The Hilbert-Samuel polynomial and its coefficients

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Certificate

It is certified that the work contained in the thesis entitled **"The Hilbert-Samuel polynomial and its coefficients"** by Kumari Saloni, a student in the Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

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Professor Anupam Saikia
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To

My Parents

Vibha Singh and S. N. Singh

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Abstract

In this thesis, we study the relations between the Hilbert coefficients of $\mathfrak{m}$-primary ideals in a Noetherian local ring $(R, \mathfrak{m})$ and the structural properties of the ring such as Cohen-Macaulayness, Buchsbaumness and having finitely generated local cohomology modules. We consider the Hilbert coefficients $g_i^K(Q)$ of an $\mathfrak{m}$-primary ideal Q with respect to an ideal K introduced by Jayanthan and Verma. Ghezzi et al. solved the Vasconcelos' negativity conjecture and characterized the Cohen-Macaulayness of an unmixed local ring in terms of the vanishing of $e_1(Q)$ for a parameter ideal Q. We generalize their result and obtain a necessary and sufficient condition for the ring to be Cohen-Macaulay in terms of the first two Hilbert coefficients $g_0^K(Q)$ (which is the usual multiplicity) and $g_1^K(Q)$. We investigate the finiteness of various sets of Hilbert coefficients $g_i^K(Q)$ and its relations with the structure of R. A number of results relating the finiteness of the sets of $e_i(Q)$ with certain properties of R due to Ghezzi et al. and Goto and Ozeki are generalized to the case of $g_i^K(Q)$. We focus much of our attention on the first and second coefficients $g_1^K(Q)$ and $g_2^K(Q)$. We prove that $g_2^K(Q) \leq \lambda_R(R/K)$ for a parameter ideal Q provided $\operatorname{depth} R$ is at least $\dim R - 1$. We further obtain necessary and sufficient conditions for the equality. In particular, we prove that if the associated graded ring has depth at least $\dim R - 2$, then $g_2^K(Q) = \lambda_R(R/K)$ implies almost maximal depth of the corresponding fiber cone. We also examine the difference between the Hilbert polynomial and the Hilbert function with respect to K. Finally, we derive uniform lower and upper bounds for $e_i(Q)$ under certain assumptions on the depth of associated graded rings. It is proved that $e_3(Q) \leq 0$ for a parameter ideal Q provided $\operatorname{depth} R$ is at least $d-1$. We also discuss a necessary condition for the vanishing of $e_d(Q)$ under certain assumptions. Consequently, vanishing of $e_2(Q)$ is characterized in rings of depth at least $\dim R - 1$.

Contents

List of notations

$R(I)$	Rees algebra of I
$R^*(I)$	Extended Rees algebra of I
$F_K(I)$	Fiber cone of I with respect to K
$G(I)$	Associated graded ring of I
$s(I)$	Analytic spread of I
$r_J(I)$	Reduction number of a reduction J of I
$\mathrm{ht}_{\mathrm{R}}(I)$	Height of an ideal $I \subseteq R$
$\mathrm{H}^i_I(M)$	i-th local cohomology module of M with support in ideal I
$\mathrm{Ass}_R(M)$	Set of associated prime ideals of an R-module M
$\lambda_R(M)$	Length of an R-module M
$H(Q,n,M)$	The Hilbert-Samuel function $\lambda_R(M/Q^nM)$ of an R-module M
$P(Q,x,M)$	The Hilbert-Samuel polynomial corresponding to $H(Q,n,M)$
$H(Q,n)$	The Hilbert-Samuel function $\lambda_R(R/Q^n)$
$P(Q,x)$	The Hilbert-Samuel polynomial corresponding to $H(Q,n)$
$H_K(Q,n)$	The Hilbert-Samuel function $\lambda_R(R/KQ^n)$ of Q with respect to K
$P_K(Q,x)$	The Hilbert-Samuel polynomial corresponding to $H_K(Q,n)$
$H(F,n)$	Hilbert function of fiber cone
$g_i^K(Q)$	Hilbert coefficients of Q with respect to K
$\Lambda_i^K(R)$	$\{g_i^K(Q) \mid Q$ is a parameter ideal of $R\}$
$\delta_i^K(R)$	$\{g_i^K(Q) \mid Q$ is a parameter ideal of R such that $Q \subseteq K\}$
$\Delta^K(R)$	$\{g_1^K(I) \mid I$ is an $\mathfrak{m}$-primary ideal of $R\}$

1 Introduction

The primary goal of the thesis is to investigate the Hilbert coefficients and their relation to the structural properties of the ring and various blow-up algebras. Certain uniform bounds for the Hilbert coefficients are given in a number of cases. Throughout this thesis, all rings are assumed to be commutative Noetherian with identity. We assume that $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d with maximal ideal $\mathfrak{m}$ and M is a finitely generated R-module of dimension r.

The study of Hilbert functions originated in 1890 from seminal work of David Hilbert in [Hil90]. Hilbert's work was motivated by the theory of invariants which requires applications of commutative algebra. One of the major results of [Hil90] is regarding certain numerical invariants of a projective variety $X = Z(I) \subseteq P_k^t$. Given that X is an intersection of hypersurfaces, it is of interest to know how many hypersurfaces of each degree contain X, i.e., to know the dimension of the vector spaces of homogeneous polynomials of degree n vanishing on X for various n. Since the dimension of the space of all homogeneous polynomial of degree n is $\binom{n+t}{t}$, the above problem is equivalent to determining the dimension of degree n piece, say R_n, of ring $k[x_0, x_1, \ldots, x_t]/I$. Hilbert showed that the function $H(n) = \dim_k(R_n)$ is asymptotically polynomial.

Samuel [Sam51] extended Hilbert's results to the case of graded modules. Let $\lambda_R(M)$ denote the length of an R-module M. Let $S = \bigoplus_{n=0}^{\infty} S_n$ be a Noetherian graded ring such that S_0 is an Artinian local ring. Let $M = \bigoplus_{n=0}^{\infty} M_n$ be a finitely generated graded S-module. Then the numerical function $\lambda_{S_0}(M_n)$, also known as the Hilbert function of M, is asymptotically polynomial. The polynomial which coincides with the Hilbert function of M for large n is known as the Hilbert polynomial of M.

The blow-up algebras of R associated with an $\mathfrak{m}$-primary ideal I, namely the associated graded ring $G(I) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ and the fiber cone $F_{\mathfrak{m}}(I) = \bigoplus_{n=0}^{\infty} I^n/\mathfrak{m}I^n$ are graded rings. This indicates a possible extension of Hilbert function to local rings. Samuel advanced the ideas of Hilbert to the case of local rings which led to the definition of Hilbert-Samuel function. Let Q be an ideal of definition for an R-module M, i.e., $\mathfrak{m}^n M \subseteq QM$ for some $n > 0$. The numerical function $H(Q,n,M) = \lambda_R(M/Q^nM)$ is called the Hilbert-Samuel function of Q with respect to M and it coincides with a polynomial $P(Q,x,M)$ of degree r for large n. The polynomial $P(Q,x,M)$ is called the Hilbert-Samuel polynomial of Q with respect to M . We may write

$$P(Q,x,M) = e_0(Q,M)\binom{x+r-1}{r} - e_1(Q,M)\binom{x+r-2}{r-1} + \ldots + (-1)^r e_r(Q,M)$$

for unique integers $e_i(Q,M)$ (see [SH06, Lemma 11.1.1]). When $M = R$, we write $P(Q,x)$ and $e_i(Q)$ instead of $P(Q,x,R)$ and $e_i(Q,R)$ respectively for brevity. The coefficients $e_i(Q)$ are called the Hilbert coefficients of Q.

There has been a significant amount of work relating the Hilbert coefficients to the structural properties of the corresponding ideals, blow-up algebras and the ring itself. For example, Nagata [Nag62] proved that in an unmixed local ring R (i.e. $\dim \hat{R}/\mathfrak{p} = \dim \hat{R}$ for all $\mathfrak{p} \in \operatorname{Ass}(\hat{R})$ where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R), if $e_0(Q) = 1$ then $Q = \mathfrak{m}$ and R is regular. The leading coefficient $e_0(Q)$ is called the multiplicity of Q and has been studied well due to its geometric significance. The other coefficients contain important informations as well. For example, Northcott [Nor60] proved that in a Cohen-Macaulay local ring, $e_1(Q) = 0$ if and only if Q is a complete intersection. However the higher coefficients are not yet explored much except when the associated graded rings have high depth and R is Cohen-Macaulay. By depth of a standard graded ring with a unique graded maximal ideal, we mean the grade of the unique maximal ideal. Marley [Mar] showed that if depth of $G(Q)$ is at least $d-1$, then the coefficients $e_i(Q)$ are non-negative for a parameter ideal Q and for $0 \leq i \leq d$.

Sally [Sal77] observed that certain conditions on Hilbert coefficients could ensure high depth of associated graded rings. Let $\mu(I)$ denote the minimal number of generators of an ideal I. Sally proved that if R is Cohen-Macaulay with $\mu(\mathfrak{m}) = e_0(\mathfrak{m}) + d - 1$, then $G(\mathfrak{m})$ is Cohen-Macaulay. Valla [Val79] extended Sally's result to $\mathfrak{m}$-primary ideals. He showed that if Q is an $\mathfrak{m}$-primary ideal of a Cohen-Macaulay local ring R with $\lambda_R(Q/Q^2) = e_0(Q) - (d-1)\lambda_R(R/Q)$, then $G(Q)$ is Cohen-Macaulay. Sally's work in [Sal79], [Sal80a], [Sal80b], [Sal83], [Sal92] and [Sal93] has centered around the same theme. Huneke [Hun87] and Ooishi [Ooi87] independently found that the depth of associated graded ring is also related to the first Hilbert coefficient. They showed that in a Cohen-Macaulay local ring the equality $e_1(Q) = e_0(Q) - \lambda_R(R/Q)$ holds if and only if there exists $x_1, \ldots, x_d \in Q$ such that $Q^2 = (x_1, \ldots, x_d)Q$. In particular, $G(Q)$ is Cohen-Macaulay. The Huneke-Ooishi result has been generalized by several authors. The most remarkable generalization is due to Huckaba and Marley [HM97] which unified many other results in this direction. They gave sharp upper and lower bounds for $e_1(Q)$ in Cohen-Macaulay local rings and proved that lower (resp. upper) bound is attained if and only if depth of $G(Q)$ is at least $d-1$ (resp. $G(Q)$ is Cohen-Macaulay). Corso, Polini and Rossi [CPR05] established an upper bound for the second Hilbert coefficient $e_2(Q)$ which is reminiscent of the upper bound on $e_1(Q)$ due to Huckaba-Marley. Furthermore, they proved that the upper bound is achieved if and only if depth $G(Q) \geq d-1$. These results indicate that the extremal behavior of Hilbert coefficients controls the depth of the corresponding associated graded ring. The relation between the Hilbert coefficients and depth properties of associated graded rings has been examined extensively in literature, e.g., [CPP98], [Eli99], [ERV96], [EV91], [Huc96], [HM97], [Ito95], [Pol00], [Ros99], [RV96], [Vas94], [Pin97], [Wan00], [Ros00], [RV96], [Val79] and the references therein.

One is further interested in conditions which could ensure high depth on the fiber cone $F_{\mathfrak{m}}(Q)$ which plays an important role in the resolution of singularities of algebraic varieties. The Cohen-Macaulay property of $F_{\mathfrak{m}}(Q)$ was studied by Shah in [Sha91]. Cortadellas and Zarzuela [CZ97], generalized the results of Shah. Further, D'Cruz, Raghavan and Verma [CRV99] also improved Shah's results. In [CZ97], [CRV99], [CGP+03] and [Got00], authors gave necessary and sufficient conditions for $F_{\mathfrak{m}}(Q)$ to be Cohen-Macaulay. In particular when $Q = \mathfrak{m}$, $G(Q)$ and $F_{\mathfrak{m}}(Q)$ coincide. So considering the results of Huckaba, Huneke, Marley and others, it is natural to expect relations between the coefficients of Hilbert function and depth of $F_{\mathfrak{m}}(Q)$. Jayanthan and Verma [JV05a] found that in order to discuss the depth properties of $F_{\mathfrak{m}}(Q)$, the appropriate numerical function to consider is $\lambda_R(R/\mathfrak{m}Q^n)$ instead of the Hilbert function $\lambda_R(Q^n/\mathfrak{m}Q^n)$ of $F_{\mathfrak{m}}(Q)$.

More generally, let K be a fixed $\mathfrak{m}$-primary ideal. We define the Hilbert-Samuel function of an $\mathfrak{m}$-primary ideal Q with respect to K as $H_K(Q,n) = \lambda_R(R/KQ^n)$. Since $\lambda_R(R/KQ^n) = \lambda_R(R/Q^n) + \lambda_R(Q^n/KQ^n)$ for all n, it is evident that $\lambda_R(R/KQ^n)$ coincides with a polynomial $P_K(Q,x)$, for large n. We may write

$$P_K(Q,x) = g_0^K(Q)\binom{x+d-1}{d} - g_1^K(Q)\binom{x+d-2}{d-1} + \ldots + (-1)^d g_d^K(Q)$$

for unique integers $g_i^K(Q)$ known as the Hilbert coefficients of Q with respect to K. The fiber cone of Q with respect to K is the graded ring $F_K(Q) = \bigoplus_{n=0}^{\infty} Q^n/KQ^n$. Jayanthan and Verma subsequently provided upper and lower bounds on the first coefficient $g_1^K(Q)$ and characterizations for $F_K(Q)$ to be Cohen-Macaulay and to have depth at least $d-1$ in terms of the extremal behavior of $g_1^K(Q)$. In [JV05a] and [JV05b], they also obtained the analogues for $g_i^K(Q)$, of various results of Goto [Got00], Huckaba and Marley [HM97], Huneke[Hun87], Rossi [Ros00] and [Ros99]. The upper bound on $e_2(Q)$, calculated in [CPR05] has been generalized by Gu, Zhu and Tang [GZT07] for $g_2^K(Q)$. They also extended the results of Huckaba [Huc96] to the coefficients $g_i^K(Q)$ under the assumption that $G(Q)$ and $F_K(Q)$ have depths at least $d-1$. A number of other results on the properties of $g_i^K(Q)$ have been found as a natural generalization of properties of $e_i(Q)$ in [Cru13], [SalK] and [ZGT08]. However, relatively less is known when R is not Cohen-Macaulay.

One of the main objectives of this thesis is to extend some recent results known for the coefficients $e_i(Q)$ to the case of $g_i^K(Q)$. A motivation for studying the invariants $g_i^K(Q)$ is their relation with the fiber coefficients (i.e. the coefficients of the Hilbert polynomial of $F_K(Q)$, see (2.11)) which provides a simple way to deduce results for the fiber coefficients. We do not need the ring to be Cohen-Macaulay for most of our results, instead we assume Q to be a parameter ideal.

Chapter 2 introduces the basic terminology and preliminary facts. We gather the results which are frequently used in the thesis.

In Chapter 3 we study the coefficients $g_i^K(Q)$ for parameter ideals Q vis-à-vis the Cohen-Macaulay property of R. Using a result of Goto et al. [GGH+15], we prove that if R is Cohen-Macaulay and $Q \subseteq K$ is a parameter ideal then $g_i^K(Q) = (-1)^i \lambda_R(R/K)$ (Theorem 3.1.2). An alternative proof can be found in [Cru13].

Ghezzi, Goto, Hong, Ozeki, Phuong, Vasconcelos and others have made significant progress of late in decoding information about R from the first Hilbert coefficient $e_1(Q)$. If R is Cohen-Macaulay, then it is well known that $e_i(Q) = 0$ for any parameter ideal Q and $i \geq 1$. For $d = 1$ case, the equality $e_1(Q) = 0$ for some parameter ideal Q is a characterization of Cohen-Macaulay property. Vasconcelos [Vas08, Conjecture 1] conjectured that if Q is a parameter ideal of an unmixed Noetherian local ring R, then $e_1(Q) < 0$ if and only if R is not Cohen-Macaulay. In the same paper, he settled the conjecture for a domain that is essentially of finite type over a field. Ghezzi, Hong and Vasconcelos [GHV09] settled it for a universally catenary Noetherian local domain containing a field. They also established the conjecture if R is a homomorphic image of a Gorenstein ring. Mandal, Singh and Verma [MSV11] proved that $e_1(Q) \leq 0$ for a parameter ideal Q in an arbitrary Noetherian local ring and $e_1(Q) < 0$ if depth $R = d-1$. The conjecture is settled, more generally for modules, by Ghezzi et al. [GGH+10] and [GGH+15].

We consider the following problem: suppose $(R, \mathfrak{m})$ is an unmixed local ring and $g_1^K(Q) = -\lambda_R(R/K)$ for a parameter ideal Q. Then does it imply that R is Cohen-Macaulay? The answer is easily found to be affirmative when $K = \mathfrak{m}$. However we will present an example with $d = 2$ which indicates that the above statement may not be true in general. Following the techniques of [GGH+10], we give a characterization of Cohen-Macaulay local rings in terms of the coefficients $g_0^K(Q)$ and $g_1^K(Q)$, provided $Q \subseteq K$ (Theorem 3.2.8).

In Chapter 4, we study the cases for which the first Hilbert coefficient $g_1^K(Q)$ is independent of the choice of parameter ideals Q, i.e., $g_1^K(Q)$ is constant for all parameter ideals Q. We consider the sets

$$\begin{aligned} \Lambda_i^K(R) &= \{g_i^K(Q) \mid Q \text{ is a parameter ideal of } R\} \text{ and} \\ \delta_i^K(R) &= \{g_i^K(Q) \mid Q \text{ is a parameter ideal of } R \text{ such that } Q \subseteq K\}. \end{aligned}$$

and relate their finiteness properties with generalized Cohen-Macaulay and Buchsbaum rings.

