as was shown in Figure 3.1, and ultimately letting $X \to \infty$ and $\delta \to 0$.

Evaluating the integral equation (4.7) over the contour following the same procedure described in Section 3.2 the results of the integral equation (4.7) will give rise the determination of ϕ_1 as:

$$\phi_1(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(x, h; \xi, \eta)\ V(x)\ dx, \tag{4.8}$$

which solves the boundary value problem for $\phi_1(x, y)$.

4.2.2 Reflection and transmission coefficients

The first order reflection and transmission coefficients R_1 and T_1 appearing in the BVP (equations (2.26)-(2.29)) are now obtained by letting $\xi \to \mp\infty$ and using the far-field condition (2.29) (with (x, y) replaced by (ξ, η)) in equation (4.8). For this, from equation (4.6) we require the result

$$G(x, y; \xi, \eta) = \frac{-4\pi i \sec\theta \cosh k_0(h-y) \cosh k_0(h-\eta) e^{i\mu|x-\xi|}}{2k_0h + \sinh 2k_0h} \text{ as } |x-\xi| \to \infty. \tag{4.9}$$

To find R_1, we note from equations (2.29) and (4.9), respectively, that

$$\phi_1(\xi, \eta) = R_1 \cosh k_0(h-\eta) e^{-i\mu\xi} \quad \text{as} \quad \xi \to -\infty\ , \tag{4.10}$$

$$G(x, h; \xi, \eta) = \frac{-4\pi i \sec\theta \cosh k_0(h-\eta) e^{i\mu(x-\xi)}}{2k_0h + \sinh 2k_0h} \quad \text{as} \quad \xi \to -\infty. \tag{4.11}$$

Substituting equations (4.10) and (4.11) in equation (4.8), we obtain

$$\begin{aligned} R_1 &= \frac{-2i \sec\theta}{2k_0h + \sinh 2k_0h} \int_{-\infty}^{\infty} e^{i\mu x}\, V(x) dx \\ &= \frac{-2ik_0^2 \sec\theta \cos 2\theta}{2k_0h + \sinh 2k_0h} \int_{-\infty}^{\infty} c(x) e^{2i\mu x} dx. \end{aligned} \tag{4.12}$$

To find T_1, we note from equations (2.29) and (4.9), respectively, that

$$\phi_1(\xi, \eta) = T_1 \cosh k_0(h-\eta) e^{i\mu\xi} \quad \text{as} \quad \xi \to \infty\ , \tag{4.13}$$

$$G(x, h; \xi, \eta) = \frac{-4\pi i \sec\theta \cosh k_0(h-\eta) e^{-i\mu(x-\xi)}}{2k_0h + \sinh 2k_0h} \quad \text{as} \quad \xi \to \infty. \tag{4.14}$$

Substituting equations (4.13) and (4.14) in equation (4.8), we obtain

$$\begin{aligned} T_1 &= \frac{-2i \sec\theta}{2k_0h + \sinh 2k_0h} \int_{-\infty}^{\infty} e^{-i\mu x}\, V(x) dx \\ &= \frac{2ik_0^2 \sec\theta}{2k_0h + \sinh 2k_0h} \int_{-\infty}^{\infty} c(x) dx. \end{aligned} \tag{4.15}$$

The results (4.12) and (4.15) may be interpreted as the results obtained by Miles [68] and Mandal and Basu [53] for the case of absence of surface tension.

As the expression of R_1 contains a $\cos 2\theta$ term, then for the oblique incidence at $\theta = \frac{\pi}{4}$ of wave train, the reflection coefficient R_1 up to the first order vanishes independently of the shape of bottom undulation, as also mentioned by Miles [68] and Mandal and Basu [53].

Also the results for normal incidence can be obtained by putting $\theta = 0$.

The reflection and transmission coefficients can be evaluated from equations (4.12) and (4.15) once the shape function $c(x)$ is known.

4.3 Solution by Fourier transform technique

In the previous section, we solved the boundary value problem (equations (2.26)-(2.29)) by using Green's function and by a suitable application of Green's integral theorem. Now we consider the same problem and solve it by a different analytical method: Fourier transform technique. Due to the presence of singularities in the integrals appearing in the first order correction of the potential, residue theorem is employed while performing the contour integration. From the solution of the first-order correction to the potential, the reflection and transmission coefficients are evaluated.

4.3.1 Solution procedure

To solve this boundary value problem for the determination of ϕ_1 we now assume that ϕ_1 is such that the Fourier transform of ϕ_1, denoted by $\overline{\phi}_1$, is given by equation (3.16) and its inverse is given by equation (3.17).

Now, taking Fourier transform of the governing equation (2.26) and the boundary conditions (2.27) and (2.28) with respect to the horizontal space variable x, we obtain

$$\frac{\partial^2\overline{\phi}_1}{\partial y^2} - \widehat{\xi}^2\overline{\phi}_1 = 0 \quad \text{in } -\infty < \xi < \infty,\ 0 \le y \le h, \tag{4.16}$$

$$\frac{\partial\overline{\phi}_1}{\partial y} + K\overline{\phi}_1 = 0 \quad \text{on } y = 0, \tag{4.17}$$

$$\frac{\partial\overline{\phi}_1}{\partial y} = \Lambda(\xi) \quad \text{on } y = h, \tag{4.18}$$

where $\widehat{\xi}^2 = \xi^2 + \nu^2$ and

$$\Lambda(\xi) = \int_{-\infty}^{\infty} V(x)e^{i\xi x}dx, \tag{4.19}$$

with $V(x)$ given by equation (2.28).

The solution of (4.16) is given by

$$\overline{\phi}_1(\xi, y) = C_3(\xi)\cosh\widehat{\xi}y + C_4(\xi)\sinh\widehat{\xi}y. \tag{4.20}$$

Applying the boundary condition (4.17) to (4.20), we obtain

$$C_4(\xi) = \frac{-KC_1}{\widehat{\xi}}\ . \tag{4.21}$$

Applying the boundary condition (4.18) to (4.20), we obtain

$$C_3(\xi) = \frac{\Lambda(\xi)}{\widehat{\xi}\sinh\xi h - K\cosh\widehat{\xi}h}\ . \tag{4.22}$$

Substituting the values of C_3 and C_4 in the equation of (4.20) we obtain

$$\overline{\phi}_1(\xi, y) = \frac{\widehat{\xi}\cosh\widehat{\xi}y - K\sinh\widehat{\xi}y}{\widehat{\xi}[\widehat{\xi}\sinh\widehat{\xi}h - K\cosh\widehat{\xi}h]}\Lambda(\xi). \tag{4.23}$$

Taking inverse Fourier transform, the solution for the velocity potential can be written in the form

$$\phi_1(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\widehat{\xi}\cosh\widehat{\xi}y - K\sinh\widehat{\xi}y}{\widehat{\xi}[\widehat{\xi}\sinh\widehat{\xi}h - K\cosh\widehat{\xi}h]}\Lambda(\xi)\, e^{-i\xi x}d\xi. \tag{4.24}$$

We now obtain the final result from (4.24) by contour integration using the residue theorem as done previously.

Here also we observe that the equation (4.24) has certain singularities (lying on the ξ-axis) other than $\widehat{\xi} = 0$. Replacing K by $\widehat{K}$ (as defined in Section 3.3) in equation (4.24), the singularities of (4.24) are displaced off the ξ-axis to the upper and the lower half planes. Hence, we write

$$\phi_1(x, y) = \lim_{\mu'\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\widehat{\xi}, y)}{G_{\mu'}(\widehat{\xi}, h)}\, e^{-i\xi x}d\xi, \tag{4.25}$$

where

$$F(\widehat{\xi}, y) = [\widehat{\xi}\cosh\widehat{\xi}y - \widehat{K}\sinh\widehat{\xi}y]\Lambda(\xi), \tag{4.26}$$

$$G_{\mu'}(\widehat{\xi}, h) = \widehat{\xi}\left[\widehat{\xi}\sinh\widehat{\xi}h - \widehat{K}\cosh\widehat{\xi}h\right]. \tag{4.27}$$

If $\widehat{K} = \widehat{K}_1 + i\widehat{K}_2$, and if $\hat{\zeta} = \alpha + i\beta$ is a zero of the expression (4.27), then $\hat{\zeta}$ can be determined as

$$\hat{\zeta} = \pm\alpha_n \pm i\beta_n, \quad \text{and} \quad \hat{\zeta} = \pm(k_0 + \gamma) \pm i\beta_n', \tag{4.28}$$

where $\alpha_n, \beta_n, \gamma$ and β_n' are given by equation (3.31).

Substituting $\hat{\zeta} = \pm\sqrt{\zeta^2 + \nu^2}$, then the roots $\hat{\zeta} = \pm\alpha_n \pm i\beta_n$ give

$$\begin{aligned} &\zeta \approx \pm\sqrt{-(\beta_n^2 + \nu^2) \pm i\widehat{K}_2(2\alpha_n'^{(1)}\beta_n)} \quad \text{up to the first order of } \widehat{K}_2 \\ \Rightarrow\quad &\zeta \approx \pm i s_n \text{ as } \mu' \to 0, \text{ where } s_n = \sqrt{\beta_n^2 + \nu^2}, \end{aligned} \tag{4.29}$$

and the roots $\hat{\zeta} = \pm(k_0 + \gamma) \pm i\beta_n'$ give

$$\zeta \approx \pm k_0\cos\theta = \pm\mu \quad \text{up to the first order of } \widehat{K}_2 \text{ and when } \mu' \to 0. \tag{4.30}$$

4.3.2 Results

We now consider the same contour as described in Section 3.3 and perform the integration along it. Then, by using the residue theorem,

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0}(-i)\left[\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\hat{\zeta},y)\ e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=is_n}\right.\\ &\quad+\left.\text{Res}\left\{\frac{F(\hat{\zeta},y)\ e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=\mu}\right] \quad \text{for } x<0,\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0} i\left[\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\hat{\zeta},y)\ e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=-is_n}\right.\\ &\quad+\left.\text{Res}\left\{\frac{F(\hat{\zeta},y)\ e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=-\mu}\right] \quad \text{for } x>0,\end{aligned} \tag{4.32}$$

which imply that

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0}(-i)\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\hat{\zeta},y)\ e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=is_n}\\ &\quad+\frac{-2ik_0^2\sec\theta\cos 2\theta}{2k_0h+\sinh 2k_0h}\left\{\int_{-\infty}^{\infty}c(x)e^{2i\mu x}dx\right\}\cosh k_0(h-y)e^{-i\mu x}\\ &\quad \text{for } x<0,\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0} i\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\hat{\zeta},y)\ e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=-is_n}\\ &\quad+\frac{2ik_0^2\sec\theta}{2k_0h+\sinh 2k_0h}\left\{\int_{-\infty}^{\infty}c(x)dx\right\}\cosh k_0(h-y)e^{i\mu x}\\ &\quad \text{for } x>0.\end{aligned} \tag{4.34}$$

The first term on right hand side of each of the equations (4.33) and (4.34) represents the non-propagating modes which decay rapidly away from the undulation and the second term represents a propagating mode from the region of the bed disturbance. Comparing equations (4.33) and (4.34) with equation (2.29), the reflection and transmission coefficients can, respectively, be written as

$$R_1=\frac{-2ik_0^2\sec\theta\cos 2\theta}{2k_0h+\sinh 2k_0h}\int_{-\infty}^{\infty}c(x)e^{2i\mu x}dx, \tag{4.35}$$

and

$$T_1 = \frac{2ik_0^2 \sec\theta}{2k_0h + \sinh 2k_0h} \int_{-\infty}^{\infty} c(x)dx. \tag{4.36}$$

These results (4.35) and (4.36) are same as the results (4.12) and (4.15) obtained by Green's function technique.

4.4 Solution by finite cosine transform technique

In the last two sections, we considered the boundary value problem (equations (2.26)-(2.29)) of scattering of surface waves by small undulation of the bottom of a laterally unbounded sea using linear water wave theory for oblique incidence and solved it by two different methods, namely Green's function technique and Fourier transform technique. Now we consider the same problem as in the previous sections and solve it by using finite cosine transform. By this method we obtain the first-order correction to the potential. Consequently, the reflection and transmission coefficients are evaluated.

4.4.1 Solution procedure

Now, the above reduced boundary value problem, described by equations (2.26)-(2.29), can be solved by introducing finite cosine transformation with respect to y, which is more appropriate than Fourier transformation which involves contour integration and related results regarding the path of integration which are cumbersome, as described in Davies and Heathershaw [19].

Define the finite cosine transformation of $\phi_1(x, y)$ with respect to y as

$$F_\kappa(x) = \int_h^0 \phi_1(x, y) \cos\kappa(h - y)dy, \tag{4.37}$$

where κ is determined by the equation (1.29).

The corresponding inverse transform (Miles [67]) is given by

$$\phi_1(x, y) = \sum_\kappa \frac{4\kappa}{2\kappa h + \sin 2\kappa h} F_\kappa(x) \cos\kappa(h - y), \tag{4.38}$$

where the summation is over the infinite, discrete set of positive real roots $\kappa = k_n$, $n = 1, 2, 3, \ldots$ of (1.29) and the imaginary root $\kappa = -ik_0$.

Applying finite cosine transform to equation (2.26) and then using the given conditions (2.27), (2.28) and (1.29), we obtain the following second order ordinary differential equations

$$F_n''(x) - (k_n^2 + \nu^2)F_n(x) = V(x), \tag{4.39a}$$

and

$$F_0''(x) - (\nu^2 - k_0^2)F_0(x) = V(x). \tag{4.39b}$$

The above ordinary differential equations (4.39a) and (4.39b) can be solved successfully in the forms

$$F_n(x) = -\frac{1}{2\beta_n}\int_{-\infty}^{\infty} e^{-\beta_n|x-\xi|}\, V(\xi)d\xi, \tag{4.40a}$$

and

$$F_0(x) = -\frac{1}{2\beta_0}\int_{-\infty}^{\infty} e^{-\beta_0|x-\xi|}\, V(\xi)d\xi, \tag{4.40b}$$

where

$$\beta_n = (k_n^2 + \nu^2)^{1/2}, \tag{4.41a}$$

and

$$\beta_0 = -i(k_0^2 - \nu^2)^{1/2} = -i\mu, \tag{4.41b}$$

if and only if we make an artificial assumption that μ possesses a small positive imaginary part which will be taken to be zero at the end of analysis.

Assuming $\mu = \mu_1 + i\mu_2$ and substituting (4.40) in (4.38), we get

$$\begin{aligned}\phi_1(x,y) &= \sum_{n=1}^{\infty}\frac{(-2k_n)}{\beta_n(2k_nh + \sin 2k_nh)}\cos k_n(h-y)\int_{-\infty}^{\infty} e^{-\beta_n|x-\xi|}\, V(\xi)d\xi \\ &+ \frac{(-2ik_0)}{(2k_0h + \sinh 2k_0h)}\cosh k_0(h-y)\frac{1}{(\mu_1 + i\mu_2)} \\ &\times \int_{-\infty}^{\infty} e^{i(\mu_1+i\mu_2)|x-\xi|}\, V(\xi)d\xi.\end{aligned} \tag{4.42}$$

$$\begin{aligned}\Rightarrow \phi_1(x,y) &= \sum_{n=1}^{\infty}\frac{(-2k_n)}{\beta_n(2k_nh + \sin 2k_nh)}\cos k_n(h-y)\int_{-\infty}^{\infty} e^{-\beta_n|x-\xi|}\, V(\xi)d\xi \\ &+ \frac{(-2ik_0)}{(2k_0h + \sinh 2k_0h)}\cosh k_0(h-y)\frac{1}{(\mu_1 + i\mu_2)} \\ &\times \int_{-\infty}^{\infty} e^{\mp i(\mu_1+i\mu_2)(x-\xi)}\, V(\xi)d\xi, \qquad x \to \mp\infty.\end{aligned} \tag{4.43}$$

4.4.2 Reflection and transmission coefficients

Now taking $\mu_2 \to 0$ and comparing equation (4.43) with equation (2.29), for $x \to -\infty$, the reflection coefficient can be written as

$$\begin{aligned} R_1 &= \frac{-2i\sec\theta}{2k_0h+\sinh 2k_0h}\int_{-\infty}^{\infty} e^{i\mu x}\, V(x)dx \\ &= \frac{-2i\sec\theta}{2k_0h+\sinh 2k_0h}(\mu^2-\nu^2)\int_{-\infty}^{\infty} c(x)e^{2i\mu x}dx \\ &= \frac{-2ik_0^2\sec\theta\cos 2\theta}{2k_0h+\sinh 2k_0h}\int_{-\infty}^{\infty} c(x)\, e^{2i\mu x}dx, \end{aligned} \tag{4.44}$$

and for $x \to +\infty$, the transmission coefficient can be written as

$$\begin{aligned} T_1 &= \frac{-2i\sec\theta}{2k_0h+\sinh 2k_0h}\int_{-\infty}^{\infty} e^{-i\mu x}\, V(x)dx \\ &= \frac{-2i\sec\theta}{2k_0h+\sinh 2k_0h}(-\mu^2-\nu^2)\int_{-\infty}^{\infty} c(x)dx \\ &= \frac{2ik_0^2\sec\theta}{2k_0h+\sinh 2k_0h}\int_{-\infty}^{\infty} c(x)dx. \end{aligned} \tag{4.45}$$

The results (4.44) and (4.45) are same as the results (4.12) and (4.15) obtained by Green's function technique.

4.5 Conclusion

In applying perturbation analysis to the scattering of water waves over an uneven bottom, the limitations that the flow is an irrotational one and the waves are of small amplitude do not cause any hindrance in obtaining results valid over some physically interesting ranges of wave and uneven bed topography. The present method of applying perturbation analysis to oblique incidence on an uneven bottom is more general in nature. Due to the nature of the waves in sea, oblique incidence is more likely to occur. After applying perturbation analysis to the governing boundary value problem a simpler boundary value problem for the first order correction of the potential the solution is obtained and this is solved by three different methods: a suitable application of Green's integral theorem; Fourier transform and finite cosine transform. By these three methods the reflection and transmission coefficients are then evaluated up to the first order of ε in terms of integrals involving the shape function $c(x)$ and it is found that the results for these coefficients are same by these three methods. The reflection and transmission coefficients can be found out numerically once the shape function $c(x)$ is known.

The problem of scattering of surface water waves by sinusoidal varying topography on the sea-bed, considered by Davies and Heathershaw [19] for normal incidence, has been extended to the oblique case by using Fourier transform technique. The present technique definitely has the advantage that it is more general in nature than Davies and Heathershaw method.

In chapter 5, we will consider different examples of the shape function to determine these coefficients numerically.

Chapter 5

Different Bed Forms and Evaluation of Coefficients

5.1 Introduction

In the last two chapters, we investigated the problem of water wave scattering by small undulation on an otherwise flat bottom of a sea and obtained the analytical expressions for the reflection and transmission coefficients in terms of integrals involving the shape function $c(x)$. These coefficients can be found out numerically once the shape function $c(x)$ is known. In the present chapter, we consider different examples of the shape function in order to evaluate these coefficients. The results are best presented graphically.

5.2 Examples

We take up the following six examples each representing small undulation on the sea-bed. After evaluating R_1 and T_1, we plot different graphs to observe the reflected and transmitted field due to the presence of the undulation.

5.2.1 Example 1

Consider the following bed profile:

$$c(x) = b\ e^{-a_0|x|}\ (a_0 > 0), \ \ -\infty < x < \infty. \tag{5.1}$$

This depth profile corresponds to an exponentially damped undulation. Here the top of the elevation lies at the point $(0, b)$, and it decreases, on either side, exponentially. Using equation (5.1) in the expressions for R_1 and T_1, respectively, in equations (3.12) and (3.15), we obtain R_1 and T_1 for normal incidence as

$$R_1 = \frac{-2ik_0^2}{2k_0h + \sinh 2k_0h} \frac{2ba_0}{4k_0^2 + a_0^2}, \tag{5.2}$$

and

$$T_1 = \frac{2ik_0^2}{2k_0h + \sinh 2k_0h} \frac{2b}{a_0}. \tag{5.3}$$

Again for oblique incidence, R_1 and T_1 are obtained from equations (4.12) and (4.15), respectively, as

$$R_1 = \frac{-2ik_0^2 \sec\theta \cos 2\theta}{2k_0h + \sinh 2k_0h} \frac{2ba_0}{4\mu^2 + a_0^2}, \tag{5.4}$$

and

$$T_1 = \frac{2ik_0^2 \sec\theta}{2k_0h + \sinh 2k_0h} \frac{2b}{a_0}. \tag{5.5}$$

Numerical result:

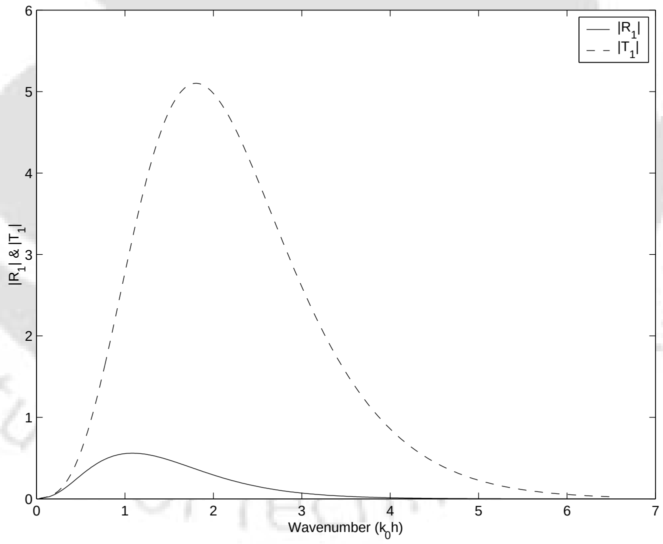

Figure 5.1: *Reflection and transmission coefficients against the wave number k_0h for $\theta = 0; b/h = 0.1; a_0h = 1$*

From Figure 5.1, we observe that the reflection coefficient is comparatively much smaller than the transmission coefficient since $|R_1| = |T_1|\ [(2k_0/a)^2 + 1]^{-1}$. As we consider small undulation (b is small, here $b/h = 0.1$) it happens so, i.e. transmission dominates reflection.