The study of Buchsbaum rings originated from a question of Buchsbaum [Buc65] concerning the invariance of the difference $I(Q; M) := \lambda_R(M/QM) - e_0(Q, M)$ for all parameter ideals Q for M. The modules for which $I(Q; M)$ is independent of parameter ideals Q for M, are known as Buchsbaum modules (see [SV78]). This notion led to the study of modules for which $I(M) := \sup\{I(Q; M) : Q \text{ is a parameter ideal for } M\}$ is finite (see [CST78]). Such modules are known as generalized Cohen-Macaulay modules. Another useful way to characterize them is through local cohomology modules. Let $H_I^i(M)$ denote the i-th local cohomology module of M with support in ideal I. A module M of dimension r is said to be generalized Cohen-Macaulay if the local cohomology modules $\mathrm{H}_{\mathfrak{m}}^i(M)$ has finite length for all $0 \leq i \leq r-1$. A ring R is called generalized Cohen-Macaulay (resp. Buchsbaum) if it is so as an R-module. A parameter ideal Q for M is said to be standard for M if $I(Q; M) = I(M)$. An ideal I with $\lambda_R(M/IM) < \infty$ is said to be M-standard ideal if every parameter ideal for M contained in I is standard for M. We refer to [Tru86] for this topic. Trung [Tru86] gave a polynomial bound for the Hilbert-Samuel function of a parameter ideal of a generalized C-M ring. As a consequence, he showed that $e_1(Q) \geq -\sum_{j=0}^{d-1} \lambda_R(\mathrm{H}_{\mathfrak{m}}^i(R))$ for a parameter ideal Q and equality holds if Q is standard for R, provided depth $R > 0$. In a Buchsbaum local ring, every parameter ideal is standard. Therefore in this case, it is clear that $e_1(Q)$ is constant and hence independent of Q. Also in view of the non-negativity of $e_1(Q)$ proved by Mandal, Singh and Verma, one can conclude the following. If R is generalized Cohen-Macaulay, then $e_1(Q)$ can assume only finite number of values. Ghezzi et al. [GGH+10] studied the converses of above properties. For $1 \leq i \leq r$, we set

$$\Lambda_i(M) = \{e_i(Q, M) \mid Q \text{ is a parameter ideal for } M\}.$$

In [GGH+10], it is proved that for an unmixed local ring R, $\Lambda_1(R)$ is finite (resp. singleton) if and only if R is generalized Cohen-Macaulay (resp. Buchsbaum). In a sequel paper [GGH+15], the authors extended these results for modules.

We intend to study the properties of coefficients $g_1^K(Q)$ along the similar lines. Suppose R is generalized Cohen-Macaulay. Using the results of [GGH+15] and [Tru86], we find uniform bounds for $g_1^K(Q)$ for a parameter ideal $Q \subseteq K$, i.e., $-\sum_{i=1}^{d-1} \binom{d-2}{i-1} \lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) - \lambda_R(R/K) \leq g_1^K(Q) \leq 0$ (Proposition 4.1.1). Consequently we get that the set $\Lambda_1^K(R)$ is finite in this case. Next, we prove that an unmixed local ring R is generalized Cohen-Macaulay if and only if the set $\Lambda_1^K(R)$ is finite (Theorem 4.1.2). It is shown that $\Lambda^1_{\mathfrak{m}}(R)$ is singleton if and only if R is Buchsbaum (Theorem 4.1.4). However we will discuss that $\Lambda_1^K(R)$ is unlikely to be singleton for an arbitrary $\mathfrak{m}$-primary ideal K in a Buchsbaum local ring R.

When R is generalized Cohen-Macaulay, Goto and Ozeki [GO11] gave uniform bounds on $e_i(Q)$. They also proved that the coefficients $e_i(Q)$ of parameter ideals Q for $1 \leq i \leq d$ can have uniform bounds simultaneously if and only if R is generalized Cohen-Macaulay. We improve upon their result and extend it to modules (Theorem 4.2.11). In the improved version, we show that the existence of uniform bounds for the first few, more precisely first $r - \operatorname{depth}_R M$, Hilbert coefficient $e_i(Q, M)$ forces M to be generalized Cohen-Macaulay. In particular, it ensures the existence of uniform bounds for all the higher coefficients. Moreover, it turns out that $e_i(Q, M)$ for $1 \leq i \leq r - \operatorname{depth}_R M$ are independent of Q if and only if $M/\mathrm{H}^0_{\mathfrak{m}}(M)$ is a Buchsbaum R-module (Theorem 4.2.14). Furthermore, the above results are generalized to the coefficients $g_i^K(Q)$. (Theorems 4.2.13 and 4.2.15).

In the last section of Chapter 4, we consider the Hilbert coefficients of $\mathfrak{m}$-primary ideals. For an R-module M, let

$$\Delta_R(M) := \{e_1(Q, M) \mid Q \text{ is an } \mathfrak{m}\text{-primary ideal of } R\}.$$

A ring R is said to be analytically unramified if its $\mathfrak{m}$-adic completion $\hat{R}$ is reduced. In [KT15], Koura and Taniguchi proved that $\Delta_R(R)$ is finite if and only if $d = 1$ and $R/\operatorname{H}^0_{\mathfrak{m}}(R)$ is analytically unramified. We generalize their result to the case of modules (Theorem 4.3.7) and the coefficients $g_1^K(Q)$ (Theorem 4.3.8).

In Chapter 5, we examine the second Hilbert coefficient $g_2^K(Q)$ for a parameter ideal Q. We prove that $g_2^K(Q)$ is bounded above by $\lambda_R(R/K)$, provided $\operatorname{depth} R \geq d-1$ (Theorem 5.2.3). As already mentioned, the Hilbert coefficients and depth of associated graded rings have certain control over each other's behavior. In particular, Mccune [Mcc13] showed that if $e_2(Q)$ vanishes in a local ring of depth at least $d-1$, then depth of $G(Q)$ is at least $d-1$. Along the similar lines, we investigate the information one can get from the equality $g_2^K(Q) = \lambda_R(R/K)$. Suppose $\operatorname{depth} R \geq d-1$. By imposing high depth on $G(Q)$, we prove that $g_2^K(Q) = \lambda_R(R/K)$ implies $\operatorname{depth} F_K(Q) \geq d-1$ (Theorem 5.2.3). When $G(Q)$ and $F_K(Q)$ have depths at least $d-1$, then it is shown that $g_i^K(Q)$ are bounded above by $(-1)^i\lambda_R(R/K)$ for $i \geq 2$ (Corollary 5.3.3). Further, if the upper bound is attained for some i then it is also attained for all the later coefficients, i.e., $g_j^K(Q) = (-1)^j\lambda_R(R/K)$ for all $j \geq i$ (Corollary 5.3.4).

In the same chapter, the difference function $P_K(Q,n) - H_K(Q,n)$ is examined for a parameter ideal Q. In a local ring of dimension one, we show that $P_K(Q,n) - H_K(Q,n)$ is non-negative for all $n \geq 0$ (Proposition 5.1.1). In case of higher dimensional rings, we obtain an analogue for the higher difference functions. For a function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, the first difference function $\Delta(f)$ is defined by $\Delta(f(n)) = f(n+1) - f(n)$. The i-th difference function $\Delta^i(f)$ is defined by $\Delta^i(f) = \Delta^{i-1}(\Delta(f))$. By convention, $\Delta^0(f) = f$. We prove that $(-1)^i\Delta^{d+1-i}(P_K(Q,n) - H_K(Q,n)) \geq 0$ for $0 \leq i \leq d+1$ and $n \geq 0$ provided $\operatorname{depth} G(Q) \geq d-1$ and $\operatorname{depth} F_K(Q) \geq d-1$ (Theorem 5.3.1).

The method of induction plays a key role in our treatment of the coefficients $g_i^K(Q)$. The most fundamental tool while applying induction is the notion of superficial elements. It provides a way to reduce the problem to lower dimensional cases. The theory of superficial elements in the context of studying $e_i(Q)$ is due to Samuel and can be found in [RV10] and [SH06]. In order to treat the coefficient $g_i^K(Q)$, an analogous theory of superficial elements in fiber cones has been developed by Jayanthan and Verma [JV05a]. The assumption $Q \subseteq K$ is required in many of our results to be able to use the results from [JV05b] while applying induction. In particular, we need an analog of Singh's formula and Sally's machine for fiber cones. In Proposition 2.2.9, we will give a general version of Singh's formula. The basic idea in the proofs of Chapter 4 is to treat K as an R-module and relate the coefficients $e_i(Q,K)$ and $g_i^K(Q)$. With this approach, we are able to make use of a number of existing results from literature which have been generalized for modules in recent years. This idea provides a simple and uniform method for the study of the coefficients $g_i^K(Q)$. However, different ideas are needed for an exhaustive study.

In Chapter 6, we restrict our focus to the coefficients $e_i(Q)$. In the classical case of an $\mathfrak{m}$-primary ideal Q in a Cohen-Macaulay local ring, relations among various Hilbert coefficients and bounds for them have been explored by several authors. Northcott's inequality [Nor60] $e_1(Q) \geq e_0(Q) - \lambda_R(R/Q)$ is one of the first results in this direction. It was improved by M. E. Rossi in [Ros99]. Several bounds on $e_1(Q)$ in terms of $e_0(Q)$ exist in literature, e.g., [Eli05], [Eli08], [HH12], [RV10] and [RV05]. For example, Rossi and Valla proved that $e_1(Q) \leq \binom{e_0(Q)-k+1}{2}$ if $I \subseteq \mathfrak{m}^k$. Such bounds are useful for examining the finiteness of Hilbert functions of ideals with fixed multiplicity, see [ST97]. It is difficult in general to find uniform bounds for $e_i(Q)$. With R and Q as above, it is known that $e_1(Q)$ and $e_2(Q)$ are non-negative, due to Northcott [Nor60] and Narita [Nar63] respectively. However the higher coefficients are not necessarily non-negative. Marley [Mar, Example 2.3] gave an example of a Cohen-Macaulay local ring and an $\mathfrak{m}$-primary ideal Q with $e_3(Q) < 0$. Itoh [Ito95] showed that $e_3(Q) \geq 0$ if Q is a normal ideal. A comprehensive account of these bounds can be found in [RV10].

The case when R is not Cohen-Macaulay is quite different. Let Q be a parameter ideal of R hereafter. Then $e_1(Q) \leq 0$ as already mentioned. Mccune [Mcc13] showed that if $\operatorname{depth} R \geq d-1$ then $e_2(Q) \leq 0$. In addition, if $\operatorname{depth} G(Q) \geq d-1$ then she proved that $e_i(Q) \leq 0$ for $2 \leq i \leq d$. We improve Mccune's result by relaxing the hypothesis to $\operatorname{depth} G(Q) \geq d-2$ (Corollary 6.2.2). If $\operatorname{depth} G(Q) \geq d-1$, we provide uniform lower bounds for $e_i(Q)$ which is independent of Q as well as i (Corollary 6.2.8). Goto and Ozeki [GO11] gave a uniform lower bound on $e_2(Q)$ in two dimensional local rings if $\operatorname{depth} R \geq 1$. We extend their result to rings of arbitrary dimension d with $\operatorname{depth} R \geq d-1$ (Theorem 6.3.4). We also discuss some equivalent conditions for the vanishing of the last coefficient $e_d(Q)$. The most notable result of this chapter is Theorem 6.2.4 where we prove that $e_3(Q) \leq 0$ provided $\operatorname{depth} R \geq d-1$.

The Rees algebra and the extended Rees algebra of an ideal I are defined as $\mathcal{R}(I) = \bigoplus_{n=0}^{\infty} I^n t^n \subseteq R[t]$ and $\mathcal{R}^*(I) = \bigoplus_{n\in\mathbb{Z}} I^n t^n \subseteq R[t, t^{-1}]$ respectively with the convention that $I^n = R$ for $n \leq 0$. Blancafort [Bla97] gave a formula for the difference between Hilbert-Samuel function and polynomial in terms of the local cohomology modules of the extended Rees algebra. We use her result for our proofs in this chapter. Our methods depend heavily on the properties of local cohomology modules for which we refer to [BS98] and [ILL+07].

The last chapter presents a summary of the results of the thesis and some problems which can be pursued in future.

2 Preliminaries

In this chapter we introduce the basic terminology and recall some preliminary facts which will be needed later. In Section 2.1, the basic definitions and notations are introduced. In Section 2.2, we recall the notion of superficial elements which is a fundamental tool while applying induction on dimension of R. It is well known that superficial elements exist if the residue field of the ring $(R, \mathfrak{m})$ is infinite. Section 2.3 briefly explains that the existence of superficial elements may be assumed for most of our results by changing R, if needed.

2.1 Definitions and notations

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and K an $\mathfrak{m}$-primary ideal of R. Let $Q \subseteq K$ be an $\mathfrak{m}$-primary ideal. Recall that the Hilbert function of Q with respect to K is defined as $H_K(Q, n) = \lambda_R(R/KQ^n)$ for $n \in \mathbb{Z}$. The Hilbert-Samuel function of Q is defined as $H(Q, n) = \lambda_R(R/Q^n)$. It is known that for $n \gg 0$, $H(Q, n)$ (resp. $H_K(Q, n)$) agrees with a polynomial $P(Q, x)$ (resp. $P_K(Q, x)$) of degree d. We can write these polynomials in the following manner.

$$P(Q, x) = \sum_{i=0}^{d} (-1)^i e_i(Q) \binom{x + d - i - 1}{d - i} \tag{2.1}$$

$$P_K(Q, x) = \sum_{i=0}^{d} (-1)^i g_i^K(Q) \binom{x + d - i - 1}{d - i} \tag{2.2}$$

for unique integers $e_i(Q)$ (resp. $g_i^K(Q)$). The Hilbert function of the fiber cone $F_K(Q)$ is given by $H(F, n) = \lambda_R(Q^n/KQ^n)$. Let $P_K(F, n)$ denote the Hilbert polynomial of

$F_K(Q)$. We can write $P_K(F,x)$ in the following way.

$$P_K(F,x) = \sum_{i=0}^{d-1}(-1)^i f_i^K(Q)\binom{x+d-i-1}{d-1-i}$$

where the coefficients $f_i^K(Q)$ are integers and are referred to as the fiber coefficients of Q with respect to K. Recall that for a finitely generated R-module M of dimension r, we write the Hilbert-Samuel polynomial of M with respect to Q as

$$P(Q,x,M) = \sum_{i=0}^{r}(-1)^i e_i(Q,M)\binom{x+r-i-1}{r-i}.$$

It is often useful to work with a module with positive depth. For a module M, let $W = \mathrm{H}^0_{\mathfrak{m}}(M)$ and $M' = M/W$. Then $\operatorname{depth} M' > 0$ and the Hilbert coefficients of M and M' are related nicely. To see this, let Q be an ideal of definition for M. We have the following exact sequence.

$$0 \longrightarrow W/(Q^nM \cap W) \longrightarrow M/Q^nM \longrightarrow M'/Q^nM' \longrightarrow 0. \tag{2.3}$$

By Artin-Rees lemma, $Q^nM \cap W \subseteq Q^{n-k}W = 0$ for some integer k and for $n \gg 0$. Hence for all $n \gg 0$,

$$\lambda_R(M/Q^nM) = \lambda_R(M'/Q^nM') + \lambda_R(W).$$

This gives

$$P(Q,n,M) = P(Q,n,M') + \lambda_R(W). \tag{2.4}$$

Hence the coefficients have the following relation.

$$e_i(Q,M) = \begin{cases} e_i(Q,M') & \text{if } \ 0 \le i \le r-1, \\ e_d(Q,M') + (-1)^d\lambda_R(W) & \text{if } \ i = r. \end{cases} \tag{2.5}$$

Remark 2.1.1. *1. It is easy to see that K as an R-module has dimension d. For this, consider the exact sequence*

$$0 \longrightarrow K \longrightarrow R \longrightarrow R/K \longrightarrow 0.$$

This gives the following long exact sequence

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(K) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(R) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(R/K) \longrightarrow R/K \xrightarrow{f} \mathrm{H}^1_{\mathfrak{m}}(K) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(R) \longrightarrow 0 \tag{2.6}$$

$$\text{and that } \mathrm{H}^i_{\mathfrak{m}}(K) \cong \mathrm{H}^i_{\mathfrak{m}}(R) \text{ for } 2 \le i \le d. \tag{2.7}$$

Since $\mathrm{H}^d_{\mathfrak{m}}(R) \neq 0$, $\mathrm{H}^d_{\mathfrak{m}}(K) \neq 0$. *Hence* $\dim K = d$.

2. *Since* $\lambda_R(R/KQ^n) = \lambda_R(R/K) + \lambda_R(K/Q^nK)$ *for all* $n \in \mathbb{Z}$. *So treating* K *as an* R*-module, we have* $P_K(Q,x) = \lambda_R(R/K) + P(Q,x,K)$. *On comparing the coefficients of both sides, we get*

$$g_0^K(Q) = e_0(Q,K) \text{ and} \tag{2.8}$$

$$g_i^K(Q) = \begin{cases} e_i(Q,K) & \text{if } 1 \le i \le d-1, \\ e_d(Q,K) + (-1)^d \lambda_R(R/K) & \text{if } i = d. \end{cases} \tag{2.9}$$

Remark 2.1.2. 1. *Since* $\lambda_R(R/KQ^n) = \lambda_R(R/Q^n) + \lambda_R(Q^n/KQ^n)$ *for all* $n \in \mathbb{Z}$, *we have* $P_K(Q,x) = P(Q,x) + P_K(F,x)$. *Thus comparing the coefficients of both sides, we get*

$$g_0^K(Q) = e_0(Q) \quad \text{and} \tag{2.10}$$

$$f_i^K(Q) = e_{i+1}(Q) - g_{i+1}^K(Q) + e_i(Q) - g_i^K(Q) \;\; \text{for } 0 \le i \le d-1. \tag{2.11}$$

2. *By putting* $K = Q$, *we see that* $P_K(Q,n) = \lambda_R(R/Q^{n+1}) = P(Q,n+1)$ *for all* $n \gg 0$. *On comparing the coefficients, we get for* $0 \le i \le d$,

$$g_i^K(Q) = e_i(Q) - e_{i-1}(Q) + \ldots + (-1)^i e_0(Q). \tag{2.12}$$

We need the following definitions. Let Q be an ideal of R. For an element $0 \neq x \in R$, let x^*(resp. x^o) denote the *initial form* of x in $G(Q)$ (resp. $F_K(Q)$) i.e. the image of x in $G(Q)_i$ (resp. $F_K(Q)_i$), where i is the unique integer such that $x \in Q^i \setminus Q^{i+1}$ (resp. $x \in Q^i \setminus KQ^i$).

Definition 2.1.3. *The postulation number for* Q *with respect to* K, *denoted by* $\eta_K(Q)$, *is defined as*

$$\eta_K(Q) := \; min\{i \mid H_K(Q,n) = P_K(Q,n) \text{ for all } n > i\}. \tag{2.13}$$

In case of $K = R$, we drop K and write $\eta(Q)$ for postulation number. By a result of Marley [Mar, Lemma 2.8], for an element $x \in Q \setminus Q^2$ and x^* a non-zero-divisor in $G(Q)$,

$$\eta(QR_1) = \eta(Q) + 1 \tag{2.14}$$

where $R_1 = R/(x)$. A similar relation for $\eta_{_K}(Q)$ in general, could be useful for method of induction. For that purpose, we define the function $H^*_K(Q,n) : \mathbb{Z} \longrightarrow \mathbb{Z}$ as follows

$$H^*_K(Q,n) = \begin{cases} \lambda_R(R/KQ^n) & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases} \tag{2.15}$$

Note that $H^*_K(Q,n) = P_K(Q,n)$ for all $n \gg 0$. We define

$$\eta^*_{_K}(Q) := \min\{i \mid H^*_K(Q,n) = P_K(Q,n) \text{ for all } n > i\}. \tag{2.16}$$

Lemma 2.1.4. *Let $(R,\mathfrak{m})$ be Noetherian local ring and K an $\mathfrak{m}$-primary ideal of R. Let $Q \subseteq K$ be an $\mathfrak{m}$-primary ideal. Let $x \in Q \setminus KQ$ be such that x^o is regular in $F_K(Q)$. Then*

$$\eta^*_{_{KR_1}}(QR_1) = \eta^*_{_K}(Q) + 1$$

where $R_1 = R/(x)$.

Proof. Consider the exact sequence

$$0 \longrightarrow \frac{(KQ^n : x)}{KQ^{n-1}} \longrightarrow R/KQ^{n-1} \xrightarrow{x} R/KQ^n \longrightarrow R/(KQ^n, x) \longrightarrow 0.$$

Since x^o is regular in $F_K(Q)$, $(KQ^n : x) = KQ^{n-1}$ for all $n \geq 1$. Hence

$$H_{KR_1}(QR_1,n) = H_K(Q,n) - H_K(Q,n-1) \text{ for all } n \geq 1 \text{ and} \tag{2.17}$$

$$H^*_{KR_1}(QR_1,n) = H^*_K(Q,n) - H^*_K(Q,n-1) \text{ for all } n \in \mathbb{Z}.$$

Therefore $P_{KR_1}(QR_1,n) = P_K(Q,n) - P_K(Q,n-1)$ for all integers n. Thus $P_{KR_1}(QR_1,n) = H^*_{KR_1}(QR_1,n)$ for all $n > \eta^*_{_K}(Q)+1$. Suppose $P_{KR_1}(QR_1,\eta^*_{_K}(Q)+1) = H^*_{KR_1}(QR_1,\eta^*_{_K}(Q)+1)$. Then $P_K(Q,\eta^*_{_K}(Q)) = H^*_K(Q,\eta^*_{_K}(Q))$ which is a contradiction to the minimality of $\eta^*_{_K}(Q)$. Therefore, $\eta^*_{_{KR_1}}(QR_1) = \eta^*_{_K}(Q) + 1$. □

Definition 2.1.5. *A reduction of an ideal Q is an ideal $J \subseteq Q$ such that $Q^{n+1} = JQ^n$ for some $n \geq 0$. A minimal reduction of Q is a reduction of Q which is minimal with respect to inclusion. For a minimal reduction J of Q, reduction number of Q with respect to J, denoted by $r_J(Q)$, is the least non-negative integer n such that $Q^{n+1} = JQ^n$.*

Definition 2.1.6. *Let M be a finitely generated R-module. Let $x_0 = 0$. A sequence of elements $x_1, \ldots, x_k$ is called a d-sequence for M if one (and hence both) of the following equivalent conditions hold.*

1. $((x_0,\dots,x_i)M :_M x_{i+1}x_j) = ((x_0,\dots,x_i)M :_M x_j)$ *for* $0 \le i \le k-1$ *and for all* $j \ge i+1$.

2. $((x_0,\dots,x_i)M :_M x_{i+1}) \cap (x_1,\dots,x_k)M = (x_0,\dots,x_i)M$ *for all* $0 \le i \le k-1$.

A regular sequence is trivially a d-sequence. We now recall the notion of generalized Cohen-Macaulay and Buchsbaum modules.

Definition 2.1.7. *A module M of dimension r is said to be generalized Cohen-Macaulay if $\mathrm{H}^i_{\mathfrak{m}}(M)$ has finite length for all $0 \le i \le r-1$.*

Remark 2.1.8. *From* (2.6) *and* (2.7)*, we have that $\mathrm{H}^i_{\mathfrak{m}}(K)$ has finite length if and only if $\mathrm{H}^i_{\mathfrak{m}}(R)$ has finite length for $1 \le i \le d-1$. Hence R is a generalized Cohen-Macaulay ring if and only if K is a generalized Cohen-Macaulay R-module.*

For a parameter ideal Q, we set

$$I(Q;M) := \lambda_R(M/QM) - e_0(Q,M) \text{ and}$$

$$I(M) := \sup\{I(Q;M) : Q \text{ is a parameter ideal for } M\}$$

It is well known that M is generalized Cohen-Macaulay if and only if $I(M) < \infty$, see [Tru86, Lemma 1.5]. In this case,

$$I(M) = \sum_{i=0}^{r-1} \binom{r-1}{i} \lambda_R(\mathrm{H}^i_{\mathfrak{m}}(M)). \tag{2.18}$$

Moreover, there exists an integer $n > 0$ such that $I(Q;M) = I(M)$ for every parameter ideal $Q \subseteq \mathfrak{m}^n$ for M.

Definition 2.1.9. 1. *A parameter ideal Q for M is said to be standard for M if $I(Q;M) = I(M)$. An ideal I with $\lambda_R(M/IM) < \infty$ is said to be M-standard ideal if every parameter ideal for M contained in I is standard for M.*

2. *An R-module M is said to be Buchsbaum if every parameter ideal for M is standard.*

Generalized Cohen-Macaulay modules have many interesting properties. We quote a few for our use.

Lemma 2.1.10. [Tru86, Lemma 1.2] *Let M be a generalized Cohen-Macaulay module of dimension $r > 0$. Then*

1. *Every parameter ideal* $Q = (x_1, \ldots, x_r)$ *for* M *is reducing i.e.*

$$I(Q; M) = \lambda_R\big(((x_1, \ldots, x_{d-1})M :_M x_d)/(x_1, \ldots, x_{d-1})M\big).$$

2. *Every parameter ideal* $Q = (x_1, \ldots, x_r)$ *for* M *is unmixed up to* $\mathfrak{m}$ *i.e.* $\dim R/\mathfrak{p} = r - i$ *for all* $\mathfrak{p} \in \operatorname{Ass}(M/Q_iM) \setminus \{\mathfrak{m}\}$ *and* $0 \leq i \leq r-1$ *where* $Q_i = (x_1, \ldots, x_i)$ *and* $Q_0 = (0)$.

3. $M_\mathfrak{p}$ *is a Cohen-Macaulay module with* $\dim M_\mathfrak{p} = r - \dim R/\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Supp}(M) \setminus \{\mathfrak{m}\}$.

The following theorem provides a polynomial bound for the Hilbert-Samuel function $\lambda_R(M/Q^nM)$ for a parameter ideal Q for M.

Theorem 2.1.11. [Tru86, Theorem 4.1] *Let* M *be a generalized Cohen-Macaulay module of dimension* $r > 0$ *and* $Q = (x_1, \ldots, x_r)$ *a parameter ideal for* M. *Then*

$$\lambda_R(M/Q^{n+1}M) \leq \binom{n+r}{r} e_0(Q, M) + \sum_{i=1}^{r} \sum_{j=0}^{r-i} \binom{n+r-i}{r-i} \binom{r-i-1}{j-1} \lambda_R(H^j_\mathfrak{m}(M))$$

for all $n \geq 0$, *where* $\binom{r-i-1}{-1} := 0$ *if* $i \neq r$ *and* $\binom{-1}{-1} := 1$. *Equality holds for some fixed* n *if and only if the following conditions are satisfied.*

1. $Q^{n+1}M \cap H^0_\mathfrak{m}(M) = 0$.

2. Q *is standard for* $M/H^0_\mathfrak{m}(M)$.

An important consequence of above theorem is the following result.

Corollary 2.1.12. [Tru86, Corollary 4.2] *A parameter ideal* Q *is standard for* M *if and only if*

$$\lambda_R(M/Q^{n+1}M) = \binom{n+r}{r} e_0(Q, M) + \sum_{i=1}^{r} \sum_{j=0}^{r-i} \binom{n+r-i}{r-i} \binom{r-i-1}{j-1} \lambda_R(H^j_\mathfrak{m}(M))$$

for all $n \geq 0$.

It is evident that for a generalized Cohen-Macaulay module M of dimension $r \geq 2$ and a parameter ideal Q for M,

$$e_1(Q, M) \geq -\sum_{j=1}^{r-1} \binom{r-2}{j-1} \lambda_R(H^j_\mathfrak{m}(M)) \tag{2.19}$$

and equality holds if Q is standard for M.

Notation. Let $\mathcal{R} = R(I)$ and $\mathcal{R}^* = R^*(I)$ denote the Rees algebra and the extended Rees algebra of an ideal I respectively. We put $\mathcal{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$ where $\mathcal{R}_+ = \bigoplus_{n>0} \mathcal{R}_n$ is the irrelevant ideal of the Rees algebra $\mathcal{R}$.

2.2 Superficial elements

In this section, we summarize the basic properties of superficial elements in $G(Q)$ and $F_K(Q)$. The notion of superficial elements provides an effective method in the study of Hilbert coefficients as it allows to reduce the problem to lower dimensional cases by applying induction on the dimension of R. The theory of superficial elements in $G(Q)$, defined below, can be found extensively in [RV10] and [SH06]. We wish to discuss the properties of the superficial elements in $F_K(Q)$ which are also well known and can be found in [JV05a]. We include the statements for easy reference. Let $Q \subseteq R$ be an ideal and M be an R-module.

Definition 2.2.1. *1. An element $x \in Q$ is said to be M-superficial in Q if there exists an integer $c > 0$ such that $(Q^nM :_M x) \cap Q^cM = Q^{n-1}M$ for all $n > c$. If x is R-superficial in Q, we say that x^* is superficial in $G(Q)$.*

2. A sequence $x_1, \ldots, x_k \in Q$ is said to be an M-superficial sequence in Q if for all $i = 1, \ldots, k$, x'_i is M_{i-1}-superficial in QR_{i-1} where $R_{i-1} = R/(x_1, \ldots, x_{i-1})$, $M_{i-1} = M/(x_1, \ldots, x_{i-1})M$ and x'_i denotes the image of x_i in R_{i-1}. If $x_1, \ldots, x_k$ is an R-superficial sequence in Q, we say that $x_1^*, \ldots, x_k^*$ is a superficial sequence in $G(Q)$.

The next proposition gives some equivalent conditions for superficial elements in $G(Q)$. These conditions hold for modules but we only need them in case of rings.

Proposition 2.2.2. [RV10, Theorem 1.2] *Let $(R, \mathfrak{m})$ be a local ring of dimension $d > 0$ and $Q \subseteq R$ be an ideal. Let $x \in Q \setminus Q^2$. Then the following statements are equivalent.*

1. x^* is superficial in $G(Q)$;

2. $(0 :_{G(Q)} x^*)_n = 0$ for $n \gg 0$;

3. $(Q^{n+1} : x) = Q^n + (0 : x)$ and $Q^n \cap (0 : x) = 0$ for $n \gg 0$.

Remark 2.2.3. *Let $x \in Q$ such that x^* is superficial in $G(Q)$. Then*

1. $x \in Q \setminus Q^2$.

2. [SH06, Remark 8.5.2] $(x^m)^*$ *is superficial in* $G(Q^m)$.

Proof. (i) Since x^* is superficial in $G(Q)$, there exists $c > 0$ such that $(Q^n : x) \cap Q^c = Q^{n-1}$ for all $n > c$. For $n = c + 2$, $(Q^{c+2} : x) \cap Q^c = Q^{c+1}$. Suppose $x \in Q^2$. Then $Q^c \subseteq (Q^{c+2} : x) \cap Q^c = Q^{c+1}$. Hence, by Nakayama Lemma, $Q^c = 0$ which is a contradiction.

□

The following results are crucial in the theory of superficial elements and used frequently in the thesis. In particular, Sally-machine is a very important tool while dealing with the depth of associated graded rings. A version of this for the fiber cones is also proved in [JV05b] which will be discussed in Chapter 5. For a ring S and an ideal $I \subseteq S$, $\operatorname{depth}(I, S)$ denotes the length of all maximal regular sequences in S contained in I.

Proposition 2.2.4. *Let $x_1, \ldots, x_k \in Q$ such that $x_1^*, \ldots, x_k^*$ is a superficial sequence in $G(Q)$. Then*

1. [RV10, Lemma 1.2] $x_1, \ldots, x_k$ *is a regular sequence in* R *if and only if* $\operatorname{depth}(Q, R) \geq k$.

2. [RV10, Lemma 1.3] $x_1^*, \ldots, x_k^*$ *is a regular sequence in* $G(Q)$ *if and only if* $\operatorname{depth} G(Q) \geq k$.

3. [RV10, Lemma 1.4] *(Sally-machine) Let R_k denote $R/(x_1, \ldots, x_k)$. Then* $\operatorname{depth} G(QR_k) \geq 1$ *if and only if* $\operatorname{depth} G(Q) \geq k + 1$.

The next proposition shows that the Hilbert coefficients behave nicely when we reduce by superficial sequences.

Proposition 2.2.5. [RV10, Proposition 1.2] *Let M be a finitely generated R-module of dimension $r > 0$ and Q an $\mathfrak{m}$-primary ideal of R. Let $x \in Q$ be an M-superficial element. Then*

1. $\dim M/xM = \dim M - 1$.

2. *$\lambda_R(0 :_M x)$ is finite and*

$$e_i(QR_1, M_1) = \begin{cases} e_i(Q, M) & \text{for } 0 \leq i \leq d-2 \\ e_{d-1}(Q, M) + (-1)^{d-1}\lambda_R(0 :_M x) & \text{for } i = d-1 \end{cases}$$

where $R_1 = R/(x)$, $M_1 = M/xM$ and $e_i(QR_1, M_1)$ denote the coefficients of the polynomial $P(QR_1, n, M_1)$.

Now we recall the notion of superficial elements in $F_K(Q)$ from [JV05a]. Let Q be an ideal of R and K an ideal with $Q \subseteq K$.

Definition 2.2.6. 1. *For an element $x \in Q$ such that $x^o \neq 0$ in $F_K(Q)$, x^o is said to be superficial in $F_K(Q)$ if there exists an integer $c > 0$ such that $(0 : x^o) \cap F_K(Q)_n = 0$ for all $n > c$.*

2. *For a sequence $x_1, \ldots, x_k \in Q$, $x_1^o, \ldots, x_k^o$ is said to be a superficial sequence in $F_K(Q)$ if for all $i = 1, \ldots, k$, $(x_i')^o$ is superficial in $F_{KR_{i-1}}(QR_{i-1})$ where $R_{i-1} = R/(x_1, \ldots, x_{i-1})$ and x_i' denotes the image of x_i in R_{i-1}.*

The existence and basic properties of superficial elements in $F_K(Q)$ are discussed in [JV05b, Proposition 2.1] and [JV05a, Section 2]. We recall the following lemma which provides a useful characterization of superficial elements in $F_K(Q)$.

Lemma 2.2.7. [JV05a, Lemma 2.3] *Let R be a Noetherian local ring of dimension $d > 0$. Let Q be an ideal of R and K an $\mathfrak{m}$-primary ideal of R such that $Q \subseteq K$. Then the following statements hold.*

1. *If there exists an integer $c > 0$ such that $(KQ^n : x) \cap Q^c = KQ^{n-1}$ for all $n > c$, then x^o is superficial in $F_K(Q)$.*

2. *If x^o is superficial in $F_K(Q)$ and x^* is superficial in $G(Q)$, then there exists an integer $c > 0$ such that $(KQ^n : x) \cap Q^c = KQ^{n-1}$ for all $n > c$. Moreover, if x is regular in R, then $(KQ^n : x) = KQ^{n-1}$ for all $n \gg 0$.*

Existence of superficial elements in $F_K(Q)$ is guaranteed by [JV05a, Proposition 2.2] when K is an $\mathfrak{m}$-primary ideal and R has infinite residue field. Indeed, with these conditions we can choose $x \in Q$ such that x^o is superficial in $F_K(Q)$ and x^* is superficial in $G(Q)$. Hence in this thesis, we use the characterization given by Lemma 2.2.7 (2) for superficial elements in $F_K(Q)$. The following lemma allows us to choose superficial elements in $F_K(Q)$ avoiding a finite set of ideals not containing Q. This result is well known for superficial elements in $G(Q)$ see [SH06, Corollary 8.5.9]. The proof is also similar, so we will not include it here.

Lemma 2.2.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. Let Q be an ideal of R and K an $\mathfrak{m}$-primary ideal of R such that $Q \subseteq K$. Let $I_1, \ldots, I_r$ be ideals in R not containing Q. Then there exists an element $x \in Q \setminus \mathfrak{m}Q$ that is not contained in any I_i such that x^* is superficial in $G(Q)$ and x^o is superficial in $F_K(Q)$.*

In particular, if Q contains a non-zero-divisor then there exists an element $x \in Q \setminus \mathfrak{m}Q$ such that x^* is superficial in $G(Q)$, x^o is superficial in $F_K(Q)$ and x is a non-zero-divisor.

The Hilbert coefficients $g_i^K(Q)$ behave nicely on reducing modulo a superficial element in $F_K(Q)$. We need a refined version of [JV05a, Lemma 3.5].

Proposition 2.2.9. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and K an $\mathfrak{m}$-primary ideal of R. Let $Q \subseteq K$ be an $\mathfrak{m}$-primary ideal and $x \in Q$ such that x^* is superficial in $G(Q)$ and x^o is superficial in $F_K(Q)$. Then*

$$g_i^{KR_1}(QR_1) = \begin{cases} g_i^K(Q) & \text{for } 0 \leq i \leq d-2 \\ g_{d-1}^K(Q) + (-1)^{d-1}\lambda_R(0:x) & \text{for } i = d-1 \end{cases}$$

where R_1 denote $R/(x)$ and $g_i^{KR_1}(QR_1)$ denote the coefficients of the polynomial $P_{KR_1}(QR_1, n)$.