5.2.2 Example 2

Consider the bed profile:

$$c(x) = b_0\ e^{-d_0 x^2}\ (d_0 > 0),\ \ -\infty < x < \infty. \tag{5.6}$$

In this case, the elevation is of the Gaussian profile, the top occurring at the point $(0, b_0)$ as in Example 1. Using equation (5.6) in the expressions for R_1 and T_1, we obtain, for normal incidence,

$$R_1 = \frac{-2ik_0^2}{2k_0h + \sinh 2k_0h} \left(\frac{\pi}{d_0}\right)^{1/2} b_0 e^{-k_0^2/d_0}, \tag{5.7}$$

and

$$T_1 = \frac{2ik_0^2}{2k_0h + \sinh 2k_0h}\ b_0 \left(\frac{\pi}{d_0}\right)^{1/2}, \tag{5.8}$$

while for oblique incidence,

$$R_1 = \frac{-2ik_0^2 \sec\theta \cos 2\theta}{2k_0h + \sinh 2k_0h} \left(\frac{\pi}{d_0}\right)^{1/2} b_0 e^{-\mu^2/d_0}, \tag{5.9}$$

and

$$T_1 = \frac{2ik_0^2 \sec\theta}{2k_0h + \sinh 2k_0h}\ b_0 \left(\frac{\pi}{d_0}\right)^{1/2}. \tag{5.10}$$

5.2.3 Example 3

Now consider the following bed profile:

$$c(x) = \begin{cases} h\left[\cos\left(\frac{2\pi x}{d}\right) - 1\right], & 0 \le x \le d, \\ 0, & \text{otherwise} \end{cases}. \tag{5.11}$$

This shape function about the mean level $y = h$ corresponds to a single hump which smoothly joins the flat bed section at $x = 0$ and $x = d$.

Substituting equation (5.11) in expressions for R_1 and T_1, the reflection and transmission coefficients are, respectively, evaluated as
for normal incidence,

$$R_1 = \frac{-k_0^2 \sec\theta \cos 2\theta}{2k_0h + \sinh 2k_0h} \frac{h\pi^2[e^{2ik_0d} - 1]}{k_0(k_0^2d^2 - \pi^2)}, \tag{5.12}$$

and

$$T_1 = \frac{2ik_0^2 \sec\theta(-h\ d)}{2k_0h + \sinh 2k_0h}, \tag{5.13}$$

while for oblique incidence,

$$R_1 = \frac{-k_0^2 \sec\theta \cos 2\theta}{2k_0h + \sinh 2k_0h} \frac{h\pi^2[e^{2i\mu d} - 1]}{\mu(\mu^2d^2 - \pi^2)}, \tag{5.14}$$

and

$$T_1 = \frac{2ik_0^2 \sec\theta(-h\ d)}{2k_0h + \sinh 2k_0h}\ . \tag{5.15}$$

Numerical result:

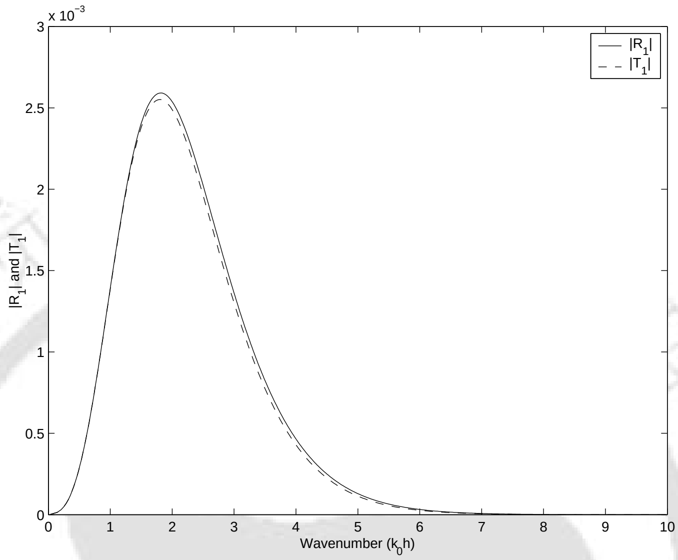

Figure 5.2: *Reflection and transmission coefficients against the wave number k_0h for $\theta = 0; d/h = 0.0001$*

Figure 5.2 corresponds to the single hump whose mathematical representation is different from Example 1. So values for different parameters are considered to get a smooth graph for reflection and transmission coefficients. For the selected set of parameters ($d/h = 0.0001$) we observe the effects of reflection and transmission are nearly same up to a certain value of the wave number after which they vanish.

The scattered fields behave differently in Examples 1 and 3 because the parameters considered in Example 3 are different from those in Example 1, and also the shapes are different although they represent single undulation. For the first example, the shape depends mainly on the parameter b (hence b/h is used for plotting) and for the third example, the shape depends mainly on the parameter d (hence d/h is used for plotting).

5.2.4 Example 4

Consider

$$c(x) = \begin{cases} h\left[\frac{x}{d} - \left(\frac{x}{d}\right)^2\right], & 0 \le x \le d, \\ 0, & \text{otherwise.} \end{cases} \tag{5.16}$$

This shape function about the mean level $y = h$ corresponds to a single hump which gives slope discontinuities at $x = 0$ and $x = d$.

Substituting equation (5.16) in equations for R_1 and T_1, the reflection and transmission coefficients are, respectively, evaluated as
for normal incidence,

$$R_1 = \frac{-2ik_0^2}{2k_0h + \sinh 2k_0h} \frac{h[(i - dk_0) - e^{2ik_0d}(i + dk_0)]}{4d^2k_0^3}, \tag{5.17}$$

and

$$T_1 = \frac{ik_0^2}{2k_0h + \sinh 2k_0h} \frac{hd}{3}, \tag{5.18}$$

while for oblique incidence,

$$R_1 = \frac{-2ik_0^2 \sec\theta \cos 2\theta}{2k_0h + \sinh 2k_0h} \frac{h[(i - d\mu) - e^{2i\mu d}(i + d\mu)]}{4d^2\mu^3}, \tag{5.19}$$

and

$$T_1 = \frac{ik_0^2 \sec\theta}{2k_0h + \sinh 2k_0h} \frac{hd}{3}. \tag{5.20}$$

5.2.5 Example 5

We now consider a special form for the shape function $c(x)$ in the form of a patch of sinusoidal ripples as the bottom undulation. This bottom undulation closely resembles some naturally occurring obstacles formed at the bottom due to sedimentation and ripple growth of sands. Hence, this type of undulation assumes more significance than the other ones.

$$c(x) = \begin{cases} c_0 \sin(lx + \delta'), & L_1 \leq x \leq L_2 \\ 0 & \text{otherwise,} \end{cases}. \tag{5.21}$$

For the continuity of bed elevation we can take

$$L_1 = \frac{-n\pi - \delta'}{l}, \quad L_2 = \frac{m\pi - \delta'}{l},$$

where c_0 the amplitude of the sinusoidal ripples, l the wave number of the sinusoidal ripples and δ' an arbitrary phase angle; and m and n positive integers. This represents a patch of sinusoidal ripples on an otherwise flat bottom, the patch consisting of $(n + m)/2$ ripples having the same wave number l.

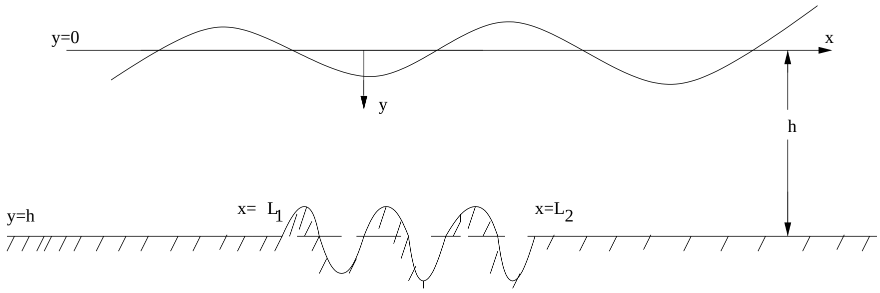

Figure 5.3: *A patch of sinusoidal ripples*

R_1 and T_1 for normal incidence:

$$R_1 = \frac{-2ik_0^2}{2k_0h + \sinh 2k_0h}\, \frac{c_0 l}{l^2 - (2k_0)^2}\Big[(-1)^n e^{2ik_0L_1} - (-1)^m e^{2ik_0L_2}\Big], \tag{5.22}$$

and

$$T_1 = \frac{2ik_0^2}{2k_0h + \sinh 2k_0h}\, \frac{(-c_0)}{l}\Big[(-1)^m - (-1)^n\Big]. \tag{5.23}$$

In the situation in which there is an integer number of ripple wavelengths in the patch $L_1 \le x \le L_2$ such that $m = n$ and $\delta' = 0$, we find R_1 and T_1, respectively, as

$$R_1 = \frac{2k_0}{2k_0h + \sinh 2k_0h}\, \frac{c_0\ (-1)^m(2k_0/l)}{(2k_0/l)^2 - 1} \sin\left(\frac{2k_0 m\pi}{l}\right), \tag{5.24}$$

and

$$T_1 = 0. \tag{5.25}$$

These results exactly match with those obtained in Davies and Heathershaw [19]. The main observation in this case is that the first order reflection coefficient is an oscillatory function in the ratio of twice the surface wave number k_0 to the ripple wave number l. Furthermore, if the bed wave number is twice the wave number of the surface waves, the theory predicts a resonant interaction between the bed and free surface. This resonant interaction is described by Davies [17, 18]. The resonant interaction over sandbars is reported by Heathershaw [32] with experiment demonstration. So we find from (5.24) that

$$R_1 = \frac{2k_0c_0}{2k_0h + \sinh(2k_0h)}\, \frac{m\pi}{2}. \tag{5.26}$$

In this case R_1 becomes a constant multiple of m, the number of ripples in the patch. Although the theory breaks down where the solution is singular ($l = 2k_0$), a large amount of reflection of the incident wave energy by the bed forms is predicted in the neighbourhood of this singularity.

The ratio $2k_0c_0/\{2k_0h+\sinh(2k_0h)\}$ indicates that the size of the reflected wave is dependent upon the ripple's amplitude and the water depth. For long waves ($k_0h \ll 1$) having wave number such that $l = 2k_0$, equation (5.26) gives rise to

$$R_1 = \frac{c_0}{2h}\frac{m\pi}{2}. \tag{5.27}$$

If $\frac{c_0}{2h} \approx \frac{1}{15}$ and $m = 10$, then $R_1 \approx 1$ which states that in the physical interesting case of ripple having long wave length with, say, $\frac{c_0}{2h} \approx \frac{1}{15}$, the theory predicts that the total reflection of wave energy will occur for $m \geq 10$.

R_1 and T_1 for oblique incidence:

$$R_1 = \frac{-2ik_0^2\sec\theta\cos 2\theta}{2k_0h+\sinh 2k_0h}\,\frac{c_0 l}{l^2-(2\mu)^2}\Big[(-1)^n\ e^{2i\mu L_1} - (-1)^m\ e^{2i\mu L_2}\Big], \tag{5.28}$$

and

$$T_1 = \frac{2ik_0^2\sec\theta}{2k_0h+\sinh 2k_0h}\,\frac{(-c_0)}{l}\Big[(-1)^m - (-1)^n\Big]. \tag{5.29}$$

In the situation in which there is an integer number of ripple wavelengths in the patch $L_1 \leq x \leq L_2$ such that $m = n$ and $\delta' = 0$, we find R_1 and T_1, respectively, as

$$R_1 = \frac{2k_0\sec\theta\cos 2\theta}{2k_0h+\sinh 2k_0h}\,\frac{c_0\ (-1)^m(2k_0/l)}{(2\mu/l)^2-1}\ \sin\left(2\mu m\pi/l\right), \tag{5.30}$$

and

$$T_1 = 0. \tag{5.31}$$

These results exactly match with those obtained by Davies and Heathershaw [19] when $\theta = 0$. Equation (5.30) illustrates that for a given number of m ripples, the first order reflection coefficient is an oscillatory function in the quotient of twice the $x-$component of the wave number and the ripple wave number. Furthermore, at the critical condition $l = 2\mu$, the theory predicts a resonant interaction between the bed and the free surface. Hence, we find from (5.30) that

$$R_1 = \frac{k_0c_0\sec^2\theta\cos 2\theta}{2k_0h+\sinh 2k_0h}\,m\pi\,. \tag{5.32}$$

In this case R_1 becomes a constant multiple of m, the number of ripples in the patch. Although the theory breaks down where the solution is singular ($l = 2\mu$), a large amount of reflection of the incident wave energy by the bed forms is predicted in the neighbourhood of this singularity.

Discussion:

As this is a non-dissipative system and since $T_1 = 0$ while R_1 may be large, there is a violation in the conservation of energy in the solution which states that the solution is required to satisfy a condition in relation to wave energy flux which precisely means that the incident component of wave energy flux on the undulating part of the bed must be balanced by the sum of reflected and transmitted energy fluxes. Here the reason for the imbalance is that the linearised analysis does not permit any attenuation of the incident waves as they travel over the region of the topography in $L_1 \leq x \leq L_2$, causing the predicted reflected wave in the perturbation solution to be overestimated and the transmitted wave to be zero. In practice, if the reflection wave is non-zero, there must be a progressive attenuation of the incident wave in $L_1 \leq x \leq L_2$. Hence, to recover the energy balance on the solution, an *ad hoc* procedure was proposed by Davies [18], in which the free surface wave amplitude in the first order solution was assumed to decrease linearly from its starting value at $x = -L$ to a new lower value at $x = +L$ and thereby establishing more accurate prediction for the magnitude of the reflected and transmitted waves.

Numerical results:

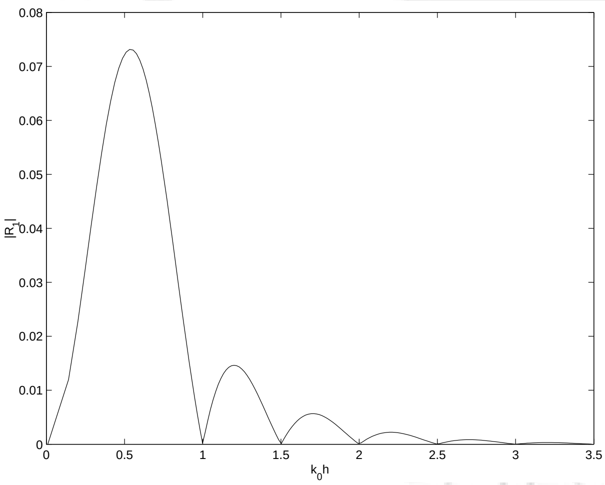

Figure 5.4: *Reflection coefficient against the wave number* k_0h *for* $\theta = 0; c_0/h = 0.1; lh = 1; m = 1$

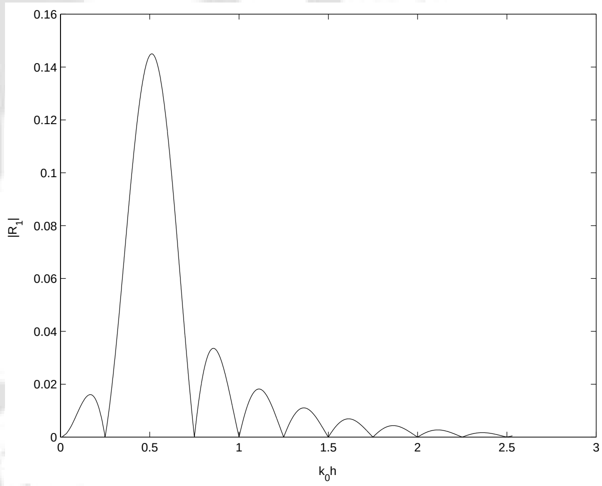

Figure 5.5: *Reflection coefficient against the wave number* k_0h *for* $\theta = 0; c_0/h = 0.1; lh = 1; m = 2$

The numerical computation is shown here for the first order reflection coefficient given by equation (5.24). In Figure 5.4, $|R_1|$ is plotted against the wave number k_0h for a single ripple and $c_0/h = 0.1, lh = 1$. From the graph it is clear that its peak value is attained when the

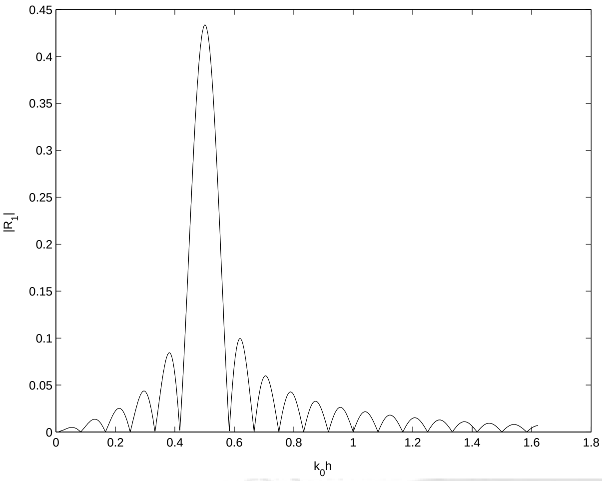

Figure 5.6: *Reflection coefficient against the wave number k_0h for $\theta = 0; c_0/h = 0.1; lh = 1; m = 6$*

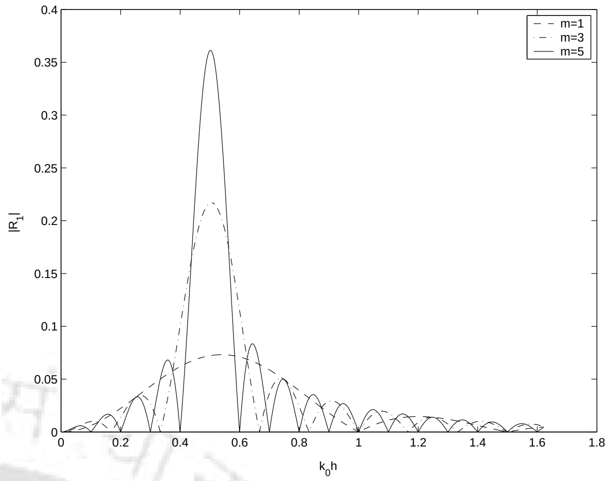

Figure 5.7: *Reflection coefficient against the wave number k_0h for $\theta = 0; c_0/h = 0.1; lh = 1; m = 1, 3,$ and 5*

wave number of the bottom undulations (lh) becomes approximately twice as large as surface wave number (k_0h). This is most evident in the curve which has its maximum value 0.0732 at $k_0h = 0.5442$. In Figure 5.5, $|R_1|$ is again plotted against k_0h for $c_0/h = 0.1, m = 2, lh = 1$. In this case the number of ripples is increased to two. The same general feature of $|R_1|$, as in Figure 5.4, is now observed with the modification that the overall values of R_1 is now increased to 0.1450 at $k_0h = 0.5104$ in comparison to the case of $m = 1$; the oscillating nature of $|R_1|$ against k_0h is more pronounced and the number of zeros of $|R_1|$ also increases. This phenomenon becomes evident when m is increased to 6 (Figure 5.6). In this case its maximum value is 0.4328 at $k_0h = 0.5$. As the number of ripples increases, the peak value of $|R_1|$ increases and it becomes more oscillatory. For proper observation, a comparison is made in Figure 5.7 which plots $|R_1|$ against k_0h for $m = 1, 3$ and 5. Now the corresponding values of $|R_1|$ for $m = 1, 3$ and 5 are 0.0732, 0.2167 and 0.3607 at $k_0h = 0.5442, 0.5103$ and 0.5 respectively.

In Figure 5.8, the reflection coefficient $|R_1|$ is also depicted against the angle of incidence θ for $k_0h = 0.03163, c_0/h = 0.1, lh = 1$ and for a single ripple, two ripples and six ripples. Also $|R_1|$ is depicted against the angle of incidence θ for $k_0h = 0.03163, c_0/h = 0.1, lh = 1$ and for a single ripple, three ripples and five ripples in Figure 5.9. From the graphs, we find that $|R_1|$ increases as the number of ripples m increases and it is clear that for $\theta = \pi/4$, $|R_1|$ vanishes independently of the shape of the function which validates the equation (4.12).