Proof. For $n \in \mathbb{Z}$, the exact sequence

$$0 \longrightarrow \frac{(KQ^n : x)}{KQ^{n-1}} \longrightarrow R/KQ^{n-1} \xrightarrow{x} R/KQ^n \longrightarrow R/(KQ^n, x) \longrightarrow 0$$

gives that $\lambda_R(R/(KQ^n, x)) = \lambda_R(R/KQ^n) - \lambda_R(R/KQ^{n-1}) + \lambda_R((KQ^n : x)/KQ^{n-1})$ for all $n \in \mathbb{Z}$.

Claim: $KQ^n \cap (0 : x) = 0$ and $(KQ^n : x) = KQ^{n-1} + (0 : x)$ for $n \gg 0$.

Proof of Claim. By Lemma 2.2.7(2), there exists an integer $c > 0$ such that for all $n > c$,

$$(KQ^n : x) \cap Q^c = KQ^{n-1}.$$

So for all $n > c$, $KQ^n \cap (0 : x) \subseteq \bigcap_{m \geq n} (KQ^m : x) \cap Q^c = \bigcap_{m \geq n} KQ^{m-1} = 0$. By the Artin-Rees lemma, there exists an integer k such that $KQ^n \cap (x) \subseteq (x)Q^{n-k}$ for all $n > k$. Let $n > k + c$ and $y \in (KQ^n : x)$. Then $yx \in KQ^n \cap (x) \subseteq xQ^{n-k}$. Suppose $yx = zx$ for some $z \in Q^{n-k}$. Then $z \in (KQ^n : x) \cap Q^c = KQ^{n-1}$. Therefore $y = z + (y - z) \in KQ^{n-1} + (0 : x)$. □

Thus $\lambda_R(R/(KQ^n, x)) = \lambda_R(R/KQ^n) - \lambda_R(R/KQ^{n-1}) + \lambda_R(0 : x)$ for all $n \gg 0$. Hence, for $n \in \mathbb{Z}$,

$$P_{KR_1}(QR_1, n) = P_K(Q, n) - P_K(Q, n - 1) + \lambda_R(0 : x). \tag{2.20}$$

Now, the result follows by comparing the coefficients of both sides of (2.20). □

2.3 The extension from R to $R(X)$

For a local ring $(R, \mathfrak{m})$, let $R(X)$ denote the ring $R[X]_{\mathfrak{m}R[X]}$. As mentioned earlier, superficial elements exist if the residue field of R is infinite. To achieve this, we may pass from R to $R(X)$. This section is devoted to explain briefly that the properties and invariants treated in this thesis remains preserved under this passage. The following lemma and proposition are basic in this regard.

Lemma 2.3.1. [Mar, Lemma A.1.1] *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $(S, \mathfrak{n})$ a faithfully flat extension of R such that $\sqrt{\mathfrak{m}S} = \mathfrak{n}$. Let M be an R-module of finite length. Then*

$$\lambda_S(M \otimes_R S) = \lambda_R(M) \cdot \lambda_S(S/\mathfrak{m}S).$$

Proposition 2.3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and Q an $\mathfrak{m}$-primary ideal of R. Let $S = R(X)$ and $\mathfrak{n} = \mathfrak{m}S$. Then*

1. $\dim R = \dim S$.
2. $\operatorname{depth} R = \operatorname{depth} S$.
3. $|S/\mathfrak{n}| = \infty$.
4. *$H(QS, n) = H(Q, n)$ and $P(QS, n) = P(Q, n)$ for all $n \in \mathbb{Z}$.*
5. $\operatorname{depth} G(Q) = \operatorname{depth} G(QS)$.

6. *Let K be an ideal containing Q. Then $H_{KS}(QS,n) = H_K(Q,n)$ and $P_{KS}(QS,n) = P_K(Q,n)$ for all $n \in \mathbb{Z}$.*

7. $\operatorname{depth} F_K(Q) = \operatorname{depth} F_{KS}(QS)$.

Proof. (1)-(5) can be found in [Mar, Proposition A.1.2]. Since $\mathfrak{m}S = \mathfrak{n}$ and $\lambda_S(S/\mathfrak{n}) = 1$, we get $\lambda_R(R/KQ^n) = \lambda_S(S/KQ^nS)$ by Lemma 2.3.1. This gives (6). We include below a proof for (7) which is analogous to part (5).

Let the irrelevant ideals of $F_K(Q)$ and $F_{KS}(QS)$ be denoted by T and N. Since $F_K(Q)$ and $F_{KS}(QS)$ are Noetherian, we have that

$$\operatorname{depth} F_K(Q) = \inf\{i \mid Ext^i_{F_K(Q)}(F_K(Q)/T, F_K(Q)) \neq 0\} \text{ and} \tag{2.21}$$

$$\operatorname{depth} F_{KS}(QS) = \inf\{i \mid Ext^i_{F_{KS}(QS)}(F_{KS}(QS)/N, F_{KS}(QS)) \neq 0\}. \tag{2.22}$$

Since S is faithfully flat R-module, $F_K(Q) \otimes_R S = F_{KS}(QS)$. Thus $F_{KS}(QS)$ is a faithfully flat $F_K(Q)$-module. As $TF_{KS}(QS) = N$, we have

$$F_{KS}(QS)/N \cong F_{KS}(QS)/TF_{KS}(QS) \cong F_{KS}(QS) \otimes_{F_K(Q)} F_K(Q)/T.$$

Hence,

$$Ext^i_{F_{KS}(QS)}(F_{KS}(QS)/N, F_{KS}(QS)) \cong Ext^i_{F_K(Q)}(F_K(Q)/T, F_K(Q)) \otimes_{F_K(Q)} F_{KS}(QS).$$

Since $F_{KS}(QS)$ is a faithfully flat $F_K(Q)$-module, we get $Ext^i_{F_{KS}(QS)}(F_{KS}(QS)/N, F_{KS}(QS)) = 0$ if and only if $Ext^i_{F_K(Q)}(F_K(Q)/T, F_K(Q)) = 0$. □

We study the Hilbert coefficients of unmixed local rings in subsequent chapters. The next proposition provides a way to assume without loss of generality that an unmixed local ring has infinite residue field. We first recall the definition of unmixed modules.

Definition 2.3.3. *For a finitely generated R-module M, we set $\operatorname{Assh}_R M = \{\mathfrak{p} \in \operatorname{Ass}_R M \mid \dim R/\mathfrak{p} = \dim M\}$. Let $(0_M) = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M} M(\mathfrak{p})$ be a primary decomposition of the submodule (0_M) in M, where $M(\mathfrak{p})$ is a $\mathfrak{p}$-primary submodule of M for each $\mathfrak{p} \in \operatorname{Ass}_R M$. The submodule $U_M(0) := \bigcap_{\mathfrak{p} \in \operatorname{Assh}_R M} M(\mathfrak{p})$ is called the unmixed component of M. M is called unmixed if $\operatorname{Assh}_{\widehat{R}} \widehat{M} = \operatorname{Ass}_{\widehat{R}} \widehat{M}$ where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R.*

If $M = R$, then we write $\operatorname{Ass}(R)$ and $\operatorname{Assh}(R)$ instead of $\operatorname{Ass}_R(R)$ and $\operatorname{Assh}_R(R)$.

Proposition 2.3.4. [McD, Theorem 2.4.3] *Let R be a local ring such that $\operatorname{Ass}(R) = \operatorname{Assh}(R)$. Suppose R is a homomorphic image of a Cohen-Macaulay ring. Then $\operatorname{Ass}(\widehat{R}) = \operatorname{Assh}(\widehat{R})$.*

Given an unmixed local ring $(R, \mathfrak{m})$ and an $\mathfrak{m}$-primary ideal Q, we may pass to $\widehat{R}$ which is unmixed i.e. $\operatorname{Ass}(\widehat{R}) = \operatorname{Assh}(\widehat{R})$. This implies $\operatorname{Ass}(\widehat{R}[X]) = \operatorname{Assh}(\widehat{R}[X])$. Further, passing to $\widehat{R}(X) := \widehat{R}[X]_{\widehat{\mathfrak{m}}\widehat{R}[X]}$, we obtain a ring with infinite residue field with $\operatorname{Ass}(\widehat{R}(X)) = \operatorname{Assh}(\widehat{R}(X))$. Since $\widehat{R}$ is a homomorphic image of Cohen-Macaulay ring by Cohen Structure Theorem, it follows that $\widehat{R}(X)$ is so. Now by Proposition 2.3.4, we get $\widehat{R}(X)$ is unmixed local ring with infinite residue field. Note that the properties like depth, dimension and Hilbert coefficients are preserved while passing from R to $R(X)$ for any local ring R. Therefore while dealing with Hilbert coefficients in unmixed local ring, we may assume without loss of generality that the ring has infinite residue field.

3

The first Hilbert coefficient and Cohen-Macaulayness

In this chapter, we relate the properties of the Hilbert coefficients $g_i^K(Q)$ with Cohen-Macaulayness of R. If R is C-M and $Q \subseteq R$ is a parameter ideal, then it is well known that $e_0(Q) = \lambda_R(R/Q)$ and $e_i(Q) = 0$ for $1 \leq i \leq d$. Vasconcelos conjectured that for every parameter ideal Q, $e_1(Q) < 0$ if and only if R is not Cohen-Macaulay. We recall the following result of Ghezzi et al. which settled the above conjecture, also known as the negativity conjecture.

Theorem 3.0.5. [GGH+10, Theorem 2.1] *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and Q a parameter ideal of R. Then the following statements are equivalent:*

(a) R is Cohen-Macaulay;

(b) R is unmixed and $e_1(Q) = 0$;

(c) R is unmixed and $e_1(Q) \geq 0$.

In Section 3.1, we calculate the values of $g_i^K(Q)$ for a parameter ideal $Q \subseteq K$ to be equal to $(-1)^i\lambda_R(R/K)$ for $1 \leq i \leq d$ in a Cohen-Macaulay local ring. Along the lines of Vasconcelos' conjecture, we ask the following question in Section 3.2: if R is unmixed local ring of dimension $d > 0$ and $g_1^K(Q) \geq -\lambda_R(R/K)$ for some parameter ideal $Q \subseteq K$ then is R Cohen-Macaulay? We give partial solution to this problem. The main result of this chapter, which is a generalization of Theorem 3.0.5, provides a necessary and sufficient condition for the ring to be Cohen-Macaulay in terms of $g_0^K(Q)$ and $g_1^K(Q)$.

3.1 Hilbert coefficients in Cohen-Macaulay rings

Suppose R is Cohen-Macaulay and $Q \subseteq K$ is a parameter ideal. Then $g_0^K(Q) = e_0(Q) = \lambda_R(R/Q)$ from (2.10). We now determine the other coefficients. Our result is an application of the following result of Goto et al. in which they have characterized the Buchsbaumness of a finite R-module M in terms of the first Hilbert coefficient $e_1(Q, M)$.

Theorem 3.1.1. [GGH+15, Theorem 5.3] *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a generalized Cohen-Macaulay R-module of dimension $r \geq 2$ and $\operatorname{depth}_R M > 0$. Let Q be a parameter ideal for M. Then the following statements are equivalent:*

(a) Q is a standard parameter ideal for M;

(b) $e_1(Q, M) = -\sum_{i=1}^{r-1} \binom{r-2}{i-1} \lambda_R(\mathrm{H}_m^i(M))$.

Theorem 3.1.2. *Let R be a Cohen-Macaulay local ring of dimension $d > 0$ and K an $\mathfrak{m}$-primary ideal of R. Let $Q \subseteq K$ be a parameter ideal of R. Then $g_0^K(Q) = \lambda_R(R/Q)$ and for $1 \leq i \leq d$,*

$$g_i^K(Q) = (-1)^i \lambda_R(R/K).$$

Furthermore, for all $n \geq 0$

$$H_K(Q, n) = P_K(Q, n).$$

Proof. We may assume that R has infinite residue field and $Q = (x_1, \ldots, x_d)$ where $x_1^*, \ldots, x_d^*$ (resp. $x_1^o, \ldots, x_d^o$) is a superficial sequence in $G(Q)$ (resp. $F_K(Q)$). R is Cohen-Macaulay implies $x_1, \ldots, x_d$ is a regular sequence in R. We first show that $g_1^K(Q) = -\lambda_R(R/K)$ by induction on d. Let $d = 1$. Since K is a Cohen-Macaulay R-module, $e_1(Q, K) = 0$. Thus $g_1^K(Q) = e_1(Q, K) - \lambda_R(R/K) = -\lambda_R(R/K)$ by (2.9). Let $d > 1$ and $R_{d-1} = R/(x_1, \ldots, x_{d-1})$. Then by Proposition 2.2.9 and induction hypothesis, $g_1^K(Q) = g_1^{KR_{d-1}}(QR_{d-1}) = -\lambda_R(R/K)$.

Suppose $d \geq 2$. Then Remark 2.1.8 along with (2.6) and (2.7) imply that K is a generalized Cohen-Macaulay R-module of dimension d with $\mathrm{H}_m^i(K) \cong \mathrm{H}_m^i(R) = 0$ for $2 \leq i \leq d-1$ and $\mathrm{H}_m^1(K) \cong R/K$. Further (2.9) gives that,

$$\begin{aligned} e_1(Q, K) = g_1^K(Q) &= -\lambda_R(R/K) \\ &= -\sum_{i=1}^{d-1} \binom{d-2}{i-1} \lambda_R(\mathrm{H}_m^i(K)). \end{aligned}$$

Therefore Q is a standard parameter ideal for K due to Theorem 3.1.1. By Corollary 2.1.12, we have

$$\begin{aligned}
\lambda_R(R/KQ^{n+1}) &= \lambda_R(R/K) + \lambda_R(K/KQ^{n+1}) \\
&= \lambda_R(R/K) + \binom{n+d}{d} e_0(Q,K) \\
&\quad + \sum_{i=1}^{d}\sum_{j=0}^{d-i}\binom{n+d-i}{d-i}\binom{d-i-1}{j-1}\lambda_R(\mathrm{H}^j_m(K)) \\
&= \binom{n+d}{d} e_0(Q) + \sum_{i=1}^{d-1}\binom{n+d-i}{d-i}\lambda_R(\mathrm{H}^1_m(K)) + \lambda_R(R/K) \\
&= \binom{n+d}{d} e_0(Q) + \sum_{i=1}^{d}\binom{n+d-i}{d-i}\lambda_R(R/K) \\
&= P_K(Q,n)
\end{aligned}$$

for all $n \geq 0$. It follows that $g_i^K(Q) = (-1)^i \lambda_R(R/K)$ for all $1 \leq i \leq d$. □

Corollary 3.1.3. *Let R be a Noetherian local ring of dimension $d > 0$ and K an $\mathfrak{m}$-primary ideal of R. Let $Q \subseteq K$ be a parameter ideal of R. Suppose $R/\mathrm{H}^0_{\mathfrak{m}}(R)$ is Cohen-Macaulay. Then $g_0^K(Q) = \lambda_R(R/(Q + \mathrm{H}^0_m(R)))$ and*

$$g_i^K(Q) = \begin{cases} (-1)^i \lambda_R(R/(K + \mathrm{H}^0_m(R))) & \text{if } 1 \leq i \leq d-1 \\ (-1)^d \Big(\lambda_R(R/(K + \mathrm{H}^0_m(R))) + \lambda_R(\mathrm{H}^0_m(R))\Big) & \text{if } i = d. \end{cases}$$

Proof. Let $W = \mathrm{H}^0_m(R)$ and $R' = R/\mathrm{H}^0_m(R)$. Consider the following exact sequence.

$$0 \longrightarrow W/(KQ^n \cap W) \longrightarrow R/KQ^n \longrightarrow R'/KQ^nR' \longrightarrow 0.$$

Since $KQ^n \cap W \subseteq Q^n \cap W \subseteq Q^{n-k}W = 0$ for some integer k and for $n \gg 0$ by Artin-Rees lemma, we get for $n \gg 0$,

$$\lambda_R(R/KQ^n) = \lambda_R(R'/KQ^nR') + \lambda_R(W).$$

This implies

$$P_K(Q,n) = P_{KR'}(QR',n) + \lambda_R(W). \tag{3.1}$$

On comparing the coefficients, we get that

$$g_i^K(Q) = \begin{cases} g_i^{KR'}(QR') & \text{if } 0 \leq i \leq d-1, \\ g_d^{KR'}(QR') + (-1)^d \lambda_R(W) & \text{if } i = d. \end{cases} \tag{3.2}$$

Now using Theorem 3.1.2, we get the result. □

Since $R/\operatorname{H}^0_m(R)$ is Cohen-Macaulay for one dimensional local ring R, we immediately get the following corollary.

Corollary 3.1.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension one and K an $\mathfrak{m}$-primary ideal. Let Q be a parameter ideal of R. Then $g_0^K(Q) = \lambda_R(R/(Q + \operatorname{H}^0_m(R))$ and $g_1^K(Q) = -\lambda_R(R/K) - \lambda_R(H^0_{\mathfrak{m}}(K))$.*

Proof. It is enough to see that by Corollary 3.1.3,

$$\begin{aligned} g_1^K(Q) &= -\big(\lambda_R(R/(K+W)) + \lambda_R(W)\big) \\ &= -\lambda_R(R/K) + \lambda_R(W/(K \cap W)) - \lambda_R(W) \\ &= -\lambda_R(R/K) + \lambda_R(\operatorname{H}^0_m(R)/\operatorname{H}^0_m(K)) - \lambda_R(\operatorname{H}^0_m(R)) \\ &= -\lambda_R(R/K) - \lambda_R(\operatorname{H}^0_m(K)). \end{aligned}$$

□

Note that (3.2) holds without R' being Cohen-Macaulay. We recall this relation at later parts of the thesis again. In the following example, we see that $g_i^K(Q)$ may attain the value $(-1)^i\lambda_R(R/K)$ even if Q is not a parameter ideal. In order to do the calculation, let us recall the following useful consequence of a version of Huneke's fundamental lemma for the Hilbert function $\lambda_R(R/KQ^n)$ given by Jayanthan and Verma in [JV05a].

Theorem 3.1.5. [JV05a, Corollary 3.3] *Let $(R, \mathfrak{m})$ be a two dimensional Cohen-Macaulay local ring and K be an $\mathfrak{m}$-primary ideal. Let $I \subseteq K$ be an $\mathfrak{m}$-primary ideal* *and $J = (x, y)$ be a minimal reduction of I. Set*

$$v_n = \begin{cases} e_0(I) & \text{if } n = 0, \\ e_0(I) - \lambda_R(R/KI) + \lambda_R(R/K) & \text{if } n = 1, \\ \lambda_R(KI^n/KJI^{n-1}) - \lambda_R((KI^{n-1} : J)/KI^{n-2}) & \text{if } n \geq 2. \end{cases}$$

Then $g_1^K(I) = \sum_{n\geq 1} v_n$ and $g_2^K(I) = \sum_{n\geq 1}(n-1)v_n + \lambda_R(R/K)$.

Example 3.1.6. *Let $R = k[[x, y]]$ be a power series ring. Let $I = (x^3, x^2y, y^3)$ and $K = \mathfrak{m}^2$. Then $J = (x^3, y^3)$ is a minimal reduction of I. Theorem 3.1.5 yields that*

$$P_K(I, n) = 9\binom{n+1}{2} + 3n + 3.$$

So, we see that $\lambda_R(R/K) = 3 = (-1)^i g_i^K(I)$ for $i = 1, 2$.

We close this section by providing an upper bound on $g_1^K(Q)$ in the rings of depth at least $d - 1$.

Proposition 3.1.7. *Let R be a Noetherian local ring of dimension $d > 0$. Let Q and K be as in Theorem 3.1.2. Suppose* $\operatorname{depth} R \geq d - 1$. *Then $g_1^K(Q) \leq -\lambda_R(R/K)$.*

Proof. We may assume that R has infinite residue field and $Q = (x_1, \ldots, x_d)$ where $x_1^*, \ldots, x_d^*$ (resp. $x_1^o, \ldots, x_d^o$) is a superficial sequence in $G(Q)$ (resp. $F_K(Q)$). We may also assume that $x_1, \ldots, x_{d-1}$ is a regular sequence in R. Therefore in view of the Proposition 2.2.9, it is enough to prove the result for $d = 1$ in which case, $g_1^K(Q) = -\lambda_R(\mathrm{H}_m^0(K)) - \lambda_R(R/K) \leq -\lambda_R(R/K)$ by Corollary 3.1.4. □

Remark 3.1.8. *We highlight that under the hypothesis of Theorem 3.1.5, the fiber cone $F_K(Q)$ is Cohen-Macaulay with the fiber coefficients $f_i^K(Q) = 0$ for $1 \leq i \leq d - 1$ by* (2.11).

3.2 A characterization of Cohen-Macaulayness

In this section, we study the problem briefly mentioned at the beginning i.e. if R is an unmixed local ring of positive dimension, then does $g_1^K(Q) \geq -\lambda_R(R/K)$ for some parameter ideal Q imply R is Cohen-Macaulay? We have an affirmative answer for the most natural choice of K i.e. the maximal ideal $\mathfrak{m}$.

Remark 3.2.1. *Suppose* $K = \mathfrak{m}$ *and* $g_1^{\mathfrak{m}}(Q) \geq -\lambda_R(R/\mathfrak{m}) = -1$ *for some parameter ideal* Q. *From* (2.10) *and* (2.11), *we get that* $e_1(Q) - g_1^K(Q) = f_0^K(Q) \geq 1$. *Hence* $e_1(Q) \geq 0$ *which implies that* R *is Cohen-Macaulay by Theorem 3.0.5.*

However we may not expect it to be true in general as evidenced by the following example, worked out in [Mcc13],

Example 3.2.2. [Mcc13, Example 3.8] *Let* $R = k[[x^5, xy^4, x^4y, y^5]] \cong k[[t_1, t_2, t_3, t_4]]/J$, *where* $J = (t_2t_3 - t_1t_4,\ t_2^4 - t_3t_4^3,\ t_1t_2^3 - t_3^2t_4^2,\ t_1^2t_2^2 - t_3^3t_4,\ t_1^3t_2 - t_3^4,\ t_3^5 - t_1^4t_4)$. *Then* $\dim R = 2$. *Let* $K = Q = (x^5, y^5)$. *We have*

$$P(Q, n) = 5\binom{n+1}{2} + 2n.$$

By (2.12), $g_1^K(Q) = e_1(Q) - e_0(Q) = -7 \geq -\lambda_R(R/K)$ *but* $\operatorname{depth} R = 1$.

For an arbitrary K, we prove in Theorem 3.2.8 that an unmixed local ring is Cohen-Macaulay if and only if $g_1^K(Q) \geq -\lambda_R(R/K) + (\lambda_R(R/Q) - g_0^K(Q))$. Observe that the quantity at right hand side is larger than $-\lambda_R(R/K)$. Hence Theorem 3.2.8 provides a partial solution to the problem mentioned earlier. The most natural approach is to devise the proof by method of induction. We may reduce our problem to the lower dimensional cases. To be more precise in this respect, it is desired that the hypothesis of the problem remains valid after reduction by superficial elements. In order to achieve this in our case, we need a superficial sequence $(x_1, \ldots, x_d)$ which is a generating set of the parameter ideal Q of an unmixed local ring R such that the reduced ring $R/(x_1, \ldots, x_i)$ remains unmixed. For that purpose, we recall the following results from [GN01].

Lemma 3.2.3. [GN01, Lemma 3.1] *Let* $(R, \mathfrak{m})$ *be a complete local ring. Let* K_R *be the canonical module of* R *and* $S = \operatorname{Hom}_R(K_R, K_R)$. *Let* $\phi : R \longrightarrow S$ *be the canonical map i.e* $\phi(r)(x) = rx$ *for* $r \in R$ *and* $x \in K_R$. *Suppose* $\operatorname{Ass}(R) \subseteq \operatorname{Assh}(R) \cup \{\mathfrak{m}\}$. *Then*

1. $\ker(\phi)$ *has finite length.*

2. $H_{\mathfrak{m}}^1(R)$ *is isomorphic to* $\mathrm{H}_m^0(\operatorname{Coker}(\phi))$. *In particular,* $H_{\mathfrak{m}}^1(R)$ *has finite length.*

Lemma 3.2.4. [GN01, Lemma 3.2] *Let* R *be a homomorphic image of a Cohen-Macaulay local ring. Suppose* $\operatorname{Ass}(R) \subseteq \operatorname{Assh}(R) \cup \{\mathfrak{m}\}$. *Then the set*

$$\mathscr{F} := \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{ht}_R \mathfrak{p} > 1 = \operatorname{depth} R_{\mathfrak{p}},\ \mathfrak{p} \neq \mathfrak{m}\}$$

is finite.

The next proposition and its proof are same as [GN01, Proposition 3.3] except the fact that we choose $x_1, \ldots, x_d \in Q$ such that $x_1^o, \ldots, x_d^o$ is a superficial sequence in $F_K(Q)$ in addition to $x_1^*, \ldots, x_d^*$ being superficial in $G(Q)$.

Proposition 3.2.5. [GN01, Proposition 3.3] *Let $(R, \mathfrak{m})$ be a homomorphic image of a Cohen-Macaulay local ring of dimension d and assume that $\operatorname{Ass}(R) \subseteq \operatorname{Assh}(R) \cup \{\mathfrak{m}\}$. Let K be an $\mathfrak{m}$-primary ideal and $Q \subseteq K$ a parameter ideal. Then there exists a system of generators $x_1, \ldots, x_d$ of Q such that such that $x_1^*, \ldots, x_d^*$ (resp. $x_1^o, \ldots, x_d^o$) is superficial in $G(Q)$ (resp. $F_K(Q)$) and $\operatorname{Ass}(R/Q_i) \subseteq \operatorname{Assh}(R/Q_i) \cup \{\mathfrak{m}\}$, where $Q_i = (x_1, \ldots, x_i)$ for $1 \le i \le d$ and $Q_0 = (0)$.*

Proof. If $d = 0$, the result obviously holds. Assume $d > 0$. By repeating the process, it is enough to prove the assertion for $i = 1$. Let $\mathscr{F}$ be as in Lemma 3.2.4. Now by Lemma 2.2.8 and prime avoidance , choose $x_1 \in Q$ such that x_1^* (resp. x_1^o) is superficial in $G(Q)$ (resp. $F_K(Q)$) and

$$x_1 \in Q \setminus \Big(\mathfrak{m}Q \bigcup \Big(\bigcup_{\mathfrak{p} \in \operatorname{Assh}(R)} \mathfrak{p} \Big) \bigcup \Big(\bigcup_{\mathfrak{p} \in \mathscr{F}} \mathfrak{p} \Big)\Big).$$

Let $\mathfrak{p} \in \operatorname{Ass}(R/(x_1))$, $\mathfrak{p} \neq \mathfrak{m}$. We show that $\dim R/\mathfrak{p} = \dim R/(x_1)$. Since $\mathfrak{p} \in \operatorname{Ass}(R/(x_1))$, $\operatorname{depth}(R_\mathfrak{p}/x_1 R_\mathfrak{p}) = 0$. However $\operatorname{depth} R_\mathfrak{p} > 0$ as $\mathfrak{p} \notin \operatorname{Ass}(R)$ by choice of x_1. Note that x_1 is a non-zero-divisor in $R_\mathfrak{p}$. This implies that $\operatorname{depth} R_\mathfrak{p} = 1$ and $\operatorname{ht}_R \mathfrak{p} = 1$ as $\mathfrak{p} \notin \mathscr{F}$. Since R is catenary and equidimensional, we get $\dim R/\mathfrak{p} = \dim R - \operatorname{ht}_R \mathfrak{p} = d - 1 = \dim R/(x_1)$. It follows that $\mathfrak{p} \in \operatorname{Assh}(R/(x_1))$. □

The following results are the key steps towards Theorem 3.2.8.

Proposition 3.2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension one and K an $\mathfrak{m}$-primary ideal of R. Let $Q \subseteq K$ be a parameter ideal. Suppose $g_0^K(Q) + g_1^K(Q) \ge -\lambda_R(R/K) + \lambda_R(R/Q)$. Then R is Cohen-Macaulay.*

Proof. Set $W = H^0_\mathfrak{m}(R)$ and $R' = R/W$. Then R' is a Cohen-Macaulay local ring of dimension one. Therefore using Corollary 3.1.3, we get

$$\begin{aligned} -\lambda_R(R/K) + \lambda_R(R/Q) &\le g_0^K(Q) + g_1^K(Q) \\ &= \lambda_R(R/(Q+W)) - \lambda_R(R/(K+W)) - \lambda_R(W) \\ &= \lambda_R((K+W)/(Q+W)) - \lambda_R(W) \end{aligned}$$

$$= \lambda_R(K/((Q+W)\cap K)) - \lambda_R(W)$$

$$\leq \lambda_R(K/Q) - \lambda_R(W)$$

$$\leq -\lambda_R(R/K) + \lambda_R(R/Q) - \lambda_R(W).$$

Hence $\lambda_R(W) \leq 0$ which implies that $W = 0$. Thus R is Cohen-Macaulay. □

Lemma 3.2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and K an $\mathfrak{m}$-primary ideal. Let Q be a parameter ideal of R. Suppose $U = U_R(0) \neq 0$ and $S = R/U$. Then the following assertions hold.*

1. $\dim U < \dim R$.
2. *We have* $g_0^K(Q) = g_0^{KS}(QS)$ *and*

$$g_1^K(Q) = \begin{cases} g_1^{KS}(QS) & \text{if } \dim U \leq d-2, \\ g_1^{KS}(QS) - s_0 & \text{if } \dim U = d-1 \end{cases}$$

   where s_0 is the multiplicity of the module $\bigoplus_{n\geq 0} U/(KQ^{n+1}\cap U)$.

3. $g_1^K(Q) \leq g_1^{KS}(QS)$ *with equality if and only if* $\dim U \leq d-2$.

Proof. (1) Let $(0) = \bigcap_{p\in \mathrm{Ass}(R)} Q(p)$ be a primary decomposition of (0) and $U = U_R(0) = \bigcap_{p\in \mathrm{Assh}(R)} Q(p)$. For all $p \in \mathrm{Ass}(R)$ and $p' \in \mathrm{Assh}(R)$ with $p \neq p'$, we have $Q(p)R_{p'} = R_{p'}$. Hence for all $p' \in \mathrm{Assh}(R)$, we get that $(0) = \bigcap_{p\in \mathrm{Ass}(R)} Q(p)R_{p'} = Q(p')R_{p'}$. Thus $UR_{p'} = \bigcap_{p\in \mathrm{Assh}(R)} Q(p)R_{p'} = Q(p')R_{p'} = (0)$. Hence $\mathrm{Assh}(R) \cap \mathrm{Supp}_R(U) = \phi$. Therefore $\dim U < \dim R$.

(2) Considering the short exact sequence

$$0 \longrightarrow U/(KQ^n \cap U) \longrightarrow R/KQ^n \longrightarrow S/KQ^nS \longrightarrow 0,$$

we get that

$$\lambda_R(R/KQ^n) = \lambda_R(S/KQ^nS) + \lambda_R(U/(KQ^n\cap U)) \text{ for all } n \in \mathbb{Z}. \tag{3.3}$$

Hence $\lambda_R(U/(KQ^n \cap U))$ agrees with a polynomial, say $T(n)$ for $n \gg 0$. We write

$$T(n) = s_0\binom{n+t}{t} - s_1\binom{n+t-1}{t-1} + \ldots + (-1)^t s_t \tag{3.4}$$

for some $t \geq 0$ and $s_i \in \mathbb{Z}$ for $0 \leq i \leq t$. We claim that $t = \dim U$. By the Artin-Rees lemma, there exists an integer k such that for all $n \gg 0$, $KQ^n \cap U \subseteq Q^n \cap U = Q^{n-k}(Q^k \cap U) \subseteq Q^{n-k}U$. Hence $\lambda_R(U/Q^{n-k}U) \leq \lambda_R(U/(KQ^n \cap U))$ for all $n \gg 0$ which implies $t \geq \dim U$. On the other hand, Since $\lambda_R(U/KQ^nU) = \lambda_R(U/Q^nU) + \lambda_R(Q^nU/KQ^nU)$, we see that $\lambda_R(U/KQ^nU)$ coincides with a polynomial of degree equals $\dim U$ for all $n \gg 0$. Therefore $\lambda_R(U/(KQ^n \cap U)) \leq \lambda_R(U/KQ^nU)$ for all $n \in \mathbb{Z}$ implies that $t = \dim U$.

From (3.3), we get

$$P_K(Q, n) = P_{KS}(QS, n) + T(n) \text{ for all } n \in \mathbb{Z}. \tag{3.5}$$

By comparing the coefficients of both sides of (3.5) and using (3.4), we get the result.

(3) follows from (2). □

We now prove the main result of this chapter which provides a characterization for the Cohen-Macaulay local rings.

Theorem 3.2.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and K an $\mathfrak{m}$-primary ideal. Let $Q \subseteq K$ be a parameter ideal of R. Then the following statements are equivalent:*

(a) R is Cohen-Macaulay;

(b) R is unmixed and $g_0^K(Q) + g_1^K(Q) = -\lambda_R(R/K) + \lambda_R(R/Q)$;

(c) R is unmixed and $g_0^K(Q) + g_1^K(Q) \geq -\lambda_R(R/K) + \lambda_R(R/Q)$;

(d) R is unmixed and $f_0^K(Q) \leq \lambda_R(R/K) + e_1(Q) + e_0(Q) - \lambda_R(R/Q)$;

(e) R is unmixed and $f_0^K(Q) = \lambda_R(R/K) + e_1(Q) + e_0(Q) - \lambda_R(R/Q)$.

Proof. Using (2.10) and (2.11), we get that (e) ⇔ (b) and (d) ⇔ (c). Hence it suffices to prove the equivalence of the first three statements.
(a) ⇒ (b) Follows from Theorem 3.1.2.
(b) ⇒ (c) This is Clear.
(c) ⇒ (a) We apply induction on d. The result is obvious for $d = 1$. Let $d \geq 2$.

We may assume that R is complete with infinite residue field. Suppose $d = 2$. Then we may assume that $Q = (x_1, x_2)$ such that x_1^o is superficial in $F_K(Q)$. Since

R is unmixed, we can choose x_1 to be a non-zero-divisor on R. Let $S = R/(x_1)$. Then $Q/(x_1)$ is a parameter ideal of S. By Proposition 2.2.9, $g_0^{KS}(QS) = g_0^K(Q)$ and $g_1^{KS}(QS) = g_1^K(Q)$. Hence

$$g_0^{KS}(QS) + g_1^{KS}(QS) \geq -\lambda_R(R/K) + \lambda_R(R/Q) = -\lambda_R(S/KS) + \lambda_R(S/QS).$$

Therefore, by Proposition 3.2.6, S is Cohen-Macaulay which implies that R is Cohen-Macaulay.

Let $d \geq 3$. By Proposition 3.2.5, there exists a system of generators $x_1, \ldots, x_d$ of Q such that x_1^o is superficial in $F_K(Q)$ and $\operatorname{Ass}(R/(x_1)) \subseteq \operatorname{Assh}(R/(x_1)) \cup \{\mathfrak{m}\}$. Let $S = R/(x_1)$ and $\bar{S} = S/U_S(0)$. Then $\bar{S}$ is an unmixed local ring of dimension $d-1$ by Proposition 2.3.4 and $Q\bar{S}$ is a parameter ideal contained in $K\bar{S}$. Since $\dim U_S(0) = 0$, we have $g_i^{K\bar{S}}(Q\bar{S}) = g_i^{KS}(QS)$ for $i = 0, 1$ due to Lemma 3.2.7(2). Therefore,

$$\begin{aligned}
g_0^{K\bar{S}}(Q\bar{S}) + g_1^{K\bar{S}}(Q\bar{S}) &= g_0^K(Q) + g_1^K(Q) \\
&\geq -\lambda_R(R/K) + \lambda_R(R/Q) \\
&= -\lambda_R(S/KS) + \lambda_R(S/QS) \\
&\geq \lambda_R(KS/((QS + U_S(0)) \cap KS)) \\
&= \lambda_R((KS + U_S(0))/(QS + U_S(0))) \\
&= -\lambda_R(S/(KS + U_S(0))) + \lambda_R(S/(QS + U_S(0))) \\
&= -\lambda_R(\bar{S}/K\bar{S}) + \lambda_R(\bar{S}/Q\bar{S}).
\end{aligned}$$

Hence by induction hypothesis, $\bar{S}$ is Cohen-Macaulay. Thus $H_{\mathfrak{m}}^i(\bar{S}) = 0$ for $0 \leq i \leq d-2$. The exact sequence

$$0 \longrightarrow U_S(0) \longrightarrow S \longrightarrow \bar{S} \longrightarrow 0$$

gives the following long exact sequence

$$\cdots \longrightarrow H_{\mathfrak{m}}^i(U_S(0)) \longrightarrow H_{\mathfrak{m}}^i(S) \longrightarrow H_{\mathfrak{m}}^i(\bar{S}) \longrightarrow \cdots.$$

Since $U_S(0)$ is Artinian, $H_{\mathfrak{m}}^0(U_S(0)) = U_S(0)$ and $H_{\mathfrak{m}}^i(U_S(0)) = 0$ for all $i \geq 1$. Therefore $H_{\mathfrak{m}}^0(S) = U_S(0)$ and $H_{\mathfrak{m}}^i(S) = 0$ for $1 \leq i \leq d-2$.