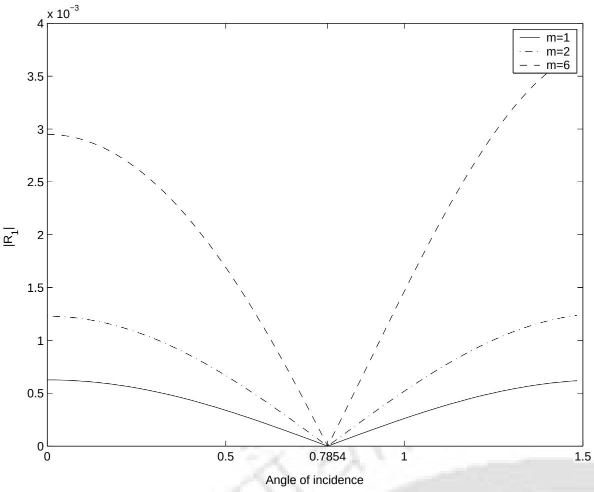

Figure 5.8: *Reflection coefficient against the angle of incidence* θ *for* $k_0h = 0.03163; c_0/h = 0.1; lh = 1; m = 1, 2,$ *and* 6

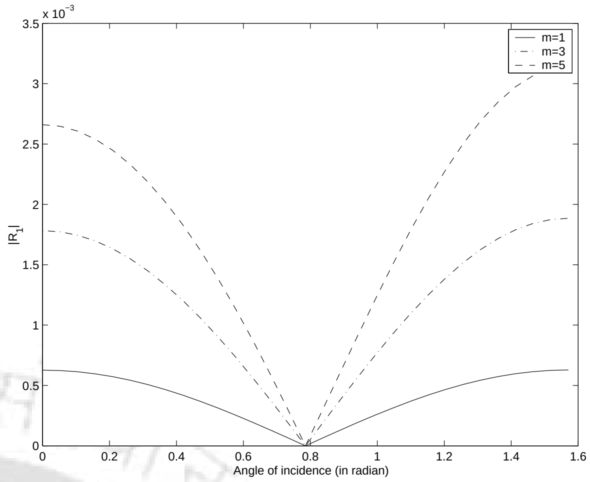

Figure 5.9: *Reflection coefficient against the angle of incidence* θ *for* $k_0h = 0.03163; c_0/h = 0.1; lh = 1; m = 1, 3$ *and* 5

5.2.6 Example 6

Now consider the shape function

$$c(x) = \begin{cases} a\sin(l_1 x), & L_3 \leq x \leq 0, \\ a\sin(l_2 x), & 0 \leq x \leq L_4, \\ 0 & \text{otherwise}, \end{cases} \tag{5.33}$$

where $L_3 = -n\pi/l_1$; $L_4 = m\pi/l_2$; m and n are integers. This also represents a patch of sinusoidal ripples on an otherwise flat bottom, the patch in $[-n\pi/l_1, 0]$ consists of $n/2$ number of ripples having the wave number l_1 and the patch in $[0, m\pi/l_2]$ consists of $m/2$ number of ripples having the wave number l_2.

R_1 and T_1 for oblique incidence:

For this type of undulation at oblique incidence, we obtain the reflection and transmission coefficients, respectively, as

$$R_1 = \frac{2ik_0^2 a\sec\theta\cos 2\theta}{2k_0h + \sinh 2k_0h}\Big[\frac{l_1}{l_1^2 - (2\mu)^2}\{1 - (-1)^n \exp(2i\mu L_3)\} + \frac{l_2}{l_2^2 - (2\mu)^2}\{(-1)^m \exp(2i\mu L_4) - 1\}\Big], \tag{5.34}$$

and

$$T_1 = \frac{-2ik_0^2 a\sec\theta}{2k_0h + \sinh 2k_0h}\Big[\frac{1}{l_1}\{1 - (-1)^n\} + \frac{1}{l_2}\{(-1)^m - 1\}\Big]. \tag{5.35}$$

At the critical conditions $l_1 = 2\mu$ and $l_2 = 2\mu$, we find from (5.34) that

$$R_1 = \frac{k_0 a \;\sec^2\theta\cos 2\theta}{2k_0h + \sinh 2k_0h}\;\frac{(m+n)\pi}{2}\;. \tag{5.36}$$

The result for normal incidence can be obtained by putting $\theta = 0$.

As in Example 5, here also it is observed that R_1 becomes a constant multiple of $(m+n)/2$, the number of ripples in the patch. Although the theory breaks down where the solution is singular ($l_1 = 2\mu$ and $l_2 = 2\mu$), a large amount of reflection of the incident wave energy by the bed forms is provided in the neighbourhood of these singularities.

If $l_1 = l_2 = l$ and $m = n$, equations (5.34) and (5.35), respectively, reduce to equations (5.30) and (5.31) in Example 5 where all the ripples have the same wave number l.

Numerical results:

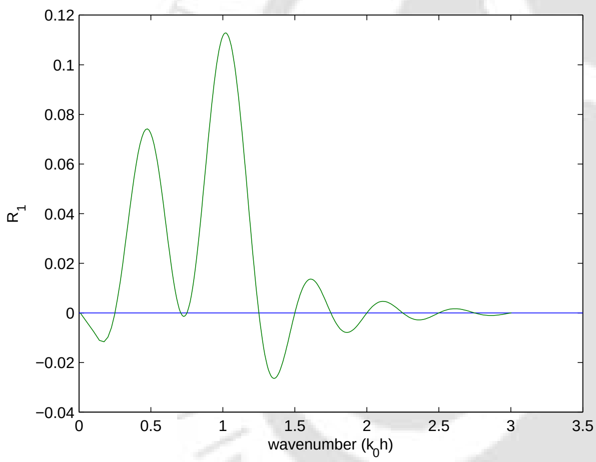

Figure 5.10: *Reflection coefficient* R_1 *against the wave number* k_0h *for* $\theta = 0; a/h = 0.1; l_1h = 2; l_2h = 1; n = 4; m = 2$.

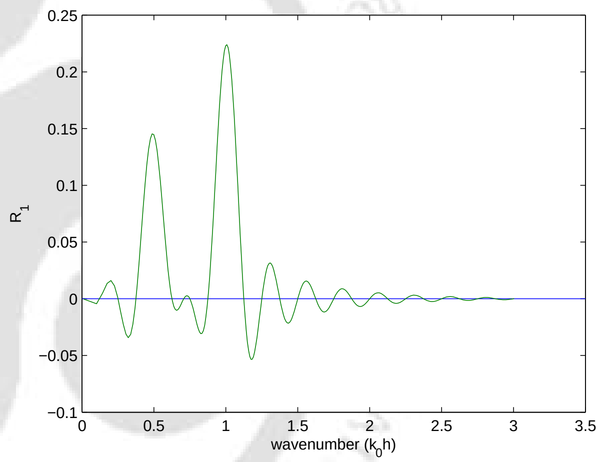

Figure 5.11: *Reflection coefficient* R_1 *against the wave number* k_0h *for* $\theta = 0; a/h = 0.1; l_1h = 2; l_2h = 1; n = 8; m = 4$.

In Example 6, a patch of sinusoidal bottom undulations with ripples having two different wave numbers l_1 and l_2 over two segments is considered which may be of relevance and interest to physical oceanographers. For this case, we consider the numerical computation for the first order reflection coefficient R_1 given by equation (5.34). In Figure 5.10, R_1 is depicted against the wave number k_0h for $\theta = 0, a/h = 0.1, l_1h = 2, l_2h = 1$ and $n = 4, m = 2$. From the graph it is clear that its peak value is attained when $l_1h = 2k_0h$. When the number of ripples increases from $n = 4, m = 2$ to $n = 8, m = 4$ (Figure 5.11), the general feature of

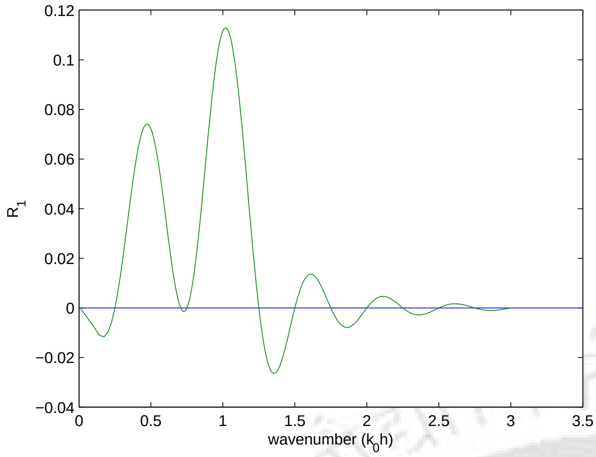

Figure 5.12: *Reflection coefficient* R_1 *against the wave number* k_0h *for* $\theta = 0; a/h = 0.1; l_1h = 1; l_2h = 2; n = 2; m = 4.$

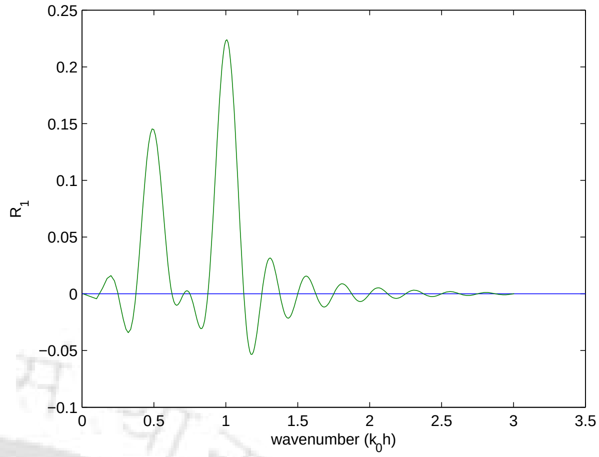

Figure 5.13: *Reflection coefficient* R_1 *against the wave number* k_0h *for* $\theta = 0; a/h = 0.1; l_1h = 1; l_2h = 2; n = 4; m = 8.$

R_1 remains the same but with the observation that the overall value of R_1 increases with increasing number of zeros of R_1.

Even if the roles of l_1 and l_2 are reversed, the same conclusion can be observed as shown by the Figures 5.12 and 5.13.

5.3 Conclusion

Different examples of bed profile are considered to evaluate the reflection and transmission coefficients numerically. The results are best presented graphically. Out of all these special cases, the particular case of a patch of sinusoidal ripples on the sea-bed is of considerable significance due to the ability of an undulating bed to reflect incident wave energy which is important in respect of both coastal protection, and of possible ripple growth if the bed is erodable. For this particular case we observe that the first order transmission coefficient vanishes identically, and the reflection coefficient becomes a constant multiple of the number of ripples in the patch when the bed wave length is half the surface wave length. Consequently, if the number of ripples is large, then there occurs a large amount of reflection of the incident wave energy. This result may be useful in the construction of an effective reflector of the incident wave energy for protecting coastal areas from the rough sea in the arctic regions. One important result that follows is that even if the ripples do not have the same wave numbers, the same conclusion can be made. The evaluation of the reflection and transmission coefficients for these different forms allow us to observe the behaviour of the scattered field associated with each undulation.

Chapter 6

Eigenfunction Expansion Method for Scattering of Surface Waves by Small Undulation

6.1 Introduction

In the previous Chapters 3 and 4, the problem of scattering of surface waves by small undulation on a sea-bed in the case of finite depth of water within the framework of linearised theory of water waves is solved by Green's function technique, Fourier transform technique and finite cosine transform technique. By these methods the analytical solutions of the reflection and transmission coefficients are obtained. In Chapter 5, we considered different shape functions to evaluate these coefficients numerically. In this chapter, we consider the same problem as the one considered in Chapter 2. Here the problem is solved by a direct method based on an eigenfunction expansion using linear water wave theory for both normal and oblique incidences. An appropriate set of orthogonal eigenfunctions depending upon a single parameter is constructed. The value of the parameter is selected in such a way that all the conditions of the boundary value problem up to the first order are satisfied completely. A patch of sinusoidal undulations is considered as an example of the bed surface to evaluate the first order reflection and transmission coefficients. The computational results of first order reflection coefficient is presented graphically and compared with the results existing in literature. Excellent agreement is observed between the present method and the known method. The results for the normal incidence are recovered from the results of oblique incidence as a special case.

6.2 Case-I: Normal incidence

6.2.1 Statement and formulation

Considering the same statement and assumptions as in the formulation for the normal incidence case, described in Section 2.2, we proceed to solve the boundary value problem (equations (2.1)-(2.3)) with the incident wave given by equation (2.4). Hence, we have the same far-field requirement as given by (2.6).

6.2.2 Solution procedure

The bottom condition (2.3) can be approximated up to first order of the small parameter ε as

$$\frac{\partial\phi}{\partial y} - \varepsilon\frac{\partial}{\partial x}\left\{c(x)\frac{\partial\phi}{\partial x}\right\} = 0 \quad \text{on} \quad y = h. \tag{6.1}$$

Thus the bottom condition, in effect, reduces approximately to a condition on $y = h$. This suggests that we may consider the governing partial differential equation for ϕ to hold in the strip $0 < y < h$, $-\infty < x < \infty$ along with boundary conditions (2.2), (6.1) and the far-field requirements (2.6). It may be noted here that R and T depend on ε.

We choose eigenfunction expansions for the regions $\phi(x, y)$ for $x < 0$ and $x > 0$ in the form

$$\phi(x,y) = \begin{cases} \sum_{n=1}^{\infty} A_n\varphi_n(y)e^{k_n x} + \cosh k_0(h-y)(e^{ik_0x} + Re^{-ik_0x}) & \text{for} \quad x < 0, \\ \sum_{n=1}^{\infty} B_n\varphi_n(y)e^{-k_n x} + T\cosh k_0(h-y)e^{-ik_0x} & \text{for} \quad x > 0, \end{cases} \tag{6.2}$$

where

$$\varphi_n = \cos k_n y - \frac{K}{k_n}\sin k_n y, \quad n = 1, 2, \ldots. \tag{6.3}$$

The constants $k_n, n = 1, 2, \ldots$, are such that $k_n > 0, n = 1, 2, \ldots$, and are *to be determined.* A_n and $B_n, n = 1, 2, \ldots$, are also unknown constants. It is obvious that the representations (6.2) for $\phi(x, y)$ satisfy equation (2.1) in $0 < y < h$, the boundary condition (2.2) and the far-field requirements (2.6).

The conditions that must be applied to $\phi(x, y)$ represented by (6.2) are as follows:

(i) ϕ must be continuous across $x = 0$ $(0 < y < h)$,

(ii) $\dfrac{\partial\phi}{\partial x}$ must be continuous across $x = 0$ $(0 < y < h)$,

(iii) ϕ must satisfy the boundary condition (6.1) (for both $x < 0, x > 0$).

The conditions (i) and (ii) produce, respectively,

$$\sum_{n=1}^{\infty}(A_n - B_n)\varphi_n = (T - 1 - R)\cosh k_0(h - y), \quad 0 < y < h, \tag{6.4}$$

and

$$\sum_{n=1}^{\infty} k_n(A_n + B_n)\varphi_n = ik_0(T - 1 + R)\cosh k_0(h - y), \quad 0 < y < h. \tag{6.5}$$

From equations (6.4) and (6.5), the constants $A_n, B_n,\ n = 1, 2, \dots,$ can be obtained if and only if the sequence $\{\varphi_n, 0 < y < h : n = 1, 2, \dots\}$ forms an orthogonal set. So we now assume that

$$\int_0^h \varphi_m(y)\varphi_n(y)dy = N_n\delta_{mn},\ m, n = 1, 2, ... \tag{6.6}$$

where δ_{mn} is the Kronecker delta and

$$N_n = \frac{1}{k_n^2}\left[\frac{k_n^2 - K^2}{4k_n}\sin 2k_nh + \frac{k_n^2 + K^2}{2}h - K\sin^2 k_nh\right] \ n = 1, 2, ... \ . \tag{6.7}$$

It is obvious that $\varphi_n(y),\ n = 1, 2, \dots$ satisfy

$$\varphi_n^{''}(y) + k_n^2\varphi_n(y) = 0, \quad 0 < y < h, \tag{6.8a}$$

with

$$K\varphi_n(y) + \varphi_n^{'}(y) = 0, \quad \text{on} \ \ y = 0, \tag{6.8b}$$

and for orthogonality in $(0, h)$, we impose the condition

$$\lambda\varphi_n(y) + \varphi_n^{'}(y) = 0, \quad \text{on} \ \ y = h, \tag{6.8c}$$

where λ is an *unknown constant* to be determined. Using the form of φ_n given by (6.3) in this condition, we obtain

$$\left(k_n + \lambda\frac{K}{k_n}\right)\tan k_nh + K - \lambda = 0,\ n = 1, 2, ... \tag{6.9}$$

Once λ is known, the unknown constants $k_n,\ n = 1, 2, ...,$ introduced in (6.3) can be obtained by solving the transcendental equation (6.9).

Using the orthogonality properties of $\varphi_n(y), n = 1, 2, ...,$ in (6.4) and (6.5), we obtain

$$A_n - B_n = (T - 1 - R)\frac{c_n}{N_n}, \tag{6.10}$$

$$A_n + B_n = (T - 1 + R)\frac{ik_0c_n}{k_nN_n}, \tag{6.11}$$

where

$$c_n = \frac{k_n \sin k_n h + K \cos k_n h}{k_0^2 + k_n^2}. \tag{6.12}$$

Thus

$$A_n = \left\{(T-1)\left(1+\frac{ik_0}{k_n}\right) - R\left(1-\frac{ik_0}{k_n}\right)\right\} d_n, \tag{6.13a}$$

$$B_n = \left\{(T-1)\left(\frac{ik_0}{k_n}-1\right) + R\left(1+\frac{ik_0}{k_n}\right)\right\} d_n, \tag{6.13b}$$

with

$$d_n = \frac{c_n}{2N_n}. \tag{6.14}$$

The boundary condition (6.1) is now used for the regions $x < 0$ and $x > 0$, respectively. Using the eigenfunction expansions (6.2) in (6.1) (for $x < 0$ and $x > 0$), we obtain, after some simplifications,

$$\sum_{n=1}^{\infty} d_n \left\{(T-1)\left(1+\frac{ik_0}{k_n}\right) - R\left(1-\frac{ik_0}{k_n}\right)\right\} \left[e_n + \varepsilon\left\{k_n^2 c(x) + k_n c'(x)\right\} s_n\right] e^{k_n x}$$
$$+\varepsilon\left[\left\{ik_0 c'(x) - k_0^2 c(x)\right\} e^{ik_0 x} - R\left\{ik_0 c'(x) + k_0^2 c(x)\right\} e^{-ik_0 x}\right] = 0 \quad \text{for} \quad x < 0, \tag{6.15}$$

$$\sum_{n=1}^{\infty} d_n \left\{(T-1)\left(\frac{ik_0}{k_n}-1\right) + R\left(1+\frac{ik_0}{k_n}\right)\right\} \left[e_n + \varepsilon\left\{k_n^2 c(x) - k_n c'(x)\right\} s_n\right] e^{-k_n x}$$
$$-\varepsilon T\left\{ik_0 c'(x) - k_0^2 c(x)\right\} e^{ik_0 x} = 0 \qquad \text{for} \quad x > 0, \tag{6.16}$$

where

$$e_n = k_n \sin k_n h + K \cos k_n h, \tag{6.17}$$
$$s_n = \cos k_n h - \frac{K}{k_n} \sin k_n h. \tag{6.18}$$

Now multiplying (6.15) by $e^{ik_0 x}$ and integrating with respect to x from $-\infty$ to 0, we obtain

$$-T(H+\varepsilon M) + R\left[-L - \varepsilon N + \varepsilon \int_{-\infty}^{0} \left\{ik_0 c'(x) + k_0^2 c(x)\right\} e^{ik_0 x} dx\right]$$
$$= -H - \varepsilon M + \varepsilon \int_{-\infty}^{0} \left\{ik_0 c'(x) - k_0^2 c(x)\right\} e^{2ik_0 x} dx, \tag{6.19}$$

where

$$H = \sum_{n=1}^{\infty} \frac{d_n e_n}{k_n}, \quad L = \sum_{n=1}^{\infty} \frac{d_n e_n (ik_0 - k_n)}{k_n (ik_0 + k_n)}, \tag{6.20}$$

$$M = \sum_{n=1}^{\infty} \frac{d_n (ik_0 + k_n)}{k_n} f_n, \quad N = \sum_{n=1}^{\infty} \frac{d_n (ik_0 - k_n)}{k_n} f_n, \tag{6.21}$$

with

$$f_n = s_n \int_{-\infty}^{0} \left\{ k_n^2 c(x) + k_n c^{'}(x) \right\} e^{(k_n + ik_0)x} dx, n = 1, 2, ... \tag{6.22}$$

Similarly, multiplying (6.16) by $e^{-ik_0 x}$ and integrating with respect to x from 0 to ∞, we obtain

$$T \left[L + \varepsilon P + \varepsilon \int_{0}^{\infty} \left\{ ik_0 c^{'}(x) - k_0^2 c(x) \right\} e^{ik_0 x} dx \right] + R(H + \varepsilon Q) = L + \varepsilon P, \tag{6.23}$$

where

$$P = \sum_{n=1}^{\infty} \frac{d_n(ik_0 - k_n)}{k_n} g_n, \quad Q = \sum_{n=1}^{\infty} \frac{d_n(ik_0 + k_n)}{k_n} g_n, \tag{6.24}$$

with

$$g_n = s_n \int_{0}^{\infty} \left\{ k_n^2 c(x) - k_n c^{'}(x) \right\} e^{-(k_n + ik_0)x} dx, \quad n = 1, 2, ... \tag{6.25}$$

It may be noted that the series represented by H, L, M, N, P and Q are convergent and can be computed numerically once the form of $c(x)$ is known and $k_n, n = 1, 2, \ldots,$ are found from (6.9).

We assume that R and T can be expanded in terms of ε in the forms

$$R = R_0 + \varepsilon R_1 + \ldots, \qquad T = T_0 + \varepsilon T_1 + \ldots \quad . \tag{6.26}$$

Substituting these expansions for R and T in equations (6.19) and (6.23) and equating the coefficients of ε^0 from both sides of (6.19) and (6.23), we obtain

$$LR_0 + HT_0 = H, \tag{6.27}$$

$$HR_0 + LT_0 = L. \tag{6.28}$$

Equations (6.27) and (6.28) produce

$$R_0 \equiv 0, \quad T_0 \equiv 1, \tag{6.29}$$

which are obvious, since when $\varepsilon = 0$, there is no bottom undulation and the incident surface wave train propagates without any hindrance resulting in no reflection.

Again, equating the coefficients of ε from both sides of equations (6.19) and (6.23) after writing $T = 1 + \varepsilon T_1 + ...$ and $R = \varepsilon R_1 + ...$, we obtain

$$LR_1 + HT_1 = - \int_{-\infty}^{0} \left\{ ik_0 c^{'}(x) - k_0^2 c(x) \right\} e^{2ik_0 x} dx, \tag{6.30}$$

$$HR_1 + LT_1 = - \int_{0}^{\infty} \left\{ ik_0 c^{'}(x) - k_0^2 c(x) \right\} dx. \tag{6.31}$$

These two equations give the first order corrections R_1 and T_1 for the reflection and transmission coefficients, respectively, in integral form as

$$R_1 = \frac{1}{L^2 - H^2}\Big[- L \int_{-\infty}^{0} \Big\{ ik_0 c^{'}(x) - k_0^2 c(x) \Big\} e^{2ik_0 x} dx + H \int_{0}^{\infty} \Big\{ ik_0 c^{'}(x) - k_0^2 c(x) \Big\} dx \Big], \tag{6.32}$$

$$T_1 = \frac{1}{L^2 - H^2}\Big[H \int_{-\infty}^{0} \Big\{ ik_0 c^{'}(x) - k_0^2 c(x) \Big\} e^{2ik_0 x} dx - L \int_{0}^{\infty} \Big\{ ik_0 c^{'}(x) - k_0^2 c(x) \Big\} dx \Big]. \tag{6.33}$$

R_1 and T_1 can easily be found out numerically once the shape function $c(x)$ is known.

6.2.3 Solution $\phi(x, y)$ in expansion form

Now substituting the values of A_n and B_n from equations (6.13) into (6.2) and then assuming $\phi = \phi_0 + \varepsilon\phi_1 + \ldots, \quad R = \varepsilon R_1 + \ldots, \quad T = 1 + \varepsilon T_1 + \ldots$, we obtain, by equating the coefficients of ε, $\phi_1(x, y)$ as

$$\phi_1(x, y) = \sum_{n=1}^{\infty} \Big\{ T_1(1 + \frac{ik_0}{k_n}) - R_1(1 - \frac{ik_0}{k_n}) \Big\} d_n \varphi_n(y) e^{k_n x} + R_1 \cosh k_0(h - y) e^{-ik_0 x} \quad \text{for } x < 0, \tag{6.34}$$

and

$$\phi_1(x, y) = \sum_{n=1}^{\infty} \Big\{ T_1(\frac{ik_0}{k_n} - 1) + R_1(1 + \frac{ik_0}{k_n}) \Big\} d_n \varphi_n(y) e^{-k_n x} + T_1 \cosh k_0(h - y) e^{ik_0 x} \quad \text{for } x > 0. \tag{6.35}$$

6.2.4 Comparison of expansion form of solution with the previous results and the available results

Davies and Heathershaw [19] solved the problem of water wave scattering by a sinusoidal varying topography on the sea-bed when the incidence was normal. In the solution method they applied Fourier transform to the governing equation and the boundary conditions. Its solution, the inverse transform of the velocity potential, is expressed in an integral form and the integral is then evaluated by contour-integration procedure for the asymptotic behaviour of the solution for the potential as $x \to -\infty$ and as $x \to +\infty$ (which corresponds to the propagating terms).

The problem of Davies and Heathershaw [19] is generalised to the problem of scattering of surface water waves by small undulation of the sea-bed which is solved in Section 3.3.

For this generalised problem the eigenfunction expansion solution for $\phi_1(x, y)$ is given by equations (3.34) and (3.35).

The obvious difference between the eigenfunction expansion solutions (6.34)-(6.35) with (3.34)-(3.35) can be observed. The difference is due to the fact that we have incorporated a new set of orthogonal functions $\{\varphi_n\}, n = 1, 2, \ldots$, as given by (6.3), containing the positive roots k_n of the new transcendental equation (6.9), in the expressions for $\phi_1(x, y)$ in equations (6.34)-(6.35).

6.2.5 A special bed surface

For comparing our results with available results, we choose the shape function $c(x)$ as

$$c(x) = \begin{cases} c_0 \sin lx & \text{for } -\dfrac{2m\pi}{l} \leq x \leq \dfrac{2m\pi}{l}, \\ 0 & \text{otherwise .} \end{cases} \tag{6.36}$$

with c_0 as the amplitude of the sinusoidal ripples, l the wave number of the sinusoidal ripples and m a positive integer. This represents a patch of sinusoidal ripples on an otherwise flat bottom, the patch consisting of $2m$ ripples having the same wave number l. For this case we obtain the reflection and transmission coefficients, respectively, as

$$R_1 = \frac{L}{L^2 - H^2} \frac{c_0 k_0^2 l}{l^2 - (2k_0)^2} \left(1 - e^{\frac{-4ik_0 m\pi}{l}}\right), \tag{6.37}$$

$$T_1 = \frac{-H}{L^2 - H^2} \frac{c_0 k_0^2 l}{l^2 - (2k_0)^2} \left(1 - e^{\frac{-4ik_0 m\pi}{l}}\right). \tag{6.38}$$

At the critical condition $l = 2k_0$, where the bed wavelength is half the surface wave length, equations (6.37) and (6.38) reveal that there is a Bragg resonance (as described by Davies [18]) between the surface waves and bed forms and consequently, at resonance,

$$R_1 = \frac{L}{L^2 - H^2} \frac{c_0 k_0 \, (-i\pi m)}{2}, \tag{6.39}$$

$$T_1 = \frac{-H}{L^2 - H^2} \frac{c_0 k_0 \, (-i\pi m)}{2}. \tag{6.40}$$

It is observed that although the theory breaks down where the solution is singular ($l = 2k_0$), a large amount of reflection of the incident wave energy by the bed forms is provided in the neighbourhood of this singularity.

However, the first order reflection coefficient R_1 and the transmission coefficient T_1 were obtained explicitly by Davies and Heathershaw [19] (also refer to Mandal and Basu [53] for the case without surface tension and $\theta = 0$). These coefficients can also be obtained from equations (3.36) and (3.37) as

$$R_1 = \begin{cases} \dfrac{4c_0k_0^2}{l(2k_0h + \sinh 2k_0h)} \dfrac{\sin(\frac{4\pi m k_0}{l})}{(\frac{2k_0}{l})^2 - 1} & \text{for } \dfrac{2k_0}{l} \neq 1, \\ \dfrac{c_0 l m\pi}{lh + \sinh lh} & \text{for } \dfrac{2k_0}{l} = 1, \end{cases} \tag{6.41}$$

and

$$T_1 = 0. \tag{6.42}$$

6.3 Case-II: Oblique incidence

6.3.1 Formulation of the problem

Considering the same statement and assumptions as in the formulation for the oblique incidence case, described in Section 2.3, we proceed to solve the boundary value problem (equations (2.13)-(2.15)) assuming the incident wave given by equation (2.16). Hence, we have the same far-field requirement as given by (2.18).

6.3.2 Solution procedure

The bottom condition (2.15) can be approximated up to the first order of the small parameter ε as

$$\frac{\partial \psi}{\partial y} - \varepsilon\left[\frac{\partial}{\partial x}\left\{c(x)\frac{\partial \psi}{\partial x}\right\} + c(x)\frac{\partial^2 \psi}{\partial z^2}\right] = 0 \quad \text{on} \quad y = h. \tag{6.43}$$

Now, in view of the geometry of the problem, i.e., because of the uniformity in the $z-$direction, $\psi(x, y, z)$ can be written as

$$\psi(x, y, z) = \phi(x, y)e^{i\nu z}. \tag{6.44}$$

Then $\phi(x, y)$ satisfies the equations

$$(\nabla^2 - \nu^2)\phi = 0 \quad \text{in} \quad -\infty < x < \infty,\ 0 < y < h, \tag{6.45}$$

$$\frac{\partial \phi}{\partial y} + K\phi = 0 \quad \text{on} \quad y = 0, \tag{6.46}$$

$$\frac{\partial \phi}{\partial y} - \varepsilon\left[\frac{\partial}{\partial x}\left\{c(x)\frac{\partial \phi}{\partial x}\right\} - \nu^2 c(x)\phi(x, y)\right] = 0 \quad \text{on} \quad y = h, \tag{6.47}$$

$$\phi(x, y) \sim \begin{cases} (e^{i\mu x} + Re^{-i\mu x})\cosh k_0(h - y) & \text{as} \quad x \to -\infty, \\ Te^{i\mu x}\cosh k_0(h - y) & \text{as} \quad x \to +\infty, \end{cases} \tag{6.48}$$

where ∇^2 is the two-dimensional Laplacian operator.

By Havelock's expansion of water wave potential, we may choose the eigenfunction expansion for $\phi(x,y)$ in the form

$$\phi(x,y) = \begin{cases} \cosh k_0(h-y)(e^{i\mu x} + Re^{-i\mu x}) + \sum_{n=1}^{\infty} A_n \varphi_n(y) e^{q_n x} & \text{for } x < 0, \\ T \cosh k_0(h-y) e^{i\mu x} + \sum_{n=1}^{\infty} B_n \varphi_n(y) e^{-q_n x} & \text{for } x > 0, \end{cases} \tag{6.49}$$

where φ_n is given by the equation (6.3) with $q_n^2 = k_n^2 + \nu^2$. The constants $k_n,\ n = 1, 2, \ldots,$ are such that $k_n > 0,\ n = 1, 2, \ldots,$ and are *to be determined*. A_n and $B_n,\ n = 1, 2, \ldots,$ are also unknown constants. It is obvious that the representation (6.49) for $\phi(x, y)$ satisfies equation (6.45) in $(0 < y < h)$, the boundary condition (6.46) and the far-field requirements (6.48).

In addition to these conditions $\phi(x, y)$ has to satisfy the conditions (i) and (ii) in Section 6.2 and it also must satisfy the boundary condition (6.47) for $x < 0, x > 0$.

The first two conditions produce, respectively,

$$\sum_{n=1}^{\infty} (A_n - B_n)\varphi_n = (T - 1 - R) \cosh k_0(h - y), \quad 0 < y < h, \tag{6.50}$$

and

$$\sum_{n=1}^{\infty} q_n(A_n + B_n)\varphi_n = i\mu(T - 1 + R) \cosh k_0(h - y), \quad 0 < y < h. \tag{6.51}$$

From equations (6.50) and (6.51), the constants $A_n, B_n,\ n = 1, 2, \ldots,$ can be obtained if and only if the sequence $\{\varphi_n, 0 < y < h : n = 1, 2, \ldots\}$ forms an orthogonal set. So we now assume that the equation (6.6) and hence φ satisfies the same set of equations (6.8a)-(6.8c). We have the same transcendental equation involving λ given by equation (6.9).

Using the orthogonality properties of $\varphi_n(y),\ n = 1, 2, \ldots,$ in equations (6.50) and (6.51), we obtain

$$A_n - B_n = (T - 1 - R)\frac{c_n}{N_n}, \tag{6.52}$$

$$A_n + B_n = (T - 1 + R)\frac{i\mu c_n}{q_n N_n}, \tag{6.53}$$

where c_n is given by equation (6.12).

Thus,

$$A_n = \left\{(T-1)\left(1 + \frac{i\mu}{q_n}\right) - R\left(1 - \frac{i\mu}{q_n}\right)\right\} d_n, \tag{6.54}$$

$$B_n = \left\{(T-1)\left(\frac{i\mu}{q_n} - 1\right) + R\left(\frac{i\mu}{q_n} + 1\right)\right\} d_n. \tag{6.55}$$

where d_n is given by equation (6.14).

The boundary condition (6.47) is now used for the regions $x < 0$ and $x > 0$ respectively. Using the eigenfunction expansions (6.49) in (6.47) (for $x < 0$ and $x > 0$), after some simplifications we obtain

$$\sum_{n=1}^{\infty} d_n \left\{ (T-1)\left(1+\frac{i\mu}{q_n}\right) - R\left(1-\frac{i\mu}{q_n}\right)\right\} \Big[e_n + \varepsilon\{q_n c'(x) + k_n^2 c(x)\}s_n\Big] e^{q_n x}$$
$$+\varepsilon\Big[\{i\mu c'(x) - k_0^2 c(x)\}e^{i\mu x} - R\{i\mu c'(x) + k_0^2 c(x)\}e^{-i\mu x}\Big] = 0 \qquad \text{for } x < 0, \qquad (6.56)$$
$$\sum_{n=1}^{\infty} d_n \left\{ (T-1)\left(\frac{i\mu}{q_n}-1\right) + R\left(\frac{i\mu}{q_n}+1\right)\right\} \Big[e_n + \varepsilon\{(-1)q_n c'(x) + k_n^2 c(x)\}s_n\Big] e^{-q_n x}$$
$$+\varepsilon T\left\{i\mu c'(x) - k_0^2 c(x)\right\} e^{i\mu x} = 0 \qquad \text{for } x > 0, \qquad (6.57)$$

where e_n and s_n are given by the equations (6.17) and (6.18) respectively.

Now multiplying (6.56) by $e^{i\mu x}$ and integrating with respect to x from $-\infty$ to 0, we obtain

$$T(H_1 + \varepsilon M_1) + R\Big[L_1 + \varepsilon N_1 - \varepsilon\int_{-\infty}^{0}\{i\mu c'(x) + k_0^2 c(x)\}dx\Big]$$
$$= H_1 + \varepsilon M_1 - \varepsilon\int_{-\infty}^{0}\{i\mu c'(x) - k_0^2 c(x)\}e^{2i\mu x}dx, \qquad (6.58)$$

where

$$H_1 = \sum_{n=1}^{\infty}\frac{d_n e_n}{q_n}, \quad L_1 = \sum_{n=1}^{\infty}\frac{d_n e_n(i\mu - q_n)}{q_n(i\mu + q_n)}, \qquad (6.59)$$
$$M_1 = \sum_{n=1}^{\infty}\frac{d_n(i\mu + q_n)}{q_n}\tilde{f}_n, \quad N_1 = \sum_{n=1}^{\infty}\frac{d_n(i\mu - q_n)}{q_n}\tilde{f}_n, \qquad (6.60)$$

with

$$\tilde{f}_n = s_n\int_{-\infty}^{0}\{q_n c'(x) + k_n^2 c(x)\}e^{(q_n + i\mu)x}dx, n = 1, 2, \ldots \qquad (6.61)$$

Similarly, multiplying (6.57) by $e^{-i\mu x}$ and integrating with respect to x from 0 to ∞, we obtain

$$T\Big[L_1 + \varepsilon P_1 + \varepsilon\int_{0}^{\infty}\{i\mu c'(x) - k_0^2 c(x)\}dx\Big] + R(H_1 + \varepsilon Q_1) = L_1 + \varepsilon P_1, \qquad (6.62)$$

where

$$P_1 = \sum_{n=1}^{\infty}\frac{d_n(i\mu - q_n)}{q_n}\tilde{g}_n, \quad Q_1 = \sum_{n=1}^{\infty}\frac{d_n(i\mu + q_n)}{q_n}\tilde{g}_n, \qquad (6.63)$$

with

$$\tilde{g}_n = s_n\int_{0}^{\infty}\{(-1)q_n c'(x) + k_n^2 c(x)\}e^{-(q_n + i\mu)x}dx, \; n = 1, 2, \ldots \qquad (6.64)$$

It may be noted that the series represented by H_1, L_1, M_1, N_1, P_1 and Q_1 are convergent and can be computed numerically once the form of $c(x)$ is known and $k_n,\ n = 1, 2, \ldots,$ are found from equation (6.9).

Substituting these expansions for R and T, given by (6.26), in (6.58) and (6.62) and equating the the coefficients of ε^0 from both sides of (6.58) and (6.62), we obtain

$$T_0 H_1 + R_0 L_1 = H_1, \tag{6.65}$$

$$T_0 L_1 + R_0 H_1 = L_1. \tag{6.66}$$

The equations (6.65) and (6.66) produce

$$R_0 \equiv 0, \quad T_0 \equiv 1. \tag{6.67}$$

Again, equating the coefficients of ε from both sides of (6.58) and (6.62), we obtain

$$L_1 R_1 + H_1 T_1 \;=\; -\int_{-\infty}^{0} \{i\mu c'(x) - k_0^2 c(x)\} e^{2i\mu x} dx, \tag{6.68}$$

$$H_1 R_1 + L_1 T_1 \;=\; -\int_{0}^{\infty} \{i\mu c'(x) - k_0^2 c(x)\} dx. \tag{6.69}$$

These two equations give the first order corrections R_1 and T_1 for the reflection and transmission coefficients, respectively, in integral form as

$$R_1 = \frac{1}{L_1^2 - H_1^2}\left[- L_1 \int_{-\infty}^{0} \left\{ i\mu c'(x) - k_0^2 c(x) \right\} e^{2i\mu x} dx + H_1 \int_{0}^{\infty} \left\{ i\mu c'(x) - k_0^2 c(x) \right\} dx \right], \tag{6.70}$$

$$T_1 = \frac{1}{L_1^2 - H_1^2}\left[H_1 \int_{-\infty}^{0} \left\{ i\mu c'(x) - k_0^2 c(x) \right\} e^{2i\mu x} dx - L_1 \int_{0}^{\infty} \left\{ i\mu c'(x) - k_0^2 c(x) \right\} dx \right]. \tag{6.71}$$

The results for normal incidence can be obtained by putting $\theta = 0$.

6.3.3 Solution $\phi(x, y)$ in expansion form

Now substituting the values of A_n and B_n from equations (6.54) and (6.55) into (6.49) and then assuming $\phi = \phi_0 + \varepsilon \phi_1 + \ldots,\quad R = \varepsilon R_1 + \ldots,\quad T = 1 + \varepsilon T_1 + \ldots,$ we obtain by equating the coefficients of ε as

$$\phi_1(x, y) = \sum_{n=1}^{\infty} \left\{ T_1 (1 + \frac{i\mu}{q_n}) - R_1 (1 - \frac{i\mu}{q_n}) \right\} d_n \varphi_n(y) e^{q_n x} + R_1 \cosh k_0 (h - y) e^{-i\mu x} \quad \text{for } x < 0, \tag{6.72}$$

and

$$\phi_1(x,y)=\sum_{n=1}^{\infty}\left\{T_1(\frac{i\mu}{q_n}-1)+R_1(1+\frac{i\mu}{q_n})\right\}d_n\varphi_n(y)e^{-q_nx}+T_1\cosh k_0(h-y)e^{i\mu x}\quad \text{for } x>0. \tag{6.73}$$

6.3.4 Comparison with previous results and extended results of Davies and Heathershaw (1984) for oblique case

The problem of water wave scattering by a sinusoidal varying topography on the sea-bed, considered by Davies and Heathershaw [19], is generalised to the case of oblique incidence in Section 4.3 in which the solution for $\phi_1(x,y)$ is given by equations (4.33) and (4.34).

The obvious difference between the eigenfunction expansion solutions (6.72)-(6.73) with (4.33)-(4.34) can be observed. The difference is due to the fact that we have incorporated a new set of orthogonal functions $\{\varphi_n\}$, $n=1,2,\ldots$, as given by (6.3), containing the positive roots k_n of the new transcendental equation (6.9), in the expressions for $\phi_1(x,y)$ in (6.72)-(6.73).

6.3.5 A special bed surface

For comparing our results with available results, we choose the same shape function $c(x)$ as considered previously (equation (6.36)). In this case, the equations (6.70)-(6.71) reduce, respectively, to

$$R_1=\frac{L_1}{L_1^2-H_1^2}\,\frac{c_0l\;k_0^2\cos 2\theta}{l^2-(2\mu)^2}\left(1-e^{\frac{-4i\mu\pi m}{l}}\right), \tag{6.74}$$

$$T_1=\frac{-H_1}{L_1^2-H_1^2}\,\frac{c_0l\;k_0^2\cos 2\theta}{l^2-(2\mu)^2}\left(1-e^{\frac{-4i\mu\pi m}{l}}\right). \tag{6.75}$$

As in the case of the normal incidence, at the critical condition $l=2\mu$, equations (6.74) and (6.75) reveal that there is a Bragg resonance between the surface waves and bed forms and hence,

$$R_1=\frac{L_1}{L_1^2-H_1^2}\;c_0k_0^2\cos 2\theta(-im\pi), \tag{6.76}$$

$$T_1=\frac{-H_1}{L_1^2-H_1^2}\;c_0k_0^2\cos 2\theta(-im\pi). \tag{6.77}$$

at resonance. It is observed that although the theory breaks down where the solution is singular ($l=2\mu$), a large amount of reflection of the incident wave energy by the bed forms is provided in the neighbourhood of this singularity.