Now considering the exact sequence

$$0 \longrightarrow R \xrightarrow{x_1} R \longrightarrow S \longrightarrow 0,$$

we get the long exact sequence

$$\cdots \longrightarrow H^{i-1}_{\mathfrak{m}}(S) \longrightarrow H^i_{\mathfrak{m}}(R) \xrightarrow{x_1} H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(S) \longrightarrow \cdots .$$

This implies that the map $H^1_{\mathfrak{m}}(R) \xrightarrow{x_1} H^1_{\mathfrak{m}}(R)$ is surjective and $H^i_{\mathfrak{m}}(R) \xrightarrow{x_1} H^i_{\mathfrak{m}}(R)$ is injective for $2 \leq i \leq d-1$. Thus $H^1_{\mathfrak{m}}(R) = x_1 H^1_{\mathfrak{m}}(R)$. Since $H^1_{\mathfrak{m}}(R)$ is finitely generated by Lemma 3.2.3, using Nakayama's Lemma we get that $H^1_{\mathfrak{m}}(R) = 0$. Since $H^i_{\mathfrak{m}}(R)$ is $\mathfrak{m}$-torsion, the injectivity of the map $H^i_{\mathfrak{m}}(R) \xrightarrow{x_1} H^i_{\mathfrak{m}}(R)$ gives that $H^i_{\mathfrak{m}}(R) = 0$ for $2 \leq i \leq d-1$. Therefore R is Cohen-Macaulay. □

We now recover Theorem 3.0.5 which settles the negativity conjecture on $e_1(Q)$. As mentioned already, Theorem 3.2.8 is more general in a sense that negativity conjecture and Theorem 3.0.5 are restricted to the case when $K = R$ and they can be recovered.

Corollary 3.2.9. *Let R be a Noetherian local ring of dimension $d > 0$ and Q a parameter ideal of R. Then the following statements are equivalent:*

(a) R is Cohen-Macaulay;

(b) R is unmixed and $e_1(Q) = 0$;

(c) R is unmixed and $e_1(Q) \geq 0$.

Proof. Let $K = Q$ in Theorem 3.2.8. From (2.12), $g^K_0(Q) + g^K_1(Q) = e_1(Q)$. Hence the result follows from Theorem 3.2.8. □

It is of natural interest to seek bounds on the Hilbert coefficients. Observe that $g^K_1(Q) \leq e_1(Q, K) \leq 0$ holds in general. Indeed, when $\operatorname{depth} R \geq d-1$, we obtained a better uniform upper bound on $g^K_1(Q)$ in Proposition 3.1.7 i.e. $g^K_1(Q) \leq -\lambda_R(R/K)$. The following result is useful in this context although it does not provide uniform bound. It also allows us to recover the fact that $e_1(Q) \leq 0$ in Corollary 3.2.11.

Corollary 3.2.10. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and K an $\mathfrak{m}$-primary ideal. Let $Q \subseteq K$ be a parameter ideal of R. Then*

1. *(a) $g^K_0(Q) + g^K_1(Q) \leq -\lambda_R(R/K) + \lambda_R(R/Q)$.*

   (b) $f^K_0(Q) \geq \lambda_R(R/K) + e_1(Q) + e_0(Q) - \lambda_R(R/Q)$.

2. *Suppose* $\operatorname{depth} R = d-1$. *Then*

(a) $g_0^K(Q) + g_1^K(Q) < -\lambda_R(R/K) + \lambda_R(R/Q)$.

(b) $f_0^K(Q) > \lambda_R(R/K) + e_1(Q) + e_0(Q) - \lambda_R(R/Q)$.

Proof. In view of (2.10) and (2.11), it suffices to prove 1(a) and 2(a).
1(a) We may assume that R is complete. Let $d = 1$. Since $g_0^K(Q) = e_0(Q) \leq \lambda_R(R/Q)$, using Corollary 3.1.4, we get that $g_0^K(Q) + g_1^K(Q) \leq -\lambda_R(R/K) + \lambda_R(R/Q)$. Suppose $d \geq 2$. Set $S = R/U_R(0)$. Then S is an unmixed local ring and QS is a parameter ideal of S. Hence

$$\begin{aligned} g_0^K(Q) + g_1^K(Q) &\leq g_0^{KS}(QS) + g_1^K(QS) && \text{[by Lemma 3.2.7(2)]} \\ &\leq -\lambda_R(S/KS) + \lambda_R(S/QS) && \text{[from Theorem 3.2.8]} \\ &\leq -\lambda_R(R/K) + \lambda_R(R/Q). \end{aligned}$$

2(a) Let $d = 1$. Then $\operatorname{depth}(R) = 0$ implies that $g_0^K(Q) = e_0(Q) < \lambda_R(R/Q)$. Hence, by Corollary 3.1.4, $g_0^K(Q) + g_1^K(Q) < -\lambda_R(R/K) + \lambda_R(R/Q)$. Suppose $d \geq 2$. Let $Q = (x_1, \ldots, x_d)$ such that $x_1^o, \ldots, x_{d-1}^o$ is a superficial sequence in $F_K(Q)$. Since $\operatorname{depth}(R) = d - 1$, we may assume that $x_1, \ldots, x_{d-1}$ is a regular sequence in R. Let $R_{d-1} = R/(x_1, \ldots, x_{d-1})$. Then, by Proposition 2.2.9, $g_0^K(Q) + g_1^K(Q) = g_0^{KR_{d-1}}(QR_{d-1}) + g_1^{KR_{d-1}}(QR_{d-1}) < -\lambda_R(R/K) + \lambda_R(R/Q)$ by induction hypothesis. □

We obtain below the negativity of $e_1(Q)$. This can also be found in [GGH+10] and [MSV11].

Corollary 3.2.11. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and Q a parameter ideal of R. Then*

1. $e_1(Q) \leq 0$
2. *If* $\operatorname{depth} R = d - 1$, *then* $e_1(Q) < 0$.

Proof. It follows from letting $K = Q$ in Corollary 3.2.10 and using (2.12). □

In the following example, we see that the inequality of Corollary 3.2.10 is strict although $\operatorname{depth} R \neq d - 1$.

Example 3.2.12. *Let* $S = k[[X,Y,Z]]$ *be a power series ring over a field* k *and* $I = (YZ, X^2Y, Y^3)$. *Consider the ring* $R = S/I = k[[x,y,z]]$. *Then* $\dim R = 2$ *and* $\operatorname{depth} R = 0$. *Let* $Q = (x, z^2)$ *and* $K = (x, y, z^2)$. *Notice that* $\mathrm{H}^0_m(R) = (Y)/I$, *hence* $R' = R/\mathrm{H}^0_m(R) = k[[X,Z]]$ *which is Cohen-Macaulay. By* (3.2) *and Theorem 3.1.2, we get* $g_0^K(Q) + g_1^K(Q) = g_0^{KR'}(QR') + g_1^{KR'}(QR') = \lambda_R(R'/QR') - \lambda_R(R'/KR') = 2-2 = 0$ *whereas* $-\lambda_R(R/K) + \lambda_R(R/Q) = -2+4 = 2$.

We can mod out the unmixed component $U_R(0)$ and move into the ring $R/U_R(0)$ to obtain a more general sufficient condition for the ring to be Cohen-Macaulay.

Theorem 3.2.13. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d > 0$ *and* K *an* $\mathfrak{m}$*-primary ideal. Let* $Q \subseteq K$ *be a parameter ideal of* R. *Suppose* R *is a homomorphic image of a Cohen-Macaulay ring. Let* $U = U_R(0)$. *Then the following are equivalent:*

(a) $g_0^K(Q) + g_1^K(Q) = -\lambda_R(R/(K+U)) + \lambda_R(R/(Q+U))$;

(b) R/U *is Cohen-Macaulay and* $\dim U \le d-2$.

Proof. (a) $\Rightarrow$ (b) Let $S = R/U$. Then S is an unmixed local ring by Proposition 2.3.4. If $U = 0$, then R is unmixed. Hence $g_0^K(Q) + g_1^K(Q) = -\lambda_R(R/K) + \lambda_R(R/Q)$ implies that R is Cohen-Macaulay by Theorem 3.2.8. Suppose $U \neq 0$. First we show that $\dim U \le d-2$. By Lemma 3.2.7 (1), $\dim U \le d-1$. Suppose $\dim U = d-1$. Then $g_0^K(Q) = g_0^{KS}(QS)$ and $g_1^K(Q) = g_1^K(QS) - s_0$ for some $s_0 \ge 1$ by Lemma 3.2.7 (2). Therefore,

$$\begin{aligned} g_0^K(Q) + g_1^K(Q) &< g_0^{KS}(QS) + g_1^{KS}(QS) \\ &\le -\lambda_R(S/KS) + \lambda_R(S/QS) \qquad \text{[by Corollary 3.2.10]} \\ &= -\lambda_R(R/(K+U)) + \lambda_R(R/(Q+U)) \end{aligned}$$

which is a contradiction. Hence $\dim U \le d-2$. By Lemma 3.2.7 (2), $g_1^K(Q) = g_1^{KS}(QS)$. Hence $g_0^{KS}(QS) + g_1^{KS}(QS) = -\lambda_R(S/KS) + \lambda_R(S/QS)$. Therefore $S = R/U$ is Cohen-Macaulay by Theorem 3.2.8.

(b) $\Rightarrow$ (a) Let $S = R/U$. Since $\dim U \le d-2$, by Lemma 3.2.7(2), $g_1^K(Q) = g_1^{KS}(QS) = -\lambda_R(S/KS)$ where the last equality holds by Theorem 3.1.2. Therefore

$$g_0^K(Q) + g_1^K(Q) = \lambda_R(S/QS) - \lambda_R(S/KS)$$

$$= \lambda_R(R/(Q+U)) - \lambda_R(R/(K+U)).$$

□

Corollary 3.2.14. *Let R, Q, K and U be as in Theorem 3.2.13. Suppose*

$$g_i^K(Q) = (-1)^i(g_0^K(Q) - \lambda_R(R/(Q+U)) + \lambda_R(R/(K+U)))$$

for $1 \le i \le d$. Then R is Cohen-Macaulay.

Proof. Since $g_1^K(Q) = -(g_0^K(Q) - \lambda_R(R/(Q+U)) + \lambda_R(R/(K+U)))$, R/U is Cohen-Macaulay and $\dim U \le d-2$ by Theorem 3.2.13. Set $S = R/U$. By Lemma 3.2.7 (2), $g_0^K(Q) = g_0^{KS}(QS) = \lambda_R(S/QS) = \lambda_R(R/(Q+U))$. From Theorem 3.1.2, we have

$$g_i^{KS}(QS) = (-1)^i \lambda_R(S/KS) \text{ for } 1 \le i \le d.$$

Hence

$$g_i^K(Q) = (-1)^i \lambda_R(R/(K+U)) = (-1)^i \lambda_R(S/KS) \text{ for } 1 \le i \le d.$$

Therefore, for $n \gg 0$,

$$\lambda_R(S/KQ^nS) = \lambda_R(S/QS)\binom{n+d-1}{d} + \lambda_R(S/KS)\binom{n+d-2}{d-1} + \cdots + \lambda_R(S/KS),$$
$$\lambda_R(R/KQ^n) = \lambda_R(S/QS)\binom{n+d-1}{d} + \lambda_R(S/KS)\binom{n+d-2}{d-1} + \cdots + \lambda_R(S/KS).$$

Thus

$$\lambda_R(U/(KQ^n \cap U)) = \lambda_R(R/KQ^n) - \lambda_R(S/KQ^nS) = 0 \quad \text{for} \quad n \gg 0.$$

This implies that $U = 0$ and hence R is Cohen-Macaulay. □

4

Finiteness of the set of the first Hilbert coefficients

The objective of this chapter is to study the finiteness properties of various sets of the Hilbert coefficients $g_i^K(Q)$ relative to different properties of the ring. We set

$$\Lambda_i(M) := \{e_i(Q, M) \mid Q \text{ is a parameter ideal for } M\}.$$

For a generalized Cohen-Macaulay (resp. Buchsbaum) module M, the set $\Lambda_1(M)$ is known to have cardinality finite (resp. one), see Corollary 3.2.11 and (2.19). In [GGH+15] authors proved that finiteness (resp. cardinality one) of $\Lambda_1(M)$ characterizes the generalized Cohen-Macaulayness (resp. Buchsbaumness) of an unmixed module M. We recall their results for our use.

Theorem 4.0.15. [GGH+15, Theorem 4.5] *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension $r \geq 2$. Let $U = U_{\widehat{M}}(0)$ where $\widehat{M}$ denote the $\mathfrak{m}$-adic completion of M. Then the following conditions are equivalent:*

1. *$\Lambda_1(M)$ is a finite set;*
2. *$\dim_{\widehat{R}} U \leq r - 2$ and $\widehat{M}/U$ is a generalized Cohen-Macaulay $\widehat{R}$-module.*

When this is the case, one has

$$-\sum_{j=1}^{r-1} \binom{r-2}{j-1} \lambda_R(H_{\mathfrak{m}}^j(\widehat{M}/U)) \leq e_1(Q, M) \leq 0$$

for every parameter ideal Q for M.

Theorem 4.0.16. [GGH+15, Theorem 5.4] *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated unmixed R-module of dimension $r \geq 2$. Then the following conditions are equivalent:*

1. M is a Buchsbaum R-module;

2. $\Lambda_1(M)$ is a singleton set.

When this is the case, one has

$$e_1(Q,M) = -\sum_{j=1}^{r-1}\binom{r-2}{j-1}\lambda_R(H^j_{\mathfrak{m}}(M))$$

for every parameter ideal Q for M.

We put

$$\Lambda_i^K(R) := \{g_i^K(Q) \mid Q \text{ is a parameter ideal of } R\} \text{ and}$$

$$\delta_i^K(R) := \{g_i^K(Q) \mid Q \text{ is a parameter ideal of } R \text{ such that } Q \subseteq K\}$$

Note that $\delta_i^K(R) \subseteq \Lambda_i^K(R)$. In Section 4.1, we investigate the sets $\Lambda_1^K(R)$ for analogous properties. We prove that an unmixed local ring R is generalized Cohen-Macaulay if and only $\Lambda_1^K(R)$ (equivalently $\delta_1^K(R)$) is finite. Next, we prove that if R is unmixed and $|\Lambda_1^K(R)| = 1$ then R is Buchsbaum where as the converse holds true for $K = \mathfrak{m}$. We also provide an explanation for why the converse may not be true in general for an arbitrary $\mathfrak{m}$-primary ideal K and a parameter ideal Q.

In Section 4.2, we study the finiteness of the sets $\Lambda_i^K(R)$ for $1 \le i \le d$. Intuitively, considering the higher coefficients in our context should provide more information. Indeed we prove that R is generalized Cohen-Macaulay if and only if $\Lambda_i^K(R)$ (equivalently $\delta_i^K(R)$) are finite for all $1 \le i \le d-1$. For the coefficients $e_i(Q,R)$, S. Goto and K. Ozeki [GO11] proved that R is generalized Cohen-Macaulay if and only if $\Lambda_i(R)$ is finite for all $1 \le i \le d$ [GO11, Theorem 1.1]. We also improve upon their result and extend it to modules.

In Section 4.3 we consider the Hilbert coefficients $g_1^K(I)$ where I is an $\mathfrak{m}$-primary ideal. We set

$$\Delta_R(M) := \{e_1(I,M) \mid I \text{ is an } \mathfrak{m}\text{-primary ideal of } R\} \text{ and}$$

$$\Delta^K(R) := \{g_1^K(I) \mid I \text{ is an } \mathfrak{m}\text{-primary ideal of } R\}.$$

Recall that a local ring $(R,\mathfrak{m})$ is called analytically unramified if its $\mathfrak{m}$-adic completion is reduced. We prove that $\Delta^K(R)$ is finite if and only if $d = 1$ and $R/\operatorname{H}^0_{\mathfrak{m}}(R)$ is analytically unramified. A similar result for $\Delta_R(R)$ can be found in [KT15]. We also generalize the result of [KT15] for modules.

Since the coefficients $g_i^K(Q)$ and $e_i(Q, K)$ are related, see Remark 2.1.1, we may treat K as an R-module and use the known results on the finiteness properties of the sets of Hilbert coefficients $e_i(Q, M)$ of modules to study the similar properties of the sets of the coefficients $g_i^K(Q)$. To be able to use this method, we extend a number of results known for rings to modules.

4.1 The set $\Lambda_1^K(R)$

In this section we study the set $\Lambda_1^K(R)$. Recall that

$$\Lambda_1^K(R) = \{g_1^K(Q) \mid Q \text{ is a parameter ideal of } R\}.$$

We obtain necessary and sufficient conditions for the finiteness of $\Lambda_1^K(R)$ in an unmixed local ring. We also consider the problem when $g_1^K(Q)$ is independent of Q i.e. the set $\Lambda_1^K(R)$ is singleton. We see that for $K = \mathfrak{m}$, this property characterizes Buchsbaum local rings. We obtain partial results for arbitrary K.

Proposition 4.1.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and K an $\mathfrak{m}$-primary ideal of R.*

1. *Suppose R is generalized Cohen-Macaulay. Then the following assertions hold.*

   (a) *For any parameter ideal Q of R,*

   $$-\sum_{i=1}^{d-1} \binom{d-2}{i-1} \lambda_R(\mathrm{H}_\mathfrak{m}^i(R)) - \lambda_R(R/K) \leq g_1^K(Q) \leq 0.$$

   In particular, $\Lambda_1^K(R)$ is finite.

   (b) *If Q is a standard parameter ideal for K, then*

   $$g_i^K(Q) = \begin{cases} (-1)^i \sum_{j=0}^{d-i} \binom{d-i-1}{j-1} \lambda_R(\mathrm{H}_\mathfrak{m}^j(K)) & \text{if } 1 \leq i \leq d-1 \\ (-1)^d(\lambda_R(\mathrm{H}_\mathfrak{m}^0(K)) + \lambda_R(R/K)) & \text{if } i = d. \end{cases}$$

2. *Suppose R is an unmixed local ring. Then K is a generalized Cohen-Macaulay module (resp. Buchsbaum module) if and only if $|\Lambda_1^K(R)| < \infty$ (resp. $|\Lambda_1^K(R)| = 1$).*

Proof. (1) Since R is a generalized Cohen-Macaulay ring, by Remark 2.1.8, K is a generalized Cohen-Macaulay R-module. Hence, by Corollary 3.2.11 and (2.19), $-\sum_{i=1}^{d-1}\binom{d-2}{i-1}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(K)) \leq e_1(Q,K) \leq 0$. Using (2.6) and (2.7), we get that $-\lambda_R(\mathrm{H}^1_{\mathfrak{m}}(R)) - \lambda_R(R/K) \leq -\lambda_R(\mathrm{H}^1_{\mathfrak{m}}(K))$ and $\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) = \lambda_R(\mathrm{H}^i_{\mathfrak{m}}(K))$ for $2 \leq i \leq d-1$. Thus, $-\sum_{i=1}^{d-1}\binom{d-2}{i-1}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) - \lambda_R(R/K) \leq e_1(Q,K) \leq 0$. Since $e_1(Q,K) = g_1^K(Q)$ by (2.9), (1a) follows.

If Q is a standard parameter ideal for K then, by Corollary 2.1.12,

$$e_i(Q,K) = (-1)^i \sum_{j=0}^{d-i}\binom{d-i-1}{j-1}\lambda_R(\mathrm{H}^j_{\mathfrak{m}}(K))$$

for $1 \leq i \leq d$. Hence (1b) follows from (2.9) again.

(2) Since R is unmixed, K is an unmixed R-module and $|\Lambda_1^K(R)| = |\Lambda_1(K)|$ by (2.9). Hence the result follows from Theorems 4.0.15 and 4.0.16. □

The following theorem provides an equivalent criterion for an unmixed local ring R to be generalized Cohen-Macaulay in terms of the set $\Lambda_1^K(R)$.

Theorem 4.1.2. *Let $(R,\mathfrak{m})$ be an unmixed Noetherian local ring of dimension $d \geq 2$ and K an $\mathfrak{m}$-primary ideal of R. Then the following statements are equivalent:*

(a) R is a generalized Cohen-Macaulay ring;

(b) $\Lambda_1^K(R)$ is a finite set;

(c) $\delta_1^K(R)$ is a finite set.

Proof. (a) $\Rightarrow$ (b) Follows from Proposition 4.1.1(1a).
(b) $\Rightarrow$ (c) Since $\delta_1^K(R) \subseteq \Lambda_1^K(R)$, the assertion follows.
(c) $\Rightarrow$ (a) We may assume that R is complete. Since $\delta_1^K(R)$ is a finite set. In view of Remark 2.1.1(2), the set $S(K) := \{e_1(Q,K) | Q$ is a parameter ideal of R and $Q \subseteq K\}$ is finite. Let l be an integer such that $\mathfrak{m}^l \subseteq K$. Then the set

$$\{e_1(Q,K) \mid Q = (x_1,\ldots,x_d) \subseteq \mathfrak{m}^l \text{ is a parameter ideal of } R \text{ and a d-sequence for } K\}$$

contained in $S(K)$ is finite. Since R is unmixed, K is an unmixed module. Therefore, by [GGH+15, Lemma 4.1], K is a generalized Cohen-Macaulay R-module. Hence by Remark 2.1.8, R is generalized Cohen-Macaulay. □

Next, we give equivalent conditions for the finiteness of the set $\Lambda_1^K(R)$ in any Noetherian local ring. We need the following lemma.

Lemma 4.1.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module with $\dim M = r \geq 2$. Let K be an $\mathfrak{m}$-primary ideal of R. Assume that there exists an integer $t \geq 0$ such that $e_1(Q, M) \geq -t$ for every parameter ideal $Q \subseteq K$ for M. Then $\dim U_M(0) \leq r - 2$.*

Proof. Let $U = U_M(0)$ and $T = M/U$. Since $U_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Assh}_R(M)$, $\dim U \leq r-1$. Suppose $\dim U = r - 1$. Choose a system of parameters $(x_1, \ldots, x_r)$ for M such that $x_r U = 0$. Since $\mathfrak{m}^l \subseteq K$ for some integer $l \geq 1$, $Q = (x_1^s, \ldots, x_r^s) \subseteq K$ for all $s \geq l$. Let $s > \max\{l, t\}$. Consider the exact sequence

$$0 \longrightarrow U/(Q^{n+1}M \cap U) \longrightarrow M/Q^{n+1}M \longrightarrow T/Q^{n+1}T \longrightarrow 0$$

which gives

$$\lambda_R(M/Q^{n+1}M) = \lambda_R(T/Q^{n+1}T) + \lambda_R(U/(Q^{n+1}M \cap U)).$$

By Artin-Rees lemma, there exists an integer $k \geq 0$ such that $Q^n M \cap U = Q^{n-k}(Q^k M \cap U)$ for all $n \geq k$. Let $U' = Q^k M \cap U$ and $q = (x_1^s, \ldots, x_{r-1}^s)$. Since $Q^{n-k}U' = q^{n-k}U'$ for $n \geq k$, we get

$$\lambda_R(M/Q^{n+1}M) = \lambda_R(T/Q^{n+1}T) + \lambda_R(U'/q^{n+1-k}U') + \lambda_R(U/U') \text{ for all } n \geq k.$$

This implies that $-t \leq e_1(Q, M) = e_1(Q, T) - e_0(q, U')$. Since $e_0(q, U') = e_0(q, U)$ and $e_1(Q, T) \leq 0$ by Corollary 3.2.11, we get

$$s \leq s^{r-1} e_0((x_1, \ldots, x_{r-1}), U) = e_0(q, U) = e_1(Q, T) - e_1(Q, M) \leq t,$$

which is a contradiction. Thus $\dim U \leq r - 2$. $\square$

Theorem 4.1.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. Let $U = U_{\widehat{R}}(0)$. Then the following conditions are equivalent:*

(a) $\dim U \leq d - 2$ *and* $\widehat{R}/U$ *is a generalized Cohen-Macaulay ring;*

(b) $\Lambda_1^K(R)$ *is a finite set;*

(c) $\delta_1^K(R)$ *is a finite set.*

When this is the case, we have

$$-\sum_{i=1}^{d-1}\binom{d-2}{i-1}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(\widehat{R}/U))-\lambda_R(R/K)\le g_1^K(Q)\le 0.$$

for every parameter ideal Q of R.

Proof. We may assume that R is complete.
(a) $\Rightarrow$ (b) Since R/U is a generalized Cohen-Macaulay ring, by Proposition 4.1.1(1a), the set $\Lambda_1^{KR/U}(R/U)$ is finite. Since $g_1^K(Q)=g_1^{KR/U}(QR/U)$ by Lemma 3.2.7, the set $\Lambda_1^K(R)$ is finite.
(b) $\Rightarrow$ (c) Since $\delta_1^K(R)\subseteq\Lambda_1^K(R)$, the assertion follows.
(c) $\Rightarrow$ (a) From (2.9), $e_1(Q,K)=g_1^K(Q)$. Thus $\delta_1^K(R)$ is finite implies that there exists an integer $t\ge 0$ such that $e_1(Q,K)\ge -t$ for every parameter ideal $Q\subseteq K$. Hence, by Lemma 4.1.3, $\dim U_K(0)\le d-2$. Note that $U\cap K=U_K(0)$. Since $\dim U=\max\{\dim(U\cap K),\dim(U/(U\cap K))\}$ and $\dim(U/(U\cap K))=0$, $\dim U\le\dim(U\cap K)=\dim(U_K(0))\le d-2$. Hence, by Lemma 3.2.7, $g_1^K(R)=g_1^{KR/U}(R/U)$. Thus $\delta_1^{KR/U}(R/U)$ is finite. Now by Theorem 4.1.2, R/U is generalized Cohen-Macaulay.

The last assertion follows from Proposition 4.1.1(1a). □

In the next theorem, we give a sufficient condition for R to be Buchsbaum. For this purpose, we first prove the following lemma which relates the properties of R and an $\mathfrak{m}$-primary ideal K as an R-module.

Lemma 4.1.5. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d\ge 1$ and K an $\mathfrak{m}$-primary ideal of R.*

1. *Suppose* depth $R>0$ *and K is a Buchsbaum R-module. Then R is a Buchsbaum ring.*
2. *If R is Buchsbaum then $\mathfrak{m}$ is a Buchsbaum R-module.*

Proof. (1) Let $Q=(x_1,\ldots,x_d)$ be an arbitrary parameter ideal of R. We show that Q is standard for R. Since depth $R>0$, (2.6) gives an exact sequence

$$0\longrightarrow R/K\longrightarrow \mathrm{H}^1_{\mathfrak{m}}(K)\longrightarrow \mathrm{H}^1_{\mathfrak{m}}(R)\longrightarrow 0. \tag{4.1}$$

Thus we have

$\lambda_R(R/QK)=\lambda_R(R/K)+\lambda_R(K/QK)$

$$= \lambda_R(R/K) + \sum_{i=1}^{d-1} \binom{d-1}{i} \lambda_R(\mathrm{H}^i_{\mathfrak{m}}(K)) + e_0(Q, K) \qquad \text{[by Corollary 2.1.12]}$$

$$= \lambda_R(R/K) + \sum_{i=1}^{d-1} \binom{d-1}{i} \lambda_R(\mathrm{H}^i_{\mathfrak{m}}(R)) + (d-1)\lambda_R(R/K) + e_0(Q, K)$$

[by (2.7) and (4.1)]

$$= e_0(Q) + I(R) + d\lambda_R(R/K) \qquad \text{[since } e_0(Q,K) = e_0(Q)\text{].}$$

Hence, by [Tru86, Corollary 4.9], Q is a standard parameter ideal for R.

(2) Let $Q = (x_1, \ldots, x_d)$ be a parameter ideal for $\mathfrak{m}$. We have

$$\begin{aligned} I(Q; \mathfrak{m}) &= \lambda_R(\mathfrak{m}/Q\mathfrak{m}) - e_0(Q, \mathfrak{m}) \\ &= \lambda_R(R/Q) + \lambda_R(Q/Q\mathfrak{m}) - \lambda_R(R/\mathfrak{m}) - e_0(Q) && \text{[as } e_0(Q,\mathfrak{m}) = e_0(Q)\text{]} \\ &= I(Q; R) + d - 1 \\ &= I(R) + d - 1 && \text{[since } Q \text{ is standard for } R\text{]} \end{aligned}$$

which is independent of Q. Hence $\mathfrak{m}$ is Buchsbaum. □

Theorem 4.1.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and K an $\mathfrak{m}$-primary ideal of R. Then the following assertions hold.*

1. *Suppose R is unmixed and $|\Lambda_1^K(R)| = 1$. Then R is Buchsbaum. Further, $|\Lambda_i^K(R)| = 1$ for all $1 \leq i \leq d$.*
2. *If R is Buchsbaum then $|\Lambda_1^{\mathfrak{m}}(R)| = 1$.*

Proof. (1) By Proposition 4.1.1(2), we get that K is a Buchsbaum R-module. Hence by Lemma 4.1.5(1), R is a Buchsbaum ring. Further, every parameter ideal of R is standard for K. Hence, by Proposition 4.1.1(1b), $|\Lambda_i^K(R)| = 1$ for all $1 \leq i \leq d$.

(2) By Lemma 4.1.5(2), $\mathfrak{m}$ is a Buchsbaum R-module. Thus every parameter ideal of R is standard for $\mathfrak{m}$. Again by Proposition 4.1.1(1b), $|\Lambda_1^{\mathfrak{m}}(R)| = 1$. □

Theorem 4.1.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. Let $U = U_{\widehat{R}}(0)$. Then the following statements are equivalent:*

(a) $\dim U \leq d - 2$ *and* $\widehat{R}/U$ *is Buchsbaum;*

(b) $|\Lambda_1^{\mathfrak{m}}(R)| = 1$.

Proof. We may assume that R is complete.

(a)$\Rightarrow$ (b) By Lemma 3.2.7 and Theorem 4.1.6(2), we get $|\Lambda_1^{\mathfrak{m}}(R)| = |\Lambda_1^{\mathfrak{m}}(R/U)| = 1$.

(b)$\Rightarrow$ (a) Since $|\Lambda_1^{\mathfrak{m}}(R)| = 1$, by (2.9), $|\Lambda_1(\mathfrak{m})| = 1$. Since $U_{\mathfrak{m}}(0) = U$, $\dim U = \dim U_{\mathfrak{m}}(0) \leq d-2$ by [GGH+15, Theorem 5.5]. Thus $g_1^K(Q) = g_1^{KR/U}(QR/U)$ by Lemma 3.2.7. Hence $|\Lambda_1^{\mathfrak{m}}(R/U)| = |\Lambda_1^{\mathfrak{m}}(R)| = 1$. Therefore R/U is Buchsbaum by Theorem 4.1.6(1). □

We discuss below that for a Buchsbaum local ring R and an arbitrary $\mathfrak{m}$-primary ideal K of R, $\Lambda_1^K(R)$ need not be singleton.

Remark 4.1.8. *Suppose $(R, \mathfrak{m})$ is a Buchsbaum local ring of dimension $d \geq 2$ and K is an $\mathfrak{m}$-primary ideal of R. Suppose $|\Lambda_1^K(R)| = 1$. Then, by (2.9), $|\Lambda_1(K)| = 1$. Further, assume that R is unmixed. Then, by Theorem 4.0.16, K is a Buchsbaum R-module. Let Q be an arbitrary parameter ideal of R. Since*

$$\lambda_R(R/Q^{n+1}K) = \lambda_R(R/K) + \lambda_R(K/Q^{n+1}K) \quad \textit{for all } n$$

and Q is standard for K, using Corollary 2.1.12, we get

$$\begin{aligned}\lambda_R(R/Q^{n+1}K) = \lambda_R(R/K) + \binom{n+d}{d} e_0(Q,K) \\ + \sum_{i=1}^{d}\sum_{j=0}^{d-i}\binom{n+d-i}{d-i}\binom{d-i-1}{j-1}\lambda_R(\mathrm{H}_{\mathfrak{m}}^j(K))\end{aligned}$$

for all $n \geq 0$. In particular, for $n = 0$,

$$\begin{aligned}\lambda_R(R/QK) &= e_0(Q,K) + \lambda_R(R/K) + \sum_{i=1}^{d-1}\sum_{j=1}^{d-i}\binom{d-i-1}{j-1}\lambda_R(\mathrm{H}_{\mathfrak{m}}^j(K)) \\ &= e_0(Q) + \lambda_R(R/K) + \sum_{j=1}^{d-1}\sum_{i=1}^{d-j}\binom{d-i-1}{j-1}\lambda_R(\mathrm{H}_{\mathfrak{m}}^j(K)) \\ &= e_0(Q) + \lambda_R(R/K) + \sum_{j=1}^{d-1}\sum_{i=j-1}^{d-2}\binom{i}{j-1}\lambda_R(\mathrm{H}_{\mathfrak{m}}^j(K)) \\ &= e_0(Q) + \lambda_R(R/K) + \sum_{j=1}^{d-1}\binom{d-1}{j}\lambda_R(\mathrm{H}_{\mathfrak{m}}^j(K)) \qquad (4.2)\end{aligned}$$

Now using (2.6) *and* (2.7), *we get*

$$\begin{aligned}\lambda_R(R/QK) &= e_0(Q) + \lambda_R(R/K) + \sum_{j=1}^{d-1}\binom{d-1}{j}\lambda_R(\mathrm{H}^j_{\mathfrak{m}}(R)) + (d-1)\lambda_R(R/K)\\ &= e_0(Q) + I(R) + d\lambda_R(R/K). \qquad (4.3)\end{aligned}$$

On the other hand,

$$\begin{aligned}\lambda_R(R/QK) &= \lambda_R(R/Q) + \lambda_R(Q/QK)\\ &= e_0(Q) + I(R) + \lambda_R(Q/QK). \qquad (4.4)\end{aligned}$$

Comparing (4.3) *and* (4.4), *we get* $\lambda_R(Q/QK) = d\lambda_R(R/K)$ *for every parameter ideal* Q *of* R. *This need not be true even in regular local ring. For example, let* $R = k[[x,y]]$ *and* $K = (x,y)^2$. *Then for* $Q = (x,y)$, $\lambda_R(Q/QK) = 5 \neq 6 = 2\lambda_R(R/K)$.

Remark 4.1.9.
1. *Suppose* R *is a generalized Cohen-Macaulay local ring. Then, by Corollary 3.2.11 and* (2.19), $\Lambda_1(R)$ *is finite. By Proposition 4.1.1*(1), $\Lambda_1^K(R)$ *is finite. Hence, from* (2.11), *the set* $\{f_0^K(Q) \mid Q$ *is a parameter ideal of* $R\}$ *is finite.*
2. *Suppose* R *is Buchsbaum. Then, by* (2.19), $e_1(Q)$ *is constant for all parameter ideals* Q *i.e.* $\Lambda_1(R)$ *is singleton. By Theorem 4.1.6*(2), $|\Lambda_1^{\mathfrak{m}}(R)| = 1$. *Hence, from* (2.11), *the set* $\{f_0^{\mathfrak{m}}(Q) \mid Q$ *is a parameter ideal of* $R\}$ *is singleton.*

4.2 The set $\Lambda_i^K(R)$

The first main result of this section provides uniform lower and upper bounds for the function $\lambda_R(M/Q^{n+1}M) - e_0(Q,M)\binom{n+r}{r}$ in terms of $I(M)$ for a generalized Cohen-Macaulay module M of dimension r and a parameter ideal Q for M. The upper bound obtained generalizes a similar bound given by Linh and Trung [LT06] in the case of ring. In Theorem 4.2.3 and its corollaries, we generalize few other results of [LT06] to the case of modules in order to obtain the lower bound mentioned above. Goto and Ozeki [GO11] gave uniform bounds for $|e_i(Q)|$ where Q is a parameter ideal of R. As a consequence, they obtained a necessary and sufficient condition for $\Lambda_i(R)$ to be finite for $1 \leq i \leq d$. We improve upon their result and generalize it for modules. Recall that

$$\Lambda_i(M) := \{e_i(Q,M) \mid Q \text{ is a parameter ideal for } M\} \text{ and}$$

$$\Lambda_i^K(R) = \{g_i^K(Q) \mid Q \text{ is a parameter ideal of } R\}.$$

The next main result of this section provides a characterization for the finiteness of $\Lambda_i^K(R)$ for $1 \leq i \leq d$. First we recall the following lemma from [Tru86].