For such sinusoidal undulations on the sea-bed, the first order reflection coefficient R_1 and transmission coefficient T_1 were obtained by Mandal and Basu [53] (without surface tension). These coefficients can also be obtained from equations (4.35) and (4.36) as

$$R_1 = \begin{cases} \dfrac{2k_0 c_0 \sec\theta \cos 2\theta}{(2k_0 h + \sinh 2k_0 h)} \dfrac{(2k_0/l)}{(2\mu/l)^2 - 1} \sin\left(\dfrac{4\pi\mu m}{l}\right), & \text{for } 2\mu \neq l, \\ \dfrac{2k_0 c_0 \sec\theta \cos 2\theta}{(2k_0 h + \sinh 2k_0 h)} \dfrac{2k_0}{l} m\pi, & \text{for } 2\mu = l, \end{cases} \tag{6.78}$$

and

$$T_1 = 0. \tag{6.79}$$

6.4 Numerical results

Corresponding to the sinusoidal bottom undulations in Sections 6.2 and 6.3, the numerical computation is shown here for the first order reflection coefficient $|R_1|$ given by equations (6.74), and (6.78). In Figure 6.1, $|R_1|$ is depicted against the wave number Kh for number of ripples $m = 1$ and for $\theta = 0, c_0/h = 0.1, lh = 1$. It is observed that the two graphs for $|R_1|$ by the present method and by Davies and Heathershaw method obtained in Figure 6.1 are almost identical.

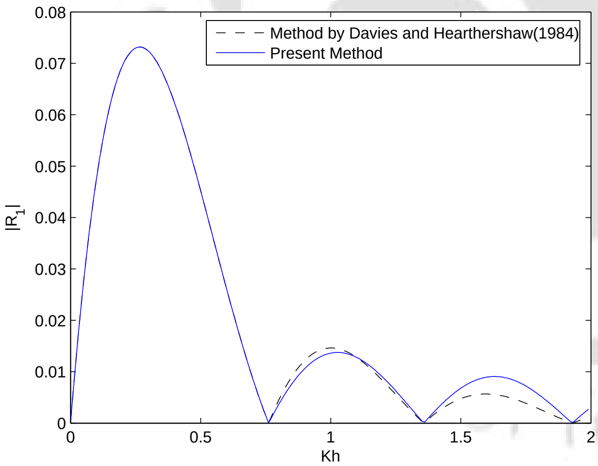

Figure 6.1: *Reflection coefficient* $|R_1|$ *against the wave number* Kh *for* $m = 1; c_0/h = 0.1; lh = 1.$

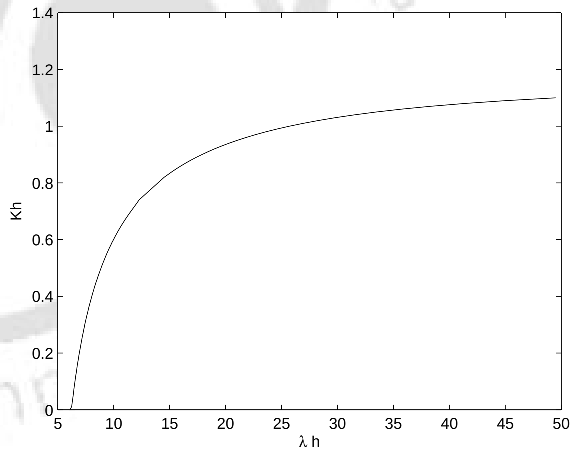

Figure 6.2: *Reflection coefficient against the wave number* Kh *for* $m = 2; c_0/h = 0.1; lh = 1$

In Figure 6.2, Kh is depicted against λh for $c_0/h = 0.1; lh = 1$ and $m = 1$. This figure shows that Kh increases as λh increases approximately up to $\lambda = 30$. As λ further increases

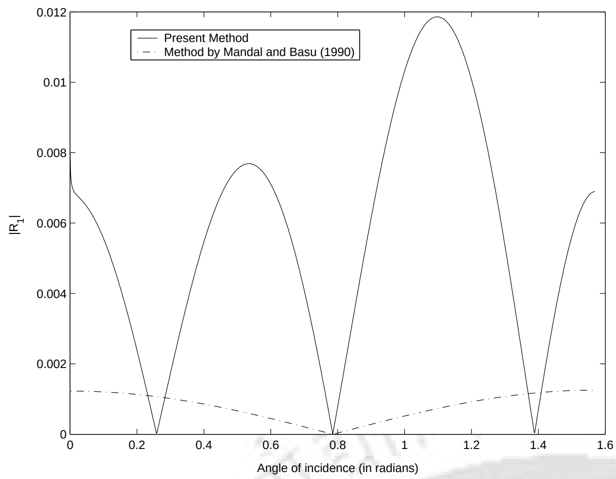

Figure 6.3: *Reflection coefficient* $|R_1|$ *against the angle of incidence* θ *for* $Kh = 0.001; kh = 0.0316; k_nh = 4.5613; c_0/h = 0.1; lh = 1; m = 1$.

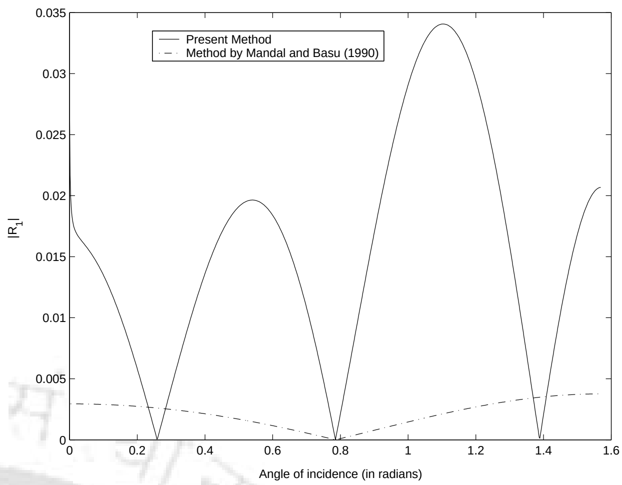

Figure 6.4: *Reflection coefficient* $|R_1|$ *against the angle of incidence* θ *for* $Kh = 0.001; kh = 0.0316; k_nh = 4.5613; c_0/h = 0.1; lh = 1; m = 3$.

Kh has the asymptotic value 1.1. This is obvious since for large λ, the equation (6.9) becomes

$$K \tan k_n h = k_n, \tag{6.80}$$

which is independent of λ.

In Figures 6.3 and 6.4, the reflection coefficient $|R_1|$ is also depicted against the angle of incidence θ for $Kh = 0.001; kh = 0.0316; k_nh = 4.5613; c_0/h = 0.1; lh = 1$ and $m = 1$ and $m = 3$ respectively. From the graphs it is clear that for $\theta = \frac{\pi}{4}$, $|R_1|$ vanishes independently of the shape function as observed by the results of Mandal and Basu [53].

The above four figures show that the method employed here produces correct numerical estimates for the reflection coefficient. For other forms of the shape function, this method is expected to yield correct results without any difficulty.

6.5 Conclusion

A direct method of eigenfunction expansion is used to solve the problem of oblique water wave scattering by small undulation on an otherwise flat bottom of a sea for normal and oblique incidences. The reflection coefficient up to first order is obtained by this method for a sinusoidal bottom undulation, which when depicted graphically against the wave number, coincides almost exactly with the result for the same obtained in Chapter 5 and also from the known result given in the literature. Though the expressions obtained for the first order velocity potential, reflection coefficient and transmission coefficient by the present method are slightly different from the results derived in earlier chapters and the results available in the literature, they still lead to very accurate numerical results. Also, the present method appears to be simple and straightforward, and avoids the use of Green's integral theorem or the Fourier transform technique employed elsewhere.

Chapter 7

Scattering of Surface Waves by Small Undulation on a Porous Sea-bed

7.1 Introduction

In Chapters 2, 3, 4 and 6, the problem of propagation of surface waves over small undulation on the sea-bed is investigated by assuming the bed to be impermeable. In the recent times, due to many interesting applications of the scattering theory of surface water waves, many researchers have turned their attention to problems related to porous bed rather than an impermeable one. In this chapter we investigate the problem of scattering of surface water waves by small undulation on a sea-bed of finite depth assuming the sea-bed to be composed of porous material of specific type. The motion of the fluid inside the porous bed is not analysed here and it is assumed that the fluid motions are such that the resulting boundary condition on the sea-bed as is used here, holds good and depends on a known parameter G', called the porosity parameter, in the analysis. A Fourier transform method is utilised in obtaining the complete solution of the mixed boundary value problem under the assumption that the undulation of the porous sea-bed is small enough so that a regular perturbation expansion in terms of a small undulation parameter is applicable. The reflection and transmission coefficients are determined and applied to the case of a patch of sinusoidal undulations on the bed. The results are presented in graphical forms.

7.2 Case-I: Normal incidence

7.2.1 Formulation of the problem

A right-handed rectangular Cartesian co-ordinate system is considered in which x-axis is the position of the undisturbed free surface of the sea and y-axis is measured positive vertically downwards from the undisturbed free surface. The sea-bed with small undulation, composed of porous material of specific type, is described by $y = h+\varepsilon c(x)$ where $c(x)$ is a function with compact support and describes the bottom undulation, h denotes the uniform finite depth of the sea far to either side of the undulation of the bottom so that $c(x) \to 0$ as $|x| \to \infty$ and the non-dimensional number $\varepsilon(\ll 1)$ a measure of smallness of the undulation. It is also assumed that the fluid is incompressible and inviscid, and the motion irrotational and time harmonic. The usual assumptions of linear water wave theory and the removal of the harmonic time dependence $\exp(-i\sigma t)$ lead to the following equations for the time-independent complex-valued potential function $\phi(x, y)$:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{in } -\infty < x < \infty, \quad 0 \le y \le h + \varepsilon c(x), \tag{7.1}$$

$$\frac{\partial\phi}{\partial y} + K\phi = 0 \quad \text{on} \quad y = 0, \tag{7.2}$$

$$\frac{\partial\phi}{\partial n} - G'\phi = 0 \quad \text{on} \quad y = h + \varepsilon c(x), \tag{7.3}$$

where $K = \sigma^2/g, \sigma$ is the angular frequency of the incoming water wave train with time dependence $e^{-i\sigma t}$, g is the acceleration due to gravity, $\partial/\partial n$ denotes the normal derivative at a point (x, y) on the bottom, and G' is the porous effect parameter corresponding to the sea-bed under consideration. In the limit of zero permeability, i.e., when the bed is impermeable, equation (7.3) becomes $\partial\phi/\partial n = 0$ on the bottom $y = h + \varepsilon c(x)$, the same equation as (2.3).

Assume $\phi_0(x, y)$ to be the progressive wave train which is incident upon the bottom undulation from the direction of $x = -\infty$. It is, then, partially reflected by and partially transmitted over the bottom undulation so that ϕ has the far-field behaviour given by

$$\phi(x, y) \sim \begin{cases} \phi_0(x, y) + R\phi_0(-x, y), & x \to -\infty, \\ T\phi_0(x, y), & x \to +\infty. \end{cases} \tag{7.4}$$

7.2.2 Solution procedure

The bottom condition (7.3) can be approximated up to first order of the small parameter ε as

$$\frac{\partial\phi}{\partial y} - \varepsilon\frac{\partial}{\partial x}\left\{c(x)\frac{\partial\phi}{\partial x}\right\} - G'\left\{\phi + \varepsilon c(x)\frac{\partial\phi}{\partial y}\right\} = 0 \quad \text{on} \quad y = h. \tag{7.5}$$

The boundary condition (7.5) suggests that ϕ, R and T introduced above can be expanded in terms of ε as

$$\left.\begin{array}{l} \phi = \phi_0 + \varepsilon\phi_1 + O(\varepsilon^2) \\ R = R_0 + \varepsilon R_1 + O(\varepsilon^2) \\ T = T_0 + \varepsilon T_1 + O(\varepsilon^2) \end{array}\right\}. \tag{7.6}$$

Substituting the expansions (7.6) in equations (7.1), (7.2), (7.5) and (7.4) we find after equating the coefficients of ε^0 and ε from both sides, that the functions $\phi_0(x,y)$ and $\phi_1(x,y)$, respectively, satisfy the following BVPs:

BVP-I $\phi_0(x,y)$ satisfies

$$\frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial^2\phi_0}{\partial y^2} = 0 \quad \text{in} \quad 0 \le y \le h, \tag{7.7}$$

$$\frac{\partial\phi_0}{\partial y} + K\phi_0 = 0 \quad \text{on} \quad y = 0, \tag{7.8}$$

$$\frac{\partial\phi_0}{\partial y} - G'\ \phi_0 = 0 \quad \text{on} \quad y = h, \tag{7.9}$$

$$\text{and} \quad R_0 = 0, \quad T_0 = 1. \tag{7.10}$$

BVP-II $\phi_1(x,y)$ satisfies

$$\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0 \quad \text{in} \quad 0 \le y \le h, \tag{7.11}$$

$$\frac{\partial\phi_1}{\partial y} + K\phi_1 = 0 \quad \text{on} \quad y = 0, \tag{7.12}$$

$$\frac{\partial\phi_1}{\partial y} - G'\phi_1 = \frac{\partial}{\partial x}\left\{c(x)\frac{\partial\phi_0}{\partial x}\right\} + G'\left\{c(x)\frac{\partial\phi_0}{\partial y}\right\} \equiv p(x) \quad \text{on} \quad y = h, \tag{7.13}$$

$$\text{and} \quad \phi_1(x,y) \sim \begin{cases} R_1\phi_0(-x,y) & \text{as} \quad x \to -\infty, \\ T_1\phi_0(x,y) & \text{as} \quad x \to +\infty. \end{cases} \tag{7.14}$$

It may be noted that the BVP-I corresponds to the surface wave propagation in water of uniform finite depth h. The solution of BVP-I can be written as

$$\phi_0(x,y) = \frac{K\sinh k_0 y - k_0\cosh k_0 y}{K\sinh k_0 h - k_0\cosh k_0 h} e^{ik_0 x}, \tag{7.15}$$

where k_0, the wave number of the incident wave, is the unique positive root (for a given G') of the equation

$$K + G' = \left(k + \frac{G'K}{k}\right)\tanh kh. \tag{7.16}$$

Hence, $p(x)$ can be written as

$$p(x) = ik_0\frac{d}{dx}\left\{c(x)e^{ik_0x}\right\} + G'k_0A\ c(x)e^{ik_0x} \quad \text{on} \quad y = h, \tag{7.17}$$

$$\text{where} \quad A = \frac{K\cosh k_0h - k_0\sinh k_0h}{K\sinh k_0h - k_0\cosh k_0h}. \tag{7.18}$$

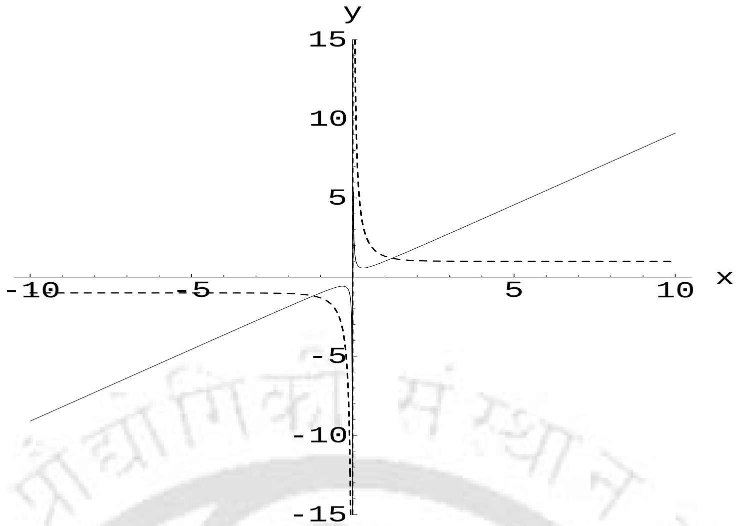

Figure 7.1: *Roots of the equation* $K + G' = (k + G'K/k)\tanh kh$ *for given* $Kh = 1; G = 0.1$

Value of G	Value of Kh	Roots of $K + G' = (k + G'K/k)\tanh kh$
0.1	0.5	0.8248, -0.8248
0.1	1.0	1.2258, -1.2258
0.1	1.5	1.6347, -1.6347
0.1	2.0	2.0712, -2.0712

Table 7.1: *Roots of* $K + G' = (k + G'K/k)\tanh kh$ *for different values of* Kh *when G=0.1*

To solve the BVP-II we now assume that ϕ_1 is such that the Fourier transform of ϕ_1 with respect to x, denoted by $\overline{\phi}_1$, exists and is given by

$$\overline{\phi}_1(\xi, y) = \int_{-\infty}^{\infty} \phi_1(x, y) e^{i\xi x} dx, \tag{7.19}$$

with its inverse

$$\phi_1(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\phi}_1(\xi, y) e^{-i\xi x} d\xi. \tag{7.20}$$

We observe that such Fourier transform exists if we make an artificial assumption that K possesses a small imaginary part, as given by $i\mu'\sigma/g$, where $\mu' > 0$ is very small which will be taken to be zero (in eliminating sense) at the end of the analysis.

Now, taking Fourier transform of equation (7.11) and the boundary conditions (7.12) and

(7.13) with respect to the horizontal space variable x, we obtain

$$\frac{\partial^2\overline{\phi}_1}{\partial y^2} - \xi^2\overline{\phi}_1 = 0 \quad \text{in} \quad -\infty < \xi < \infty,\ 0 \le y \le h, \tag{7.21}$$

$$\frac{\partial\overline{\phi}_1}{\partial y} + K\overline{\phi}_1 = 0 \quad \text{on} \quad y = 0, \tag{7.22}$$

$$\frac{\partial\overline{\phi}_1}{\partial y} - G'\overline{\phi}_1 = \overline{V}(\xi) \quad \text{on} \quad y = h, \tag{7.23}$$

where

$$\overline{V}(\xi) = \int_{-\infty}^{\infty} p(x)e^{i\xi x}dx. \tag{7.24}$$

The solution of (7.21) subject to the boundary conditions (7.22) and (7.23) can be written as

$$\overline{\phi}_1(\xi, y) = \frac{(\xi\cosh\xi y - K\sinh\xi y)\overline{V}(\xi)}{\xi[(\xi + KG'/\xi)\sinh\xi h - (K + G')\cosh\xi h]}. \tag{7.25}$$

Taking inverse Fourier transform, the solution for the velocity potential can be written in the form

$$\phi_1(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(\xi\cosh\xi y - K\sinh\xi y)\overline{V}(\xi)e^{-i\xi x}}{\xi[(\xi + KG'/\xi)\sinh\xi h - (K + G')\cosh\xi h]}\,d\xi. \tag{7.26}$$

We now obtain the final result from (7.26) by contour integration using the residue theorem.

We observe that equation (7.26) also has certain singularities (lying on the ξ-axis) other than $\xi = 0$. Replacing K by $\widehat{K} = (\sigma^2 + i\mu'\sigma)/g$ in equation (7.26), the singularities are displaced off the ξ-axis to the upper and the lower half planes. Hence, we write

$$\phi_1(x, y) = \lim_{\mu'\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\xi, y)}{G_{\mu'}(\xi, h)}d\xi, \tag{7.27}$$

where

$$F(\xi, y) = [\xi\cosh\xi y - \widehat{K}\sinh\xi y]\overline{V}(\xi)e^{-i\xi x}, \tag{7.28}$$

$$G_{\mu'}(\xi, h) = \xi[(\xi + \widehat{K}G'/\xi)\sinh\xi h - (\widehat{K} + G')\cosh\xi h]. \tag{7.29}$$

If $\widehat{K} = \widehat{K}_1 + i\widehat{K}_2$, then $\widehat{K}_2 = \mu' K/\sigma$ which is very small and if $\zeta = \alpha + i\beta$ is a zero of the expression (7.29), then ζ (up to first order of ε) can be determined as

$$\zeta = \pm\alpha_n \pm i\beta_n, \quad \text{and} \quad \zeta = \pm(k_0 + \gamma) \pm i\beta'_n, \tag{7.30}$$

where

$$\left.\begin{array}{lll}
\alpha_n & = & \widehat{K}_2\alpha_n^{(1)}, \quad \alpha_n^{(1)} = \dfrac{C}{D} \\
C & = & -[\cos\beta_n h - \widehat{K}_2 G'(\pm\beta_n)^{-1}\sin(\pm\beta_n h)] \\
D & = & (\widehat{K}_1 h + G'h - 1 - \widehat{K}_1 G'\beta_n^{-2})\sin(\pm\beta_n h) \\
 & & -\{(\pm\beta_n) - \widehat{K}_1 G'(\pm\beta_n)^{-1}\}h\cos\beta_n h, \text{ for } \beta_n > 0 \\
\beta_n\text{'s are roots of} & & (\beta - \widehat{K}_1 G'/\beta)\tan\beta h + (\widehat{K}_1 + G') = 0, \\
\gamma & = & \widehat{K}_2\alpha_n'^{(1)}, \quad \alpha_n'^{(1)} = \dfrac{C_1}{C_2 - C_3} \\
C_1 & = & -(\widehat{K}_1 + G')\cosh k_0 h + \{\pm k_0 + \widehat{K}_1 G'(\pm k_0)^{-1}\}\sinh(\pm k_0 h) \\
C_2 & = & \widehat{K}_2(\widehat{K}_1 h + G'h - 1 - \widehat{K}_1 G' k_0^{-2})\sinh(\pm k_0 h) \\
C_3 & = & \widehat{K}_2 h\{\pm k_0 + \widehat{K}_1 G'(\pm k_0)^{-1}\}\cosh k_0 h \\
\beta_n' & = & \widehat{K}_2\beta_n'^{(1)}, \quad \beta_n'^{(1)} = \dfrac{C_4}{C_5} \\
C_4 & = & \widehat{K}_2\cosh(k_0 h) + \{\widehat{K}_1 + G' - (\pm k_0)^{-1}\widehat{K}_2 G'\}\sinh(\pm k_0 h), \\
C_5 & = & \widehat{K}_2[\{\pm k_0 + \widehat{K}_1 G'(\pm k_0)^{-1}\}h\cosh(k_0 h) + (1 - \widehat{K}_1 G' k_0^{-2})\sinh(\pm k_0 h)]
\end{array}\right\}. \tag{7.31}$$

Here the contour consists of the portion $-R$ to R on the real ξ-axis and a semicircle centred at the origin and having a large radius R. The semicircle must be taken in the upper half ζ-plane ($\zeta = \xi + i\eta$) in anticlockwise direction or in the lower half plane in clockwise direction, according as $x < 0$ or $x > 0$. In the limit as $R \to \infty$, the required range of integration is recovered, since the integration along the semicircle makes a zero contribution. Hence, by using the residue theorem,

$$\begin{aligned}
\phi_1(x,y) &= \lim_{\mu'\to 0}(-i)\left[\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\zeta,y)}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=\alpha_n+i\beta_n}\right. \\
&\quad \left.+\text{Res}\left\{\frac{F(\zeta,y)}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=(k_0+\gamma)+i\beta_n'}\right] \quad \text{for } x<0,
\end{aligned} \tag{7.32}$$

and

$$\begin{aligned}
\phi_1(x,y) &= \lim_{\mu'\to 0} i\left[\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\zeta,y)}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=-\alpha_n-i\beta_n}\right. \\
&\quad \left.+\text{Res}\left\{\frac{F(\zeta,y)}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=-(k_0+\gamma)-i\beta_n'}\right] \quad \text{for } x>0,
\end{aligned} \tag{7.33}$$

which imply that

$$\begin{aligned}
\phi_1(x,y) &= \lim_{\mu'\to 0}(-i)\left[\sum_{n=1}^{\infty}\text{Res}\left\{\frac{F(\zeta,y)}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=\alpha_n+i\beta_n}\right] \\
&+ \frac{-2iB(k_0^2 + G'k_0 A)\left\{\int_{-\infty}^{\infty} c(x)e^{2ik_0 x}dx\right\}}{k_0[2(k_0 + G'K/k_0)h + (1 - G'K/k_0^2)\sinh 2k_0 h]} \\
&\times\ (k_0\cosh k_0(h-y) - G'\sinh k_0(h-y))\,e^{-ik_0 x} \quad \text{for } x<0,
\end{aligned} \tag{7.34}$$

and

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0} i\left[\sum_{n=1}^{\infty} \text{Res}\left\{\frac{F(\zeta,y)}{G_{\mu'}(\zeta,h)}\right\}\Bigg|_{\zeta=-\alpha_n-i\beta_n}\right] \\ &+ \frac{-2iB(-k_0^2+G'k_0A)\left\{\int_{-\infty}^{\infty} c(x)dx\right\}}{k_0[2(k_0+G'K/k_0)h+(1-G'K/k_0^2)\sinh 2k_0h]} \\ &\times \ (k_0\cosh k_0(h-y) - G'\sinh k_0(h-y))\, e^{ik_0x} \text{ for } x>0, \end{aligned} \tag{7.35}$$

where

$$B = \frac{k_0}{k_0 - G'\tanh k_0h}. \tag{7.36}$$

The first term on right hand side of each of equations (7.34)-(7.35) represents the non-propagating modes which decay rapidly away from the undulation and the second term represents a propagating mode from the region of the bed disturbance. Comparing equations (7.34)-(7.35) with equation (7.14), the reflection and transmission coefficients can, respectively, be written as

$$R_1 = \frac{-2iB(k_0^2+G'k_0A)}{2(k_0+G'K/k_0)h+(1-G'K/k_0^2)\sinh 2k_0h}\int_{-\infty}^{\infty} c(x)e^{2ik_0x}dx, \tag{7.37}$$

$$\text{and} \quad T_1 = \frac{-2iB(-k_0^2+G'k_0A)}{2(k_0+G'K/k_0)h+(1-G'K/k_0^2)\sinh 2k_0h}\int_{-\infty}^{\infty} c(x)dx. \tag{7.38}$$

For beds with zero porosity, results (7.37) and (7.38) can be observed as the results (3.12) and (3.15) respectively.

R_1 and T_1 in equations (7.37)-(7.38) can be evaluated once the shape function $c(x)$ is known. In the next section we consider a special form of the function $c(x)$.

7.2.3 A special bed surface

Consider the interaction of progressive surface waves with a patch of sinusoidal ripples on the porous sea-bed, and the ripples do not imply any restriction on the bed wave number. The bed surface is given by

$$c(x) = \begin{cases} a\sin(lx+\delta'), & L_1 \le x \le L_2, \\ 0 & \text{otherwise}, \end{cases} \tag{7.39}$$

where

$$L_1 = \frac{-n\pi - \delta'}{l}, \quad L_2 = \frac{m\pi - \delta'}{l},$$

with a as the amplitude of the sinusoidal ripples, l the wave number of the sinusoidal ripples and δ' an arbitrary phase angle; and m and n positive integers. This represents a patch of sinusoidal ripples on an otherwise flat bottom, the patch consisting of $(n+m)/2$ ripples having the same wave number l. For this case we obtain the reflection and transmission coefficients, respectively, as

$$R_1 = \frac{-2iB(k_0^2+G'k_0A)}{2(k_0+G'K/k_0)h+(1-G'K/k_0^2)\sinh 2k_0h}\,\frac{al}{l^2-(2k_0)^2} \times\Big[(-1)^n e^{2ik_0L_1}-(-1)^m e^{2ik_0L_2}\Big], \tag{7.40}$$

and

$$T_1 = \frac{-2iB(-k_0^2+G'k_0A)}{2(k_0+G'K/k_0)h+(1-G'K/k_0^2)\sinh 2k_0h}\,\frac{(-a)}{l}\Big[(-1)^m-(-1)^n\Big]. \tag{7.41}$$

In the situation in which there is an integer number of ripple wavelengths in the patch $L_1 \le x \le L_2$ such that $m=n$ and $\delta'=0$, we find R_1 and T_1, respectively, as

$$R_1 = \frac{4B(k_0^2+G'k_0A)}{2(k_0+G'K/k_0)h+(1-G'K/k_0^2)\sinh 2k_0h}\,\frac{(-1)^{m+1}a\,l}{l^2-(2k_0)^2}\sin\left(\frac{2k_0m\pi}{l}\right), \tag{7.42}$$

and

$$T_1 = 0. \tag{7.43}$$

These results match exactly with those obtained by Davies and Heathershaw [19] when the bed has zero porosity. Equation (7.42) illustrates that for a given number of m ripples, the first order wave reflection coefficient is an oscillatory function in the quotient of twice the surface wavenumber to the ripple wavenumber. Furthermore, if the bed wave number is equal to twice the surface wavenumber ($2k_0=l$), then equation (7.42) reveals that there is a resonant Bragg-type interaction between the surface waves and bed forms. This resonant interaction is described by Davies [17, 18] and reported by Heathershaw [32] with experiment demonstration. Hence, at resonance we find from equation (7.42) that

$$R_1 = \frac{2B(k_0^2+G'k_0A)}{2(k_0+G'K/k_0)h+(1-G'K/k_0^2)\sinh 2k_0h}\,\frac{am\pi}{l}, \tag{7.44}$$

from which we find that the reflection coefficient becomes a constant multiple of the number of ripples in the patch. It indicates that relatively few bottom undulations with its wave number equal to approximately twice the surface wave number, may give rise to a very substantial reflected wave. A possible consequence of this is a coupling between ripple growth and wave reflection, which may be important in the problems of coastal protection.

7.3 Case-II: Oblique incidence

7.3.1 Formulation of the problem

We consider a right-handed rectangular Cartesian co-ordinate system where xz-plane is the position of the undisturbed free surface of the sea and the y-axis is measured positive vertically downwards from the undisturbed free surface. Considering the same small undulation and other assumptions as in the formulation for the normal incidence case as described in Section 7.2, we proceed to handle the case of oblique incidence. The usual assumptions of linear water wave theory and the removal of the harmonic time dependence $\exp(-i\sigma t)$ lead to equations for time-independent complex valued potential function $\psi(x, y, z)$ describing small motion in water satisfying

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \quad \text{in} \quad 0 \le y \le h + \varepsilon c(x), \tag{7.45}$$

$$\frac{\partial\psi}{\partial y} + K\psi = 0 \quad \text{on} \quad y = 0, \tag{7.46}$$

$$\frac{\partial\psi}{\partial n} - G'\psi = 0 \quad \text{on} \quad y = h + \varepsilon c(x), \tag{7.47}$$

where $\partial/\partial n$ denotes the normal derivative at a point (x, y, z) on the bottom.

Assume $\psi_0(x, y, z)$ to be the progressive wave train which is incident upon the bottom undulation from the direction of negative infinity. It is, then, partially reflected by and partially transmitted over the bottom undulation so that ψ has a far-field behaviour given by

$$\psi(x, y, z) \sim \begin{cases} \psi_0(x, y, z) + R\psi_0(-x, y, z), & \text{as} \quad x \to -\infty, \\ T\psi_0(x, y, z), & \text{as} \quad x \to +\infty. \end{cases} \tag{7.48}$$

7.3.2 Solution procedure

The bottom condition (7.47) can be approximated up to the first order of the small parameter ε as

$$\frac{\partial\psi}{\partial y} - \varepsilon\left[\frac{\partial}{\partial x}\left\{c(x)\frac{\partial\psi}{\partial x}\right\} + c(x)\frac{\partial^2\psi}{\partial z^2}\right] - G'\left[\psi + \varepsilon c(x)\frac{\partial\psi}{\partial y}\right] = 0 \quad \text{on} \quad y = h. \tag{7.49}$$

Now, in view of the geometry of the problem, i.e, because of the uniformity in the $z-$direction, we can assume $\psi(x, y, z)$ as

$$\psi(x, y, z) = \phi(x, y)e^{i\nu z}. \tag{7.50}$$

Then $\phi(x,y)$ satisfies the equations

$$(\nabla^2-\nu^2)\phi=0 \quad \text{in} \quad -\infty<x<\infty,\ 0\le y\le h, \tag{7.51}$$
$$\frac{\partial\phi}{\partial y}+K\phi=0 \quad \text{on} \quad y=0, \tag{7.52}$$
$$\frac{\partial\phi}{\partial y}-\varepsilon\left[\frac{\partial}{\partial x}\left\{c(x)\frac{\partial\phi}{\partial x}\right\}-\nu^2c(x)\phi(x,y)\right]-G'\left[\phi+\varepsilon c(x)\frac{\partial\phi}{\partial y}\right]=0 \quad \text{on} \quad y=h, \tag{7.53}$$
$$\phi(x,y)\sim\begin{cases}\phi_0(x,y)+R\phi_0(-x,y), & x\to-\infty,\\ T\phi_0(x,y), & x\to+\infty.\end{cases} \tag{7.54}$$

In view of the boundary condition (7.53) and the fact that a wave train propagating in uniform finite depth experiences no reflection, we can express ϕ, R and T in terms of the perturbation parameter ε as given by equation (7.6).

Applying the expansions (7.6) in (7.51)-(7.54), we find, after equating the coefficients of ε^0 and ε from both sides, that the functions $\phi_0(x,y)$ and $\phi_1(x,y)$, respectively, satisfy the following BVPs:

BVP-III $\phi_0(x,y)$ satisfies

$$(\nabla^2-\nu^2)\phi_0=0 \quad \text{in} \quad 0\le y\le h, \tag{7.55}$$
$$\frac{\partial\phi_0}{\partial y}+K\phi_0=0 \quad \text{on} \quad y=0, \tag{7.56}$$
$$\frac{\partial\phi_0}{\partial y}-G'\ \phi_0=0 \quad \text{on} \quad y=h, \tag{7.57}$$

and

$$R_0=0, \quad T_0=1. \tag{7.58}$$

BVP-IV $\phi_1(x,y)$ satisfies

$$(\nabla^2-\nu^2)\phi_1=0 \quad \text{in} \quad 0\le y\le h, \tag{7.59}$$
$$\frac{\partial\phi_1}{\partial y}+K\phi_1=0 \quad \text{on}\ y=0, \tag{7.60}$$
$$\frac{\partial\phi_1}{\partial y}-G'\phi_1=\frac{\partial}{\partial x}\left\{c(x)\frac{\partial\phi_0}{\partial x}\right\}-\nu^2c(x)\phi_0+G'c(x)\frac{\partial\phi_0}{\partial y}\equiv V(x) \quad \text{on}\ y=h, \tag{7.61}$$

and

$$\phi_1(x,y)\sim\begin{cases}R_1\phi_0(-x,y), & \text{as} \quad x\to-\infty,\\ T_1\phi_0(x,y), & \text{as} \quad x\to+\infty.\end{cases} \tag{7.62}$$

It may be noted that the BVP-III corresponds to the surface wave propagation in water of uniform finite depth h. The solution of BVP-III can be written as

$$\phi_0(x,y) = \frac{K\sinh k_0 y - k_0\cosh k_0 y}{K\sinh k_0 h - k_0\cosh k_0 h} e^{i\mu x}, \tag{7.63}$$

where k_0, the wave number of the incident wave, is the unique positive root of the equation (7.16) and

$$\mu = k_0\cos\theta, \quad \nu = k_0\sin\theta \quad (0 \le \theta < \pi/2), \tag{7.64}$$

where θ is the angle of incidence of the wave train ($\theta = 0$ corresponds to normal incidence), μ and ν are, respectively, the x and z components of k_0.
Hence, $V(x)$ can be written as

$$V(x) = i\mu\frac{d}{dx}\left\{c(x)e^{i\mu x}\right\} - \nu^2 c(x)e^{i\mu x} + G'k_0 A\, c(x)e^{i\mu x} \quad \text{on} \quad y = h, \tag{7.65}$$

where A is given by equation (7.18).

To solve the BVP-IV we employ Fourier transform of ϕ_1 with respect to x, denoted by $\overline{\phi}_1$ and given by equation (7.19) with the inverse given by equation (7.20).

Now, taking Fourier transform of the governing equation (7.59) and the boundary conditions (7.60) and (7.61) with respect to the horizontal space variable x, we obtain

$$\frac{\partial^2\overline{\phi}_1}{\partial y^2} - \widehat{\xi}^2\overline{\phi}_1 = 0 \quad \text{in} \ -\infty < \xi < \infty,\ 0 \le y \le h, \tag{7.66}$$

$$\frac{\partial\overline{\phi}_1}{\partial y} + K\overline{\phi}_1 = 0 \quad \text{on} \quad y = 0, \tag{7.67}$$

$$\frac{\partial\overline{\phi}_1}{\partial y} - G'\overline{\phi}_1 = \overline{V}(\xi) \quad \text{on} \quad y = h, \tag{7.68}$$

where $\widehat{\xi}^2 = \xi^2 + \nu^2$ and

$$\overline{V}(\xi) = \int_{-\infty}^{\infty} V(x)e^{i\xi x}dx. \tag{7.69}$$

The solution of (7.66) subject to the boundary conditions (7.67) and (7.68) can be written as

$$\overline{\phi}_1(\widehat{\xi},y) = \frac{\widehat{\xi}\cosh\widehat{\xi}y - K\sinh\widehat{\xi}y}{\widehat{\xi}[(\widehat{\xi} + KG'/\widehat{\xi})\sinh\widehat{\xi}h - (K+G')\cosh\widehat{\xi}h]}\overline{V}(\xi). \tag{7.70}$$

Taking inverse Fourier transform, the solution for the velocity potential can be written in the form

$$\phi_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\widehat{\xi}\cosh\widehat{\xi}y - K\sinh\widehat{\xi}y}{\widehat{\xi}[(\widehat{\xi} + KG'/\widehat{\xi})\sinh\widehat{\xi}h - (K+G')\cosh\widehat{\xi}h]}\overline{V}(\xi)e^{-i\xi x}d\xi. \tag{7.71}$$

We now obtain the final result from (7.71) by contour integration using the residue theorem.

Here also we observe that the equation (7.71) has certain singularities (lying on the ξ-axis) other than $\widehat{\xi} = 0$. Replacing K by $\widehat{K} = (\sigma^2 + i\mu'\sigma)/g$ in the equation (7.71), the singularities of (7.71) are displaced off the ξ-axis to the upper and the lower half planes. Hence, we write

$$\phi_1(x,y) = \lim_{\mu'\to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(\widehat{\xi},y)}{G_{\mu'}(\widehat{\xi},h)}\, e^{-i\xi x} d\xi, \tag{7.72}$$

where

$$F(\widehat{\xi},y) = [\widehat{\xi}\cosh\widehat{\xi}y - \widehat{K}\sinh\widehat{\xi}y]\overline{V}(\xi), \tag{7.73}$$
$$G_{\mu'}(\widehat{\xi},h) = \widehat{\xi}[(\widehat{\xi} + \widehat{K}G'/\widehat{\xi})\sinh\widehat{\xi}h - (\widehat{K} + G')\cosh\widehat{\xi}h]. \tag{7.74}$$

If $\widehat{K} = \widehat{K}_1 + i\widehat{K}_2$, then $\widehat{K}_2 = \mu' K/\sigma$ which is very small and if $\hat{\zeta} = \alpha + i\beta$ is a zero of the expression (7.74), then $\hat{\zeta}$ (up to first order of ε) can be determined as

$$\hat{\zeta} = \pm\alpha_n \pm i\beta_n, \quad \text{and} \quad \hat{\zeta} = \pm(k_0 + \gamma) \pm i\beta'_n, \tag{7.75}$$

where $\alpha_n, \beta_n, \gamma, \beta'_n$ are given by (7.31).

Substituting $\hat{\zeta} = \pm\sqrt{\zeta^2 + \nu^2}$, then the roots $\hat{\zeta} = \pm\alpha_n \pm i\beta_n$ give

$$\zeta \approx \pm\sqrt{-(\beta_n^2 + \nu^2) \pm i\widehat{K}_2(2\alpha_n'^{(1)}\beta_n)} \quad \text{up to the first order of } \widehat{K}_2$$
$$\Rightarrow \quad \zeta \approx \pm i s_n \text{ as } \mu' \to 0, \text{ where } s_n = \sqrt{\beta_n^2 + \nu^2}, \tag{7.76}$$

and the roots $\hat{\zeta} = \pm(k_0 + \gamma) \pm i\beta'_n$ give

$$\zeta \approx \pm k_0 \cos\theta = \pm\mu \quad \text{up to the first order of } \widehat{K}_2 \text{ and when } \mu' \to 0. \tag{7.77}$$

Here the contour consists of the portion $-R$ to R on the real ξ-axis and a semicircle centered at the origin and having a large radius R. The semicircle must be taken in the upper half ζ-plane ($\zeta = \xi + i\eta$) in anticlockwise direction or in the lower half plane in clockwise direction, according as $x < 0$ or $x > 0$. In the limit as $R \to \infty$, the required range of integration is recovered, since the integration along the semicircle makes a zero contribution. Hence, by using the residue theorem,

$$\phi_1(x,y) = \lim_{\mu'\to 0}(-i)\left[\sum_{n=1}^{\infty} \text{Res}\left\{\frac{F(\hat{\zeta},y)\, e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta = is_n} + \text{Res}\left\{\frac{F(\hat{\zeta},y)\, e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=\mu}\right] \quad \text{for } x < 0, \tag{7.78}$$

and

$$\phi_1(x,y) = \lim_{\mu'\to 0} i\left[\sum_{n=1}^{\infty} \text{Res}\left\{\frac{F(\hat{\zeta},y)\ e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=-is_n} + \text{Res}\left\{\frac{F(\hat{\zeta},y)\ e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=-\mu}\right] \quad \text{for } x>0, \tag{7.79}$$

which imply that

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0}(-i)\sum_{n=1}^{\infty} \text{Res}\left\{\frac{F(\hat{\zeta},y)\ e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=is_n} \\ &+ \frac{-2iB\sec\theta(k_0^2\cos 2\theta + G'k_0A)\left\{\int_{-\infty}^{\infty} c(x)e^{2i\mu x}dx\right\}}{k_0[2(k_0+G'K/k_0)h + (1-G'K/k_0^2)\sinh 2k_0h]} \\ &\times\ (k_0\cosh k_0(h-y) - G'\sinh k_0(h-y))\, e^{-i\mu x} \quad \text{for } x<0,\end{aligned} \tag{7.80}$$

and

$$\begin{aligned}\phi_1(x,y) &= \lim_{\mu'\to 0} i\left[\sum_{n=1}^{\infty} \text{Res}\left\{\frac{F(\hat{\zeta},y)\ e^{-i\zeta x}}{G_{\mu'}(\hat{\zeta},h)}\right\}\Bigg|_{\zeta=-is_n}\right. \\ &+ \frac{-2iB\sec\theta(-k_0^2 + G'k_0A)\left\{\int_{-\infty}^{\infty} c(x)dx\right\}}{k_0[2(k_0+G'K/k_0)h + (1-G'K/k_0^2)\sinh 2k_0h]} \\ &\times\ (k_0\cosh k_0(h-y) - G'\sinh k_0(h-y))\, e^{i\mu x} \quad \text{for } x>0,\end{aligned} \tag{7.81}$$

where B is given by (7.36).

Comparing equations (7.80) and (7.81) with equation (7.62), the reflection and transmission coefficients can, respectively, be written as

$$R_1 = \frac{-2iB\sec\theta(k_0^2\cos 2\theta + G'k_0A)}{2(k_0+G'K/k_0)h + (1-G'K/k_0^2)\sinh 2k_0h}\int_{-\infty}^{\infty} c(x)e^{2i\mu x}dx, \tag{7.82}$$

and

$$T_1 = \frac{-2iB\sec\theta(-k_0^2 + G'k_0A)}{2(k_0+G'K/k_0)h + (1-G'K/k_0^2)\sinh 2k_0h}\int_{-\infty}^{\infty} c(x)dx. \tag{7.83}$$

When the bed has zero porosity, the results (7.82) and (7.83) may be observed as the results (4.12) and (4.15), respectively. For this case, as the expression of R_1 contains a $\cos 2\theta$ term, then for oblique incidence at $\theta = \frac{\pi}{4}$ of a wave train, the reflected wave-field vanishes independently of the shape of bottom undulation. Also the results for normal incidence can be obtained by putting $\theta = 0$.

The reflection and transmission coefficients can be evaluated from equations (7.82) and (7.83) once the shape function $c(x)$ is known. In the next section we consider a special form for the function $c(x)$.

7.3.3 A special bed surface

Consider the interaction of progressive surface waves with a patch of sinusoidal ripples on the porous sea-bed given by equation (7.39).

For this case we obtain the reflection and transmission coefficients, respectively, as

$$\begin{aligned} R_1 &= \frac{-2iB\sec\theta(k_0^2\cos 2\theta + G'k_0A)}{2(k_0+G'K/k_0)h + (1-G'K/k_0^2)\sinh 2k_0h}\frac{al}{l^2-(2\mu)^2} \\ &\quad\times\Big[(-1)^n e^{2i\mu L_1} - (-1)^m e^{2i\mu L_2}\Big], \end{aligned} \tag{7.84}$$

and

$$T_1 = \frac{-2iB\sec\theta(-k_0^2 + G'k_0A)}{2(k_0+G'K/k_0)h + (1-G'K/k_0^2)\sinh 2k_0h}\frac{(-a)}{l}\Big[(-1)^m - (-1)^n\Big]. \tag{7.85}$$

In the situation in which there is an integer number of ripple wave lengths in the patch $L_1 \le x \le L_2$ such that $m = n$ and $\delta' = 0$, we find T_1 and R_1 respectively as

$$R_1 = \frac{4B\sec\theta(k_0^2\cos 2\theta + G'k_0A)}{2(k_0+G'K/k_0)h + (1-G'K/k_0^2)\sinh 2k_0h}\frac{(-1)^{m+1}al}{l^2-(2\mu)^2}\sin\left(\frac{2\mu m\pi}{l}\right), \tag{7.86}$$

and

$$T_1 = 0. \tag{7.87}$$

These results match exactly with those obtained by Davies and Heathershaw [19] for zero porosity and $\theta = 0$. Equation (7.86) illustrates that for a given number of m ripples, the first order wave reflection coefficient is an oscillatory function in the quotient of twice the component of the wave number along x-axis and the ripple wave number. Furthermore, if the bed wave number is twice the component of the wave number along x-axis ($l = 2\mu$), then equation (7.86) reveals that there is a resonant Bragg-type interaction between the surface waves and bed forms. Hence, at resonance we find from equation (7.86) that

$$R_1 = \frac{2B\sec\theta(k_0^2\cos 2\theta + G'k_0A)}{2(k_0+G'K/k_0)h + (1-G'K/k_0^2)\sinh 2k_0h}\frac{am\pi}{l}, \tag{7.88}$$

from which we find that R_1 becomes a constant multiple of m, the number of ripples in the patch. It indicates that relatively few bottom undulations with its wavenumber equal to approximately twice the $x-$component of the surface wavenumber, may give rise to a very substantial reflected wave. A possible consequence of this is a coupling between ripple growth and wave reflection, which may be important in the problems of coastal protection.

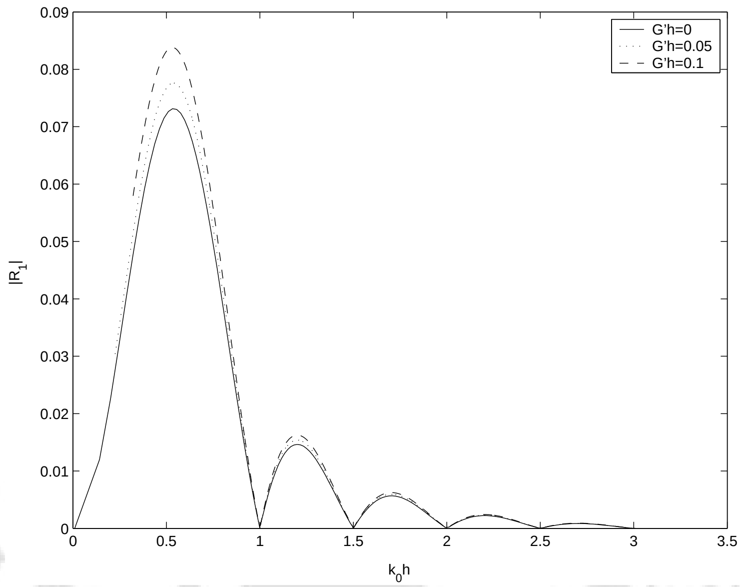

Figure 7.2: *Reflection coefficient against the wave number* k_0h *for* $\theta = 0; a/h = 0.1; lh = 1; m = 1$.

7.4 Numerical results

The numerical computation is shown here for the first order reflection coefficient given by equation (7.86). In Figure 7.2, $|R_1|$ is plotted against the wave number k_0h for a single ripple with $a/h = 0.1, lh = 1, \theta = 0$ and for different porous-effect parameters $G'h = 0, 0.05$ and 0.1. From the graph it is clear that its peak value is attained when the wave number of the bottom undulations (lh) becomes approximately twice the surface wave number (k_0h), and $|R_1|$ has an oscillating nature. Also, at Bragg resonance the value of the reflection coefficient increases as the porous effect parameter increases. In Figure 7.3, $|R_1|$ is again depicted against k_0h with $a/h = 0.1, m = 2, lh = 1, \theta = 0$ and the same set values of $G'h$. The same general feature of $|R_1|$, as in Figure 7.2, is now observed with the modification that the overall values of R_1 is now increased in comparison to the case of single ripple; the oscillating nature of $|R_1|$ against k_0h is more pronounced and the number of zeros of $|R_1|$ also increases. Like in the single ripple case, at Bragg resonance the value of the reflection coefficient also increases as the porous effect parameter increases.

In Figures 7.4 and 7.5, the reflection coefficient $|R_1|$ is plotted against the angle of incidence θ for $Kh = 0.1, a/h = 0.1, lh = 1$ and for ripples $m = 1$, and in Figure 7.6 $|R_1|$ is plotted against the angle of incidence θ for $Kh = 0.1, a/h = 0.1, lh = 1$ and $m = 2$. From these three graphs it is clear that for $G'h = 0$, $|R_1|$ vanishes independently of the shape of the function at $\theta = \frac{\pi}{4}$ which validates the equation (7.82). From the Figure 7.6 it is found

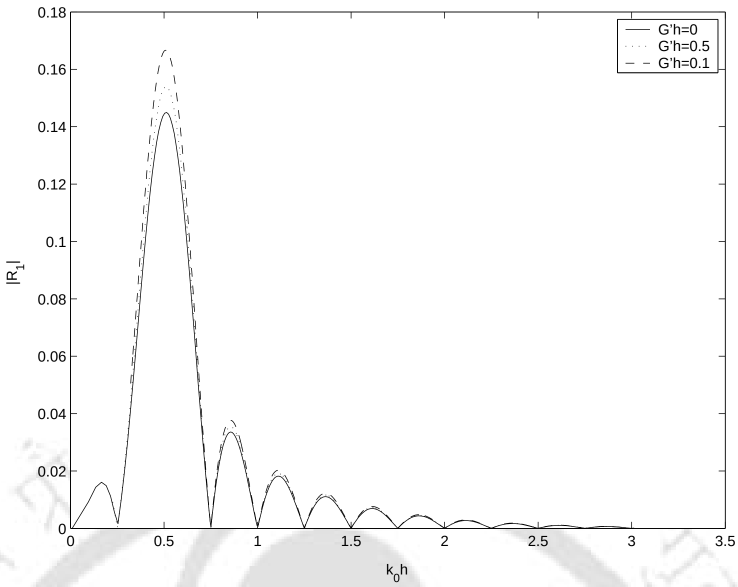

Figure 7.3: *Reflection coefficient against the wave number* k_0h *for* $\theta = 0; a/h = 0.1; lh = 1; m = 2.$

that even for very small values of $G'h$, $|R_1|$ does not vanish independently of the shape of the function at $\theta = \frac{\pi}{4}$.

7.5 Conclusion

Fourier transform method is used to solve the problem of water wave scattering by a small undulation on an otherwise flat bottom of a porous sea-bed for normal and oblique incidence. The presence of the singularities lead us to use the residue theorem to evaluate the integral appearing in the first-order correction of the potential. After deriving the velocity potential, the reflection and transmission coefficients up to the first order are obtained. Application of these results for a sinusoidal bottom undulations yields results which coincide exactly with the results for the same obtained earlier in this thesis when the bed has no porous effect and for normal incidence. From the computational results it is observed that the reflection coefficient increases with increasing porous effect.

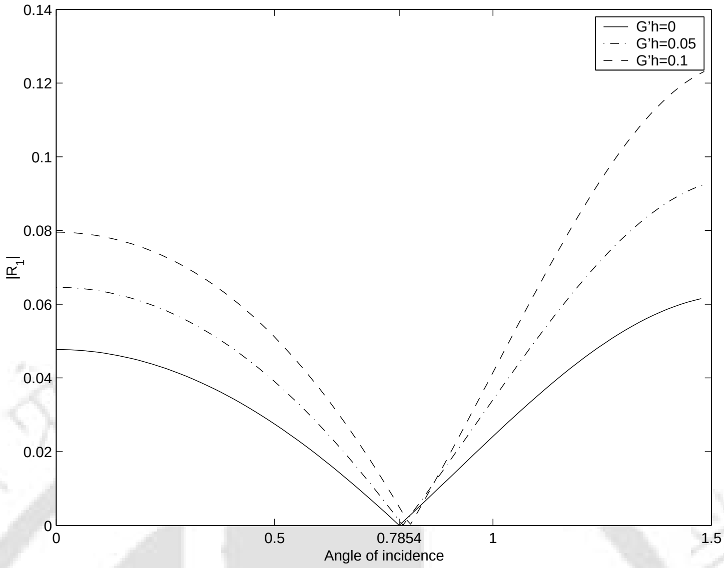

Figure 7.4: *Reflection coefficient against the angle of incidence* θ *for* $Kh = 0.1; a/h = 0.1; lh = 1; m = 1.$

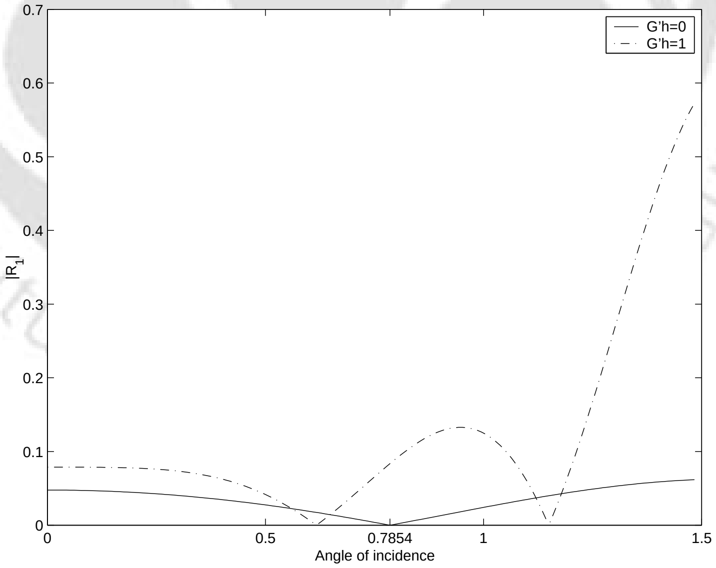

Figure 7.5: *Reflection coefficient against the angle of incidence* θ *for* $Kh = 0.1; a/h = 0.1; lh = 1; m = 1.$

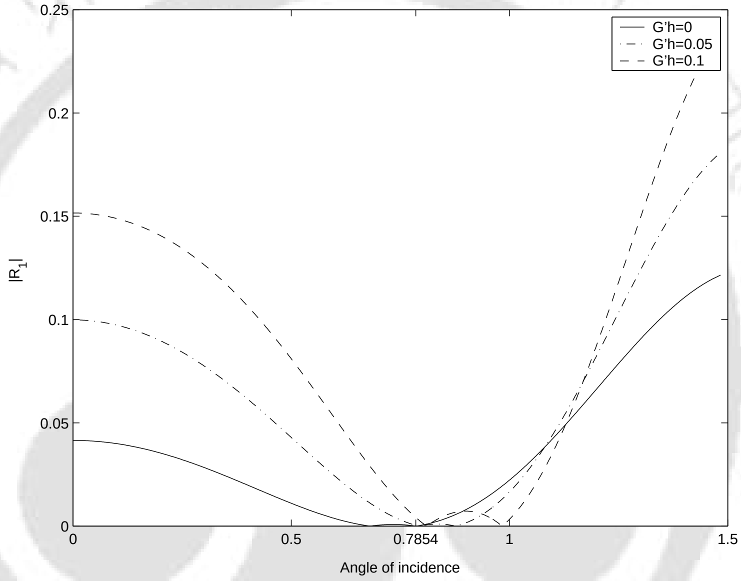

Figure 7.6: *Reflection coefficient against the angle of incidence* θ *for* $Kh = 0.1; a/h = 0.1; lh = 1; m = 2$.

Chapter 8

Summary and Further Work

This chapter is devoted to a brief summary of the results highlighting the contributions made by this thesis and techniques used in deriving these. It also provides information for the scope of possible extensions and future investigations.

8.1 Summary of results

In this thesis the scattering of a train of small amplitude harmonic surface water waves by small undulation using linear water wave theory has been investigated.

In Chapter 2, applying perturbation analysis, which involves a small parameter ε present in the representation of the small undulation of the sea-bed, we set up the boundary value problems to be satisfied by the velocity potential for the scattering of waves by small undulating topography for normal and oblique incidences.

Chapters 3 is concerned with the solution of the velocity potential for the boundary value problem established in Chapter 2 for normal incidence. The solution is obtained by three different techniques, namely, the Green's integral theorem, Fourier transform technique and finite cosine transform. Using this solution the reflection and transmission coefficients are found which involve the shape function $c(x)$ and the results may be interpreted as the results obtained by Miles [68] for normal incidence and Mandal and Basu [53] for normal incidence in absence of surface tension. The Fourier transform technique employed to solve the problem has a more general approach than that employed by Davies and Heathershaw [19] to the problem of scattering of water waves by sinusoidal undulations on an otherwise flat bed.

The boundary value problem for oblique incidence in Chapter 2 is also solved by the above three techniques in Chapter 4. The solution for the velocity potential is obtained from which the reflection and transmission coefficients are evaluated which involve the shape function $c(x)$. These results may be interpreted as the results obtained by Miles [68] and Mandal and Basu [53] in absence of surface tension. Here by putting $\theta = 0$ the results for normal incidence in Chapter 3 can be obtained.

To evaluate these coefficients numerically, different shape functions are considered in Chapter 5 and the results for both normal and oblique incidence are presented graphically. Among those cases the particular case of sinusoidal ripples on the sea-bed is of considerable significance due to the ability of an undulating bed to reflect incident wave energy which is important in respect of both coastal protection, and of possible ripple growth if the bed is erodable. For this particular case we observe that a large amount of reflection of the incident wave energy is produced for Bragg resonance. This result may be useful in the construction of an effective reflector of the incident wave energy for protecting coastal areas from the rough sea in the arctic regions. The same conclusion can be observed even if all the ripples in the patch do not have the same wave number.

In Chapter 6, the same physical problem is solved for both normal and oblique incidence by a direct method, quite different from the other three methods described in Chapters 3 and 4. This method is based on an eigenfunction expansion that includes both decaying and progressive wave mode terms. The analytical results for the reflection and transmission coefficients obtained by this method are different from the results obtained in Chapters 3 and 4. This is due to the solution approach of this direct method containing an appropriate set of orthogonal eigenfunctions which depends upon a single parameter. However, for a patch of sinusoidal undulations on the bottom these results are computed and compared with the results obtained by other methods. Excellent agreement is observed between the numerical estimates obtained by the present method and those by the known method.

Chapter 7 is concerned with the investigation of the problem of scattering of surface water waves by small undulation on a sea-bed of finite depth by assuming the sea-bed to be composed of porous material of specific type. The boundary condition on the porous sea-bed is derived by taking the porosity effect into the account. Fourier transform technique is employed to obtain the complete solution of the mixed boundary value problem from which the reflection and transmission coefficients are determined which involve the shape function $c(x)$. It is observed that with zero porosity, the results for these coefficients might be interpreted as the results obtained in Chapters 3 and 4 for normal and oblique incidence respectively. These results are applied to the case of a patch of sinusoidal undulations on the bed to evaluate the corresponding coefficients numerically and then the results are presented graphically.

It is observed that the methods presented in the thesis in obtaining the first order potential, and hence the reflection and transmission coefficients, reduce the workload to a large extent. These methods lead to a computationally more tractable form of the solution for the scattered field.

8.2 Scope for future work

We make some informal observation pertaining to the possible extensions of our results to different problems. Now we briefly outline some interesting problems which can be taken up in future.

1. The problem of scattering of water waves in a two-layer fluid with undulating topography can be considered. If satisfactory results are obtained, this can be extended for more layers.

2. Consideration of propagation of surface gravity waves over small undulation of a permeable sea-bed with a current will be a step forward.

3. Diffraction of surface waves by a two or three dimensional structure on an undulating bed will be an interesting addition.

4. Based on the results obtained for the examples in the thesis, some more physically occurring obstacles at the sea-bed can be considered.

5. The same scattering problem can be extended to include the second-order correction for the reflection and transmission coefficients, i.e., R_2 and T_2.

6. The analysis of the effect of the perturbation parameter ε on reflection and transmission coefficients can be taken up, which will throw more light on the application side.

Bibliography

[1] G. A. Athanassoulis and K. A. Belibassakis, *A consistent couple-mode theory for the propagation of small amplitude water waves over variable bathymetry regions*, J. Fluid Mech., 389 (1999), 275-301.

[2] J. A. Bailard, J. W. DeVries and J. T. Kirby, *Consideration in using Bragg reflection for storm erosion protection*, J. Waterway, Port, Coastal, and Ocean Eng., 118(1) (1992), 62-74.

[3] U. Basu and B. N. Mandal, *Diffraction of water waves by a deformation of the bottom*, Indian J. Pure Appl. Math., 22 (1991), no. 9, 781-786.

[4] M. Belzons, E. Guazzelli and O. Parodi, *Gravity waves on a rough bottom: experimental evidence of one-dimensional localization*, J. Fluid Mech., 186 (1988), 539-558.

[5] J. C. W. Berkhoff, *Computation of combined refraction-diffraction*, Proc. 13th Conf. on Coastal Engng., July 1972, Vancouver, Canada, ASCE., 2, (1973) 471-490.

[6] J. C. W. Berkhoff, *Mathematical model for simple harmonic linear waves wave diffraction and refraction*, Delft Hydr. Rep., W (1976), 154-IV.

[7] N. Booij, *A note on the accuracy of the mild-slope equation*, Coastal Engng., 7 (1983), 191-203.

[8] G. D. Crapper, *Introduction to Water Waves*, Ellis Horwood, 1984.

[9] G. F. Carrier, *Gravity waves on water of variable depth*, J. Fluid Mech., 24(4) (1966), 641-659.

[10] P. G. Chamberlain, *Wave scattering over uneven depth using the mild-slope equation*, Wave Motion, 17 (1993), 267-285.

[11] P. G. Chamberlain, *Symmetry relations and decomposition for the mild slope equation*, J. Engng. Math., 29 (1995), 121-140.

[12] P. G. Chamberlain, and D. Porter, *The modified mild-slope equations*, J. Fluid Mech., 291 (1995), 393-407.

[13] P. G. Chamberlain and D. Porter, *Decomposition methods for wave scattering by topography with application to ripple beds*, Wave Motion, 22 (1995), 201-214.

[14] P. G. Chamberlain and D. Porter, *Scattering and near-trapping of water waves by axi-symmetric topography*, J. Fluid Mech., 388 (1999), 335-354.

[15] G. J. M. Copeland, *A practical alternative to the "mild-slope" wave equation*, Coastal Engng., 9 (1985), 125-149.

[16] R. A. Dalrymple, M. A. Losada, and P. A. Martin, *Reflection and transmission from porous structures under oblique wave attack*, J. Fluid Mech., 224 (1991), 625-644.

[17] A. G. Davies, *On the interaction between surface waves and undulations on the sea bed*, J. Marine Res., 40 (1982), 331-368.

[18] A. G. Davies, *The reflection of wave energy by undulations of the sea bed*, Dyn. Atms. and Oceans, 6 (1982), 207-232.

[19] A. G. Davies and A. D. Heathershaw, *Surface-wave propagation over sinusoidallly varying topography*, J. Fluid Mech., 144 (1984) 419-443.

[20] A. G. Davies, E. Guazzelli and M. Belzons, *The propagation of long waves over an undulating bed*, Physics of Fluids A, 1(8) (1989), 1331-1340.

[21] R. G. Dean and R. A. Dalrymple, *Water Wave Mechanics For Engineering and Scientists*, World Scientific Publishing Co. Pvt. Ltd., 2000.

[22] P. Devillard, F. Dun-lop and B. Sou, *Localization of Gravity waves on a channel with a random bottom*, J. Fluid Mech., 186 (1988), 521-538.

[23] C. Eckart, *The propagation of gravity waves from deep to shallow water*, In Gravity Waves: (Proc. NBS Semicentennial Symp. on Gravity Waves, NBS, June 15-18, 1951), Washington: National Bureau of Standards, 165-175.

[24] D. V. Evans, *The application of a new source potential to the problem of transmission of water waves over a shelf of arbitrary profile*, Proc. Camb. Phil. Soc., 71 (1972), 391-410.

[25] D. V. Evans, *Mechanisms of the generation of edge waves over a sloping beach*, J. Fluid Mech., 186 (1988), 379-391.

[26] D. V. Evans and C. M. Linton, *On step approximations for water-wave problems*, J. of Fluid Mech., 278 (1994), 229-249.

[27] G. F. Fitz-Gerlad, *The reflexion of the plane gravity waves travelling in water of varibale depth*, Phil. Trans. R. Soc. London A, 284 (1976), no. A, 49-89.

[28] Z. Gu and H. Wang, *Gravity waves over porous bottoms*, Coastl Engng., 15 (1991), 497-524.

[29] J. Hamilton, *Differential equations for long-period gravity waves on fluid of rapidly varying depth*, J. Fluid Mech., 83 (1977), no. 2, 289-310.

[30] E. Guazzelli, V. Rey, and M. Belzons, *Higher-order bragg reflection of gravity surface waves by periodic beds*, J. Fluid Mech., 245 (1992), 301-317.

[31] T. Hara and C. C. Mei, *Bragg scattering of surface waves by periodic bars: theory and experiment*, J. Fluid Mech., 178 (1987), 221-241.

[32] A. D. Heathershaw, *Seabed-wave resonance and sand bar growth*, Nature, 296 (1982), 343-345.

[33] A. D. Heathershaw and A. G. Davies, *Resonant wave reflection by transverse bedforms and its reflection to beachs and offshore bars*, Marine Geology, 62 (1985), 321-338.

[34] A. E. Heins, *Water waves over a channel of finite depth with a dock*, American J. Mathematics, 70 (1948) 730-748 .

[35] E. R. Johnson, *The low-frequency scattering of Kelvin waves by stepped topography*, J. Fluid Mech., 215 (1990), 23-44.

[36] M. Kanoria, D. P. Dolai and B. N Mandal, *Water wave scattering by thick barriers*, J. Engng. Math., 35 (1999), 361-384.

[37] J. B. Keller, *Shallow water theory for arbitrary slopes of the bottom*, J. Fluid Mech., 489 (2003), 345-348.

[38] S. M. Killen and R. S. Johnson, *Propagation of axi-symmetric nonlinear shallow water waves over slowly varying depth*, Mathematics and Computers in Simulation, 55 (2001), no. 4-6, 463-472.

[39] J. T. Kirby, *A note on linear surface wave-current interaction over slowly varying topography*, J. Geophysical Res., 89 (1984), no. C1, 745-747.

[40] J. T. Kirby, *A general wave equation for waves over rippled beds*, J. Fluid Mech., 162 (1986), 171-186.

[41] J. T. Kirby, *On the gradual reflection of weakly nonlinear Stokes waves in regions with varying topography*, J. Fluid Mech., 162 (1986), 187-209.

[42] J. T. Kirby, *A note on Bragg scattering of surface waves by sinusoidal bars*, Physics of Fluids, 5 (1993), no.2, 380-386.

[43] G. Kreisel, *Surface waves*, Q. J. Appl. Math., 7 (1949), 21-44.

[44] H. Lamb, *Hydrodynamics*, Camb. Univ. Press, 1932.

[45] M. J. Lighthill, *Waves in Fluids*, Cambridge University Press, 1978.

[46] R. B. Long, *Scattering of surface waves by an irregular bottom*, J. Geophysics Res., 78 (1973), no.33, 7861-7870.

[47] Y. Liu, and D. K. P. Yue, *On generalized Bragg scattering of surface waves by bottom ripples*, J Fluid Mech., 356 (1998), 297-326.

[48] O. S. Madsen, *Wave transmission through porous structures*, J. Waterway, Port, Coastal, and Ocean Engng, Am. Soc. Civ. Engng, 100 (1974), 169-188.

[49] P. A. Madsen, *Wave reflection from a vertical permeable wave absorber*, Coastal Engng., 7 (1983), 381-396.

[50] P. A. Madsen and J. Larsen, *An efficient finite-difference approach to the mild-slope equation*, Coastal Engng., 11 (1987), 329 - 351.

[51] V. Mallayachari and V. Sundar, *Reflection characterstics of permeable seawalls*, Coastl Engng., 23 (1994), 135-150.

[52] B. N. Mandal and A. Chakrabarti, *A note on diffraction of water waves by a nearly vertical barrier*, IMA J. Appl. Math., 43 (1989), 157-165.

[53] B. N. Mandal and U. Basu, *A note on oblique water-wave diffraction by a cylindrical deformation of the bottom in the presence of surface tension*, Arch. Mech., 42 (6) (1990), 723-727.

[54] B. N. Mandal and U. Basu, *Free Surface Flow over a small geometrical disturbance at the bottom*, Bull. Cal. Math. Soc., 88 (1996), 355-362.

[55] B. N. Mandal and A. Chakrabarti, *Water Wave Scattering by Barriers*, WIT Press, 2000.

[56] B. N. Mandal and U. Basu, Wave diffraction by a small elevation of the bottom on an ocean with an ice cover, Arch. Appl. Mech., 73 (2004), 812-822.

[57] S. C. Martha and S. N. Bora, *Water wave diffraction by a small deformation of the ocean bottom for oblique incidence*, Acta Mechanica, 185 (3-4) (2006), 165-177.

[58] S. C. Martha and S. N. Bora, *Reflection and transmission coefficients for water wave scattering by a sea-bed with small undulation*, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), 87(4) (2007), 314-321.

[59] H. Mase and K. Takeba, *Bragg scattering of gravity waves over a porous ripple bed*, In: Proc. 24th ICCE. ASCE, kobe, Japan, (1994), 635-649.

[60] S. R. Massel, *Extended refraction-diffraction equation for surface waves*, Coastal Engng., 19 (1993), 97-126.

[61] C. C. Mei, *The Applied Dynamics Ocean Surface Waves*, John Wiley, 1983.

[62] C. C. Mei, *Resonant reflection of surface water waves by periodic sand-bars*, J. Fluid Mech., 152 (1985), 315-335.

[63] C. C. Mei and J. L. Black, *Scattering of surface waves by rectangular obstacles in waters of finite depth*, J. Fluid Mech., 38 (3) (1969), 499-511.

[64] C. C. Mei, T. Hara and M. Nachari, *Note on Bragg scattering of water waves by parallel bars in the sea bed*, J. Fluid Mech., 186 (1988), 147-162.

[65] C. C. Mei and B. Le Méhauté, *Note on the equations of long waves over an uneven bottom*, J. Geophysical Res., 71 (1966), no. 2, 393-400.

[66] J. W. Miles, *Surface-wave scattering matrix for a shelf*, J. Fluid Mech., 28 (1967) 755-767.

[67] J. W. Miles, *Integral Transforms in Applied Mathematics*, Cambridge University Press, Cambridge, 1971.

[68] J. W. Miles, *Oblique surface wave diffraction by a cylindrical obstacle*, Dyn. Atms. and Oceans, 6 (1981), 121-123.

[69] J. W. Miles, *Variational approximations for gravity waves in water of variable depth*, J. Fluid Mech., 232 (1991), 681-688.

[70] J. W. Miles, *On gravity-wave scattering by non-secular changes in depth*, J. Fluid Mech., 376 (1998), 53-60.

[71] J. W. Miles and P. G. Chamberlain, *Topographical scattering of gravity waves*, J. Fluid Mech., 361 (1998), 175-188.

[72] M. Nachari and C. C. Mei, *Bragg scattering of water waves by doubly periodic sea bed*, J. Fluid Mech., 192 (1988), 51-74.

[73] A. Nachbin and G. C. Papanicolaou, *Water waves in shallow channels of rapidly varying depth*, J. Fluid Mech., 241 (1992), 311-332.

[74] T. J. O'Hare and A. G. Davies, *A new model for surface wave propagation over undulating topography*, Coastal Engng., 18 (1992), 251-266.

[75] T. J. O'Hare and A. G. Davies, *A comparison of two models for surface-wave propagation over rapidly varying topography*, Appl. Ocean Res., 15 (1993), 1-11.

[76] D. Porter, *The Mild-Slope Equations*, J. Fluid Mech., 494 (2003), 51-63.

[77] D. Porter and P. G. Chamberlain, *Linear wave scattering by two-dimensional topography. Chapter-2 In Gravity waves in water of finite depth*, (ed. J.N.Hunt), Southampton, Computational Mechanics, 1997, 13-53.

[78] D. Porter and D. J. Staziker, *Extensions of the mild-slope equation*, J. Fluid Mech., 300 (1995), 367-382

[79] R. Porter and D. Porter, *Water wave scattering by a step of arbitrary profile*, J. Fluid Mech., 411 (2000), 131-164.

[80] R. Porter and D. Porter, *Interaction of water waves with three-dimensional periodic topography*, J. Fluid Mech., 434 (2001), 301-335.

[81] R. Porter and D. Porter, *Scattered and free waves over periodic beds*, J. Fluid Mech., 483 (2003), 129-163 .

[82] S. Rojanakamthorn, M. Isobe and A. Watanabe, *A mathematical model of wave transformation over submerged breakwaters*, Coastal Engng. Jpn., 32 (1989), no. 2, 209-234.

[83] S. Rojanakamthorn, M. Isobe and A. Watanabe, *Modelling of wave transformation on submerged breakwaters*, Proc. 22nd ICCE. ASCE, New York, (1990), 1060-1073.

[84] R. Silva, P. Salles and A. Palacio, *Linear wave propagating over a rapidly varying finite porous bed*, Coastal Engng., 44 (2002), 239-260.

[85] R. Smith and T. Sprinks, *Scattering of surface waves by a conical island*, J. Fluid Mech., 72(2) (1975), 373-384.

[86] C. K. Sollitt and R. H. Cross, *Wave transmission through permeable breakwaters*, In: Proc. 13th Coast. Engng. Conf., ASCE, (1972), 1827-1846.

[87] D. J. Staziker, D. Porter and D. S. G. Stirling, *The Scattering of surface waves by local bed elevations*, Appl. Ocean Res., 18 (1996), 283-291.

[88] J. J. Stoker, *Water Waves*, Interscience, NewYork, 1957.

[89] C. Ting, M. Lin and C. Kuo, *Bragg scattering of surface waves over permeable rippled beds with current*, Physics of Fluid, 12 (2000), no. 6, 1382-1388.

[90] T.-K. Tsay, W. Zhu and P. L.-F. Liu, *A finite element model for wave refraction, diffrcation, reflection and dissipation*, Appl. Ocean Res., 11 (1989), no. 1, 33-38.

[91] J. V. Wehausen, E. V. Laitone, *Surface waves*, Handbuch Der Physik, Springer Verlag, Berlin, 1960.

[92] X. Yu and A. T. Chwang, *Wave motion through porous structures*, J. Engng. Mech. Div., Am. Soc. Civ. Engng., 120 (1994), 989-1008.

[93] S. Zhu, *Water waves within a porous medium on an undulating bed*, Coastal Engng., 42 (2001), 87-101.

[94] S. Zhu and A. T. Chwang, *Wave reflection from a porous absorber*, Proc. 12th Engng. Mech. Conf., ASCE, (1998), 1605-1608.

[95] S. Zhu and A. T. Chwang, *Analytical study of a porous wave absorber*, J. Engng. Mech. Div., Am. Soc. Civ. Engng., 127 (2001), no.4, 326-332.

Appendix I

Derivation of Bernoulli's Equation for Potential Flow

We present a classical method in deriving the Bernoulli's equation applicable to water waves mechanics in the xyz-plane. Euler's equation of motion is given by

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\vec{q} = F - \frac{1}{\rho}\nabla p$$

$$\Rightarrow \frac{\partial}{\partial t}(u\hat{i} + v\hat{j} + w\hat{k}) + \left\{(u\hat{i} + v\hat{j} + w\hat{k}).(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k})\right\}(u\hat{i} + v\hat{j} + w\hat{k})$$

$$= (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) - \frac{1}{\rho}\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)p$$

$$\Rightarrow \left(\frac{\partial u}{\partial t}\hat{i} + \frac{\partial v}{\partial t}\hat{j} + \frac{\partial w}{\partial t}\hat{k}\right) + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u\hat{i} + v\hat{j} + w\hat{k})$$

$$= (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}) - \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right)$$

where $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is the body force per unit mass. This equation in the x direction can be explicitly written as (equating the coefficient of i):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{(Here } X = F_1\text{)}, \tag{I.1}$$

in the y direction it can be explicitly written as (equating the coefficient of j):

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{(Here } Y = F_2\text{)}, \tag{I.2}$$

and in the z direction it can be explicitly written as (equating the coefficient of k):

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \quad \text{(Here } Z = F_3\text{)}, \tag{I.3}$$

Assuming that only external force acting on the fluid particle is the gravitational body force per unit mass, then we may write

$$X = 0,\ Z = 0,\ Y = g,\ \text{where } g \text{ is the gravitational constant.}$$

Since we know that the gravitational force can be derived from a potential, we may write

$$Y = g = \frac{\partial}{\partial y}(gy)$$

Under the assumption that the motion is irrotational, we have

$$\begin{aligned} & \nabla \times \overrightarrow{q} = 0 \Leftrightarrow \overrightarrow{q} = \nabla\Phi \\ \Rightarrow\ & u = \frac{\partial \Phi}{\partial x},\ v = \frac{\partial \Phi}{\partial y},\ w = \frac{\partial \Phi}{\partial z} \\ \Rightarrow\ & \frac{\partial u}{\partial t} = \frac{\partial^2 \Phi}{\partial t \partial x};\ \frac{\partial v}{\partial t} = \frac{\partial^2 \Phi}{\partial t \partial y};\ \frac{\partial w}{\partial t} = \frac{\partial^2 \Phi}{\partial t \partial w} \end{aligned} \tag{I.4}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x};\ \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x};\ \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y} \tag{I.5}$$

Substituting (I.4) and (I.5) in equations (I.1), (I.2) and (I.3) we obtained respectively as

$$\frac{\partial^2 \Phi}{\partial t \partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{I.6}$$

$$\frac{\partial^2 \Phi}{\partial t \partial y} + u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(gy) - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{I.7}$$

$$\text{and}\quad \frac{\partial^2 \Phi}{\partial t \partial z} + u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{I.8}$$

Integrating equations (I.6), (I.7) and (I.8) w.r.t. x, y and z, respectively we obtained

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} = F_1(y, z, t), \tag{I.9}$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} - gy = F_2(x, z, t) \tag{I.10}$$

$$\text{and}\quad \frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} = F_3(x, y, t) \tag{I.11}$$

Subtracting (I.9) from (I.11) we have

$$F_1(y, z, t) = F_3(x, y, t) \tag{I.12}$$

Subtracting (I.9) from (I.10) we have

$$-gy = F_2(x,z,t) - F_1(y,z,t) \tag{I.13}$$

Since left hand side is a function of y, so right hand side should be a function of y alone for which $F_2(x,z,t)$ will be a function of z,t. Thus,

$$F_2(x,z,t) = F_2(z,t) \tag{I.14}$$

and hence from (I.13),

$$F_1(y,z,t) = F_2(z,t) + gy \tag{I.15}$$

Thus by the help of equations (I.12), (I.14) and (I.15), equations (I.9), (I.10) and (I.11) reduces to the single equation

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} - gy = F_2(z,t) \tag{I.16}$$

which is **Bernoulli's equation** in two dimensional fluid flow problem. Without loss of generality, $F_2(z,t)$ can be combined with the velocity potential, $\Phi(x,y,z,t)$ so that **Bernoulli's equation becomes**

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} - gy = 0 \tag{I.17}$$

which is nothing but equation (1.5).

Appendix J

Derivation of Bottom Boundary Condition

The bottom of a sea with small undulation is described by $y = h + \varepsilon c(x)$ where $c(x)$ is a function with compact support and describes the bottom undulation, h denotes the uniform finite depth of sea far to either side of the undulation of the bottom so that $c(x) \to 0$ as $|x| \to \infty$ and the non-dimensional number $\varepsilon(\ll 1)$ a measure of smallness of the undulation. The condition on the bottom of sea-bed (equation (2.3)) is given by

$$\frac{\partial \phi}{\partial n} = 0, \quad \text{on} \quad y = h + \varepsilon c(x)$$

where $\partial/\partial n$ denotes the normal derivative at a point (x, y) on the bottom. Now,

$$\tan\theta = \frac{dy}{dx} = \varepsilon c'(x)$$

$$\cos\theta = \frac{1}{[1+\varepsilon^2 c'^2(x)]^{1/2}}$$

$$\sin\theta = \frac{\varepsilon c'(x)}{[1+\varepsilon^2 c'^2(x)]^{1/2}}$$

$$\begin{aligned}
\frac{\partial \phi}{\partial n} &= \frac{\partial \phi}{\partial x}\cos(x,n) + \frac{\partial \phi}{\partial y}\cos(y,n) \\
&= \frac{\partial \phi}{\partial x}\cos\left(\frac{\pi}{2}+\theta\right) + \frac{\partial \phi}{\partial y}\cos\theta \\
&= \frac{\partial \phi}{\partial x}(-\sin\theta) + \frac{\partial \phi}{\partial y}\cos\theta \\
&= \frac{-\partial \phi}{\partial x}\varepsilon c'(x)\left[1 - \frac{1}{2}\varepsilon^2 c'^2(x) + \ldots\right] + \frac{\partial \phi}{\partial y}\left[1 - \frac{1}{2}\varepsilon^2 c'^2(x) + \ldots\right] \\
&= -\varepsilon c'(x)\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + O(\varepsilon^2) \quad \text{on} \quad y = h + \varepsilon c(x) \\
&= \frac{\partial \phi}{\partial y} - \varepsilon\frac{\partial}{\partial x}\left\{c(x)\frac{\partial \phi}{\partial x}\right\} + O(\varepsilon^2) = 0 \quad \text{on} \quad y = h \text{ (by the help of Taylor's Expansion)}
\end{aligned}$$

which is nothing but equation (2.7).

The following are the outcomes of the work carried out in the thesis.

List of Published/Accepted/Communicated Papers

Journals

1. **S. C. Martha** and S. N. Bora, Water wave diffraction by a small deformation of the ocean bottom for oblique incidence, *Acta Mechanica*, 185 (3-4), 165-177 (2006).
2. **S. C. Martha** and S. N. Bora, Reflection and transmission coefficients for water wave scattering by a sea-bed with small undulation, *Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM)*, 87(4) (2007), 314-321.
3. **S. C. Martha** and S. N. Bora, Oblique surface wave propagation over small undulation of the bottom of an ocean, *Geophysical and Astrophysical Fluid Dynamics, (In Press, May 2007)*.
4. **S. C. Martha**, S. N. Bora and A. Chakrabarti, Oblique water wave scattering by small undulations on a porous sea-bed *(Second revised version submitted to Applied Ocean Research)*.
5. **S. C. Martha**, S. N. Bora and A. Chakrabarti, Scattering of surface waves by small undulations on a porous sea-bed *(Under revision in Archive of Applied Mechanics)*.
6. **S. C. Martha**, S. N. Bora and A. Chakrabarti, Fourier transform method for scattering of water waves by small bottom undulation on a sea-bed *(Communicated to Mathematical Methods in the Applied Sciences)*.
7. **S. C. Martha**, S. N. Bora, S. De, A. Chakrabarti, and B.N. Mandal, Eigenfunction expansion method for the scattering of surface waves by small undulation *(under preparation)*.

Conferences Proceedings

1. **S. C. Martha** and S. N. Bora, Transformation technique for water wave scattering by a variable bottom, *Proceedings, Third International Conference on Theoretical, Applied, Computational and Experimental Mechanics (ICTACEM)*, Indian Institute of Technology Kharagpur, India, December 28-30, 2004, 162-164.
2. **S. C. Martha** and S. N. Bora, Water wave scattering by ocean bed of small deformation, *Some Aspects of Environmental Fluid Mechanics published by Allied Publishers Pvt. Ltd. India: Proc. International Conference on Environmental Fluid Mechanics (ICEFM'05)*, Indian Institute of Technology Guwahati, March 3-5, 2005, 132-137.
3. **S. C. Martha**, S. N. Bora and A. Chakrabarti, Reflection of wave energy by small undulation of the sea-bed, *Recent Advances in Computational Mechanics and Simulations published by I.K. International Publishing House Pvt. Ltd. India: Proc. 2nd International Congress on Computational Mechanics and Simulation,* Indian Institute of Technology Guwahati, December 8-10, 2006, Vol.II, 1483-1489.
4. **S. C. Martha**, S. N. Bora and A. Chakrabarti, Oblique Surface-wave Propagation over Sinusoidally Varying Topography, *Proceedings of 51st Congress of The Indian Society of Theoretical and Applied Mechanics (ISTAM)*, College of Engineering, Andhra University, Visakhapatnam, India, December 18-21, 2006, pages 23-30.