Lemma 4.2.1. [Tru86, Lemma 1.7] *Let M be a generalized Cohen-Macaulay module of dimension r and $Q = (x_1, \ldots, x_r)$ a parameter ideal for M. Then M/x_1M is a generalized Cohen-Macaulay module and $I(M/x_1M) \leq I(M)$.*

The following lemma is a straight generalization of [LT06, Lemma 1.1].

Lemma 4.2.2. *Let M be a generalized Cohen-Macaulay module of dimension $r > 0$. Let Q be a parameter ideal for M. Then for all $n \geq 0$,*

$$\lambda_R(M/Q^{n+1}M) - e_0(Q, M)\binom{n+r}{r} \leq \binom{n+r-1}{r-1} I(M).$$

Proof. We apply induction on r. For $r = 1$, let $M' = M/H^0_{\mathfrak{m}}(M)$. Then M' is a Cohen-Macaulay module and $e_0(Q, M') = e_0(Q, M)$ by (2.5). For all $n \geq 0$,

$$\begin{aligned}\lambda_R(M/Q^{n+1}M) &= \lambda_R(M'/Q^{n+1}M') + \lambda_R(H^0_{\mathfrak{m}}(M)/(H^0_{\mathfrak{m}}(M) \cap Q^{n+1}M)) \\ &= e_0(Q, M)(n+1) + \lambda_R(H^0_{\mathfrak{m}}(M)/(H^0_{\mathfrak{m}}(M) \cap Q^{n+1}M)) \qquad (4.5) \\ &\leq e_0(Q, M)(n+1) + \lambda_R((H^0_{\mathfrak{m}}(M)).\end{aligned}$$

Let $d > 1$ and $Q = (x_1, \ldots, x_r)$ such that $x_1, \ldots, x_r$ is an M-superficial sequence in Q. We put $M_1 = M/x_1M$. Since $e_0(Q, M) = e_0(Q, M_1)$ and $I(M_1) \leq I(M)$ by Lemma 4.2.1, from the exact sequence

$$0 \longrightarrow (Q^{t+1}M :_M x_1)/Q^tM \longrightarrow M/Q^tM \xrightarrow{x_1} M/Q^{t+1}M \longrightarrow M_1/Q^{t+1}M_1 \longrightarrow 0,$$

we get

$$\begin{aligned}\lambda_R(Q^tM/Q^{t+1}M) &= \lambda_R(M_1/Q^{t+1}M_1) - \lambda_R((Q^{t+1}M :_M x_1)/Q^tM) \qquad (4.6) \\ &\leq \lambda_R(M_1/Q^{t+1}M_1) \\ &\leq e_0(Q, M)\binom{t+r-1}{r-1} + \binom{t+r-2}{r-2} I(M)\end{aligned}$$

where the last step follows from induction. This gives

$$\begin{aligned}\lambda_R(M/Q^{n+1}M) &= \sum_{t=0}^{n} \lambda_R(Q^tM/Q^{t+1}M)\\ &\leq e_0(Q,M)\binom{n+r}{r} + \binom{n+r-1}{r-1} I(M).\end{aligned}$$

□

In [LT06, Corollary 1.4], authors proved that if R is generalized Cohen-Macaulay ring and Q is parameter ideal of R, then for $n \geq 0$, $\lambda_R((Q^{n+1} : x_1)/Q^n) \leq \binom{n+d-2}{d-2} I(R)$. Indeed, we easily generalize it for generalized Cohen-Macaulay modules in Corollary 4.2.5.

Theorem 4.2.3. *Let M be a generalized Cohen-Macaulay module of dimension $r > 0$. Let $x_1, \ldots, x_i$ be a subsystem of parameters for M and $J = (x_1, \ldots, x_i)$, $0 < i < r$. Then $M/J^{n+1}M$ is generalized Cohen-Macaulay module with*

$$I(M/J^{n+1}M) \leq \binom{n+i-1}{i-1} I(M).$$

Proof. Let $Q = (x_1, \ldots, x_i, x_{i+1}, \ldots, x_r)$ be a parameter ideal for M and $M' = M/(x_{i+1}, \ldots, x_r)M$. Then M' is a generalized Cohen-Macaulay module. Using Lemma 2.1.10(1) for M and M', we get

$$\begin{aligned}I(Q;M) &= \lambda_R\Big(((x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_r)M : x_i)/(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_r)M\Big)\\ &= \lambda_R\Big(((x_1,\ldots,x_{i-1})M' : x_i)/(x_1,\ldots,x_{i-1})M'\Big)\\ &= I(J;M').\end{aligned}$$

Hence $\lambda_R(M/QM) - e_0(Q,M) = \lambda_R(M'/JM') - e_0(J,M')$ which implies $e_0(Q,M) = e_0(J,M')$. By similar argument, $e_0(Q,M) = e_0((x_{i+1},\ldots,x_r),M)$. By Lemma 4.2.2, for all $n \geq 0$,

$$\begin{aligned}\lambda_R(M/(J^{n+1}M,(x_{i+1},\ldots,x_r)M)) &= \lambda_R(M'/J^{n+1}M')\\ &\leq \binom{n+i}{i} e_0(J,M') + \binom{n+i-1}{i-1} I(M')\\ &\leq \binom{n+i}{i} e_0(Q,M) + \binom{n+i-1}{i-1} I(M). \qquad (4.7)\end{aligned}$$

Now by the associative formula for multiplicity,

$$e_0((x_{i+1},\ldots,x_r),M/J^{n+1}M)=\sum_{\mathfrak{p}}\lambda_{R_\mathfrak{p}}(M_\mathfrak{p}/J^{n+1}M_\mathfrak{p})e_0((x_{i+1},\ldots,x_r),R/\mathfrak{p}) \qquad (4.8)$$

where $\mathfrak{p}$ runs over all the minimal primes of R such that $\dim R/\mathfrak{p}=r-i$ and $\mathfrak{p}\in \operatorname{Supp}(M)$. By Lemma 2.1.10(3) $M_\mathfrak{p}$ is a Cohen-Macaulay module of dimension i. Note that for any prime ideal $\mathfrak{q}\subsetneq\mathfrak{p}$, $\dim R/\mathfrak{q}>r-i$. Since $\dim M/JM=\dim R/\mathfrak{p}=r-i$, $\mathfrak{q}\notin\operatorname{Supp}(M/JM)$. This implies $\operatorname{Supp}_{R_\mathfrak{p}}(M_\mathfrak{p}/JM_\mathfrak{p})=\{\mathfrak{p}R_\mathfrak{p}\}$. Thus $JR_\mathfrak{p}$ is a parameter ideal for $M_\mathfrak{p}$ and

$$\lambda_R(M_\mathfrak{p}/J^{n+1}M_\mathfrak{p})=\binom{n+i}{i}\lambda_{R_\mathfrak{p}}(M_\mathfrak{p}/JM_\mathfrak{p}). \qquad (4.9)$$

Therefore from (4.8) and (4.9),

$$\begin{aligned}e_0((x_{i+1},\ldots,x_r),M/J^{n+1}M)&=\binom{n+i}{i}\sum_{\mathfrak{p}}\lambda_{R_\mathfrak{p}}(M_\mathfrak{p}/JM_\mathfrak{p})e_0((x_{i+1},\ldots,x_r),R/\mathfrak{p})\\&=\binom{n+i}{i}e_0((x_{i+1},\ldots x_r),M/JM)\\&=\binom{n+i}{i}e_0(Q,M).\end{aligned}$$

Hence using (4.7), we get

$$\begin{aligned}I((x_{i+1},\ldots,x_r),M/J^{n+1}M)&=\lambda_R(M/(J^{n+1}M,(x_{i+1},\ldots,x_r)M))\\&\qquad-e_0((x_{i+1},\ldots,x_r),M/J^{n+1}M)\\&\le\binom{n+i-1}{i-1}I(M).\end{aligned}$$

Therefore,

$$\begin{aligned}I(M/J^{n+1}M)&=\sup I((x_{i+1},\ldots,x_r),M/J^{n+1}M)\\&\le\binom{n+i-1}{i-1}I(M).\end{aligned}$$

□

Corollary 4.2.4. *Let M be a generalized Cohen-Macaulay module of dimension $r>0$. Let $x_1,\ldots,x_i$ be a subsystem of parameters for M and $J=(x_1,\ldots,x_i)$, $0<i<r$. Then for all $n\ge 0$ and $m\ge 1$,*

$$\lambda_R\Big((J^{n+1}M:_M x_{i+1}^m)/J^{n+1}M\Big)\le\binom{n+i-1}{i-1}I(M).$$

Proof. Let $\mathfrak{p} \in \operatorname{Ass}(M/J^{n+1}M) \setminus \mathfrak{m}$. Since $M/J^{n+1}M$ is generalized Cohen-Macaulay by Theorem 4.2.3, $\dim R/\mathfrak{p} = \dim M/J^{n+1}M = r - i$ by Lemma 2.1.10(2). So $x_{i+1} \notin \mathfrak{p}$. It is easy to see that

$$(J^{n+1}M :_M x_{i+1}^m)/J^{n+1}M \subseteq H^0_{\mathfrak{m}}(M/J^{n+1}M).$$

To see this, let $J^{n+1}M = N_1 \cap \ldots \cap N_k$ be an irredundant primary decomposition of $J^{n+1}M$ with $\operatorname{Ass}(M/N_i) = \{P_i\}$. let $x_{i+1}^m \alpha \in J^{n+1}M \subseteq N_i$ for some $\alpha \in M$. Then $x_{i+1}^m \bar{\alpha} = 0$ in M/N_i and $x_{i+1}^m \notin P_i$ for all $P_i \neq \mathfrak{m}$. So $\bar{\alpha} = 0$ in M/N_i. For $P_i = \mathfrak{m}$, $P_i = \sqrt{(\operatorname{Ann}(M/N_i))}$ implies $\mathfrak{m}^k \subseteq \operatorname{Ann}(M/N_i)$ for some k. Therefore, $\mathfrak{m}^k \alpha \subseteq N_i$ which implies $\mathfrak{m}^k \alpha \subseteq N_1 \cap \ldots \cap N_k = J^{n+1}M$.

Hence,

$$\begin{aligned}
\lambda_R\Big((J^{n+1}M :_M x_{i+1}^m)/J^{n+1}M\Big) &\leq \lambda_R\Big(H^0_{\mathfrak{m}}(M/J^{n+1}M)\Big) \\
&\leq I(M/J^{n+1}M) \\
&\leq \binom{n+i-1}{i-1} I(M)
\end{aligned}$$

where the last equality follows from Theorem 4.2.3. □

Corollary 4.2.5. *Let M be a generalized Cohen-Macaulay module of dimension $r \geq 2$ and $Q = (x_1, \ldots, x_r)$ be a parameter ideal for M. Then for all $n \geq 0$ and $m \geq 1$,*

$$\lambda_R\Big((Q^{n+m}M :_M x_r^m)/Q^nM\Big) \leq \binom{n+r-2}{r-2} I(M).$$

Proof. Let $J = (x_1, \ldots, x_{r-1})$. Since $Q^{n+m} \subseteq J^{n+1} + x_r^m Q^n$,

$$\begin{aligned}
\lambda_R\Big((Q^{n+m}M :_M x_r^m)/Q^nM\Big) &\leq \lambda_R\Big((J^{n+1}M + x_r^m Q^nM) :_M x_r^m/Q^nM\Big) \\
&= \lambda_R\Big(((J^{n+1}M :_M x_r^m) + Q^nM)/Q^nM\Big) \\
&= \lambda_R\Big((J^{n+1}M :_M x_r^m)/((J^{n+1}M :_M x_r^m) \cap Q^nM)\Big) \\
&\leq \lambda_R\Big((J^{n+1}M :_M x_r^m)/J^{n+1}M\Big) \\
&\leq \binom{n+r-2}{r-2} I(M)
\end{aligned}$$

by Corollary 4.2.4. □

In the next lemma, we give a lower bound on the function $\lambda_R(M/Q^{n+1}M) - e_0(Q,M)\binom{n+r}{r}$ in terms of $I(M)$ for a generalized Cohen-Macaulay module M and a parameter ideal Q for M.

Lemma 4.2.6. *Let M be a generalized Cohen-Macaulay module of dimension $r > 0$ and Q a parameter ideal for M. Then for all $n \geq 0$,*

$$-r\binom{n+r-1}{r-1}I(M) \leq \lambda_R(M/Q^{n+1}M) - e_0(Q,M)\binom{n+r}{r} \leq \binom{n+r-1}{r-1}I(M).$$

Proof. In view of Lemma 4.2.2, we only need to prove the inequality at left hand side. We apply induction on r. For $r = 1$, it is evident from (4.5) that for all $n \geq 0$,

$$\lambda_R(M/Q^{n+1}M) - e_0(Q,M)\binom{n+r}{r} \geq 0.$$

Now let $r \geq 2$ and $Q = (x_1, \ldots, x_r)$. Set $M_1 = M/x_1M$. Then from the exact sequence

$$0 \longrightarrow (Q^{t+1}M :_M x_1)/Q^tM \longrightarrow M/Q^tM \xrightarrow{x}_1 M/Q^{t+1}M \longrightarrow M_1/Q^{t+1}M_1 \longrightarrow 0,$$

we have

$$\lambda_R(Q^tM/Q^{t+1}M) = \lambda_R(M_1/Q^{t+1}M_1) - \lambda_R((Q^{t+1}M :_M x_1)/Q^tM) \tag{4.10}$$

for all $t \geq 0$. By Lemma 2.1.10(1),

$$\begin{aligned} e_0((x_2,\ldots,x_r),M_1) &= \lambda_R(M/QM) - I((x_2,\ldots,x_r);M_1) \\ &= \lambda_R(M/QM) - \lambda_R\big((x_2,\ldots,x_{r-1})M_1 :_{M_1} x_r)/(x_2,\ldots,x_{r-1})M_1\big) \\ &= \lambda_R(M/QM) - \lambda_R\big((x_1,\ldots,x_{r-1})M :_M x_r)/(x_1,\ldots,x_{r-1})M\big) \\ &= \lambda_R(M/QM) - I(Q;M) \\ &= e_0(Q,M) \end{aligned} \tag{4.11}$$

Thus ,

$$\begin{aligned} \lambda_R(M/Q^{n+1}M) &= \sum_{t=0}^{n} \lambda_R(Q^tM/Q^{t+1}M) \\ &= \sum_{t=0}^{n} \Big(\lambda_R(M_1/Q^{t+1}M_1) - \lambda_R((Q^{t+1}M : x_1)/Q^tM)\Big) \end{aligned} \tag{4.12}$$

$$
\begin{aligned}
&\geq \sum_{t=0}^{n} \left(\lambda_R(M_1/Q^{t+1}M_1) - \binom{t+r-2}{r-2} I(M) \right) && \text{[by Corollary 4.2.5]} \\
&\geq \sum_{t=0}^{n} \left(e_0(Q,M)\binom{t+r-1}{r-1} - (r-1)\binom{t+r-2}{r-2} I(M_1) \right. \\
&\qquad \left. - \binom{t+r-2}{r-2} I(M) \right) && \text{[by induction hypothesis and (4.11)]} \\
&\geq e_0(Q,M)\binom{n+r}{r} - (r-1)\binom{n+r-1}{r-1} I(M) - \binom{n+r-1}{r-1} I(M) \\
&\geq e_0(Q,M)\binom{n+r}{r} - r\binom{n+r-1}{r-1} I(M)
\end{aligned}
$$

where the last two inequalities hold since $I(M_1) \leq I(M)$, see Lemma 4.2.1. □

Let $G_Q(M) = \bigoplus_{n \geq 0} Q^n M/Q^{n+1}M$ be the associated graded module of M with respect to Q. For $i \in \mathbb{Z}$, let

$$a_i(G_Q(M)) = \sup\{n \in \mathbb{Z} : [\mathrm{H}^i_{\mathcal{M}}(G_Q(M))]_n \neq 0\}.$$

Recall that

$$\mathrm{reg}(G_Q(M)) = \sup\{a_i(G_Q(M)) + i : i \in \mathbb{Z}\}$$

is the Castelnuovo-Mumford regularity of the graded module $G_Q(M)$. We need the following lemma in order to obtain uniform bounds on the coefficients $e_i(Q,M)$ in terms of $\mathrm{reg}(G_Q(M))$. We skip the proof of the following lemma as it is similar to [GO11, Lemma 2.3].

Lemma 4.2.7. *Let M be a finitely generated module of dimension $r > 0$ and Q a parameter ideal for M.*

1. *Let $M' = M/\mathrm{H}^0_{\mathfrak{m}}(M)$. Then $\mathrm{reg}(G_Q(M)) \geq \mathrm{reg}(G_Q(M'))$.*

2. *Assume that $r \geq 2$ and $x \in Q$ is an M-superficial element in Q. Let $R_1 = R/(x)$ and $M_1 = M/xM$. Then $\mathrm{reg}(G_Q(M)) \geq \mathrm{reg}(G_{QR_1}(M_1))$.*

The following theorem provides bounds on $e_i(Q,M)$ in terms of $\mathrm{reg}(G_Q(M))$ when M is a generalized Cohen-Macaulay module. These bounds may not be tight but the significance lies in the fact that by a result of Cuong et al. i.e. Theorem 4.2.10, $\mathrm{reg}(G_Q(M))$ has a uniform bound. Thus the sets $\Lambda_i(M)$ are finite for a generalized Cohen-Macaulay module M. We further use this fact to get the finiteness of $\Lambda_i^K(R)$ when R is generalized Cohen-Macaulay.

Theorem 4.2.8. *Let M be a generalized Cohen-Macaulay module of dimension $r > 0$ and Q a parameter ideal for M. Put $\kappa = \operatorname{reg}(G_Q(M))$. Then*

1. $|e_1(Q,M)| \le I(M)$.
2. $|e_i(Q,M)| \le (r+1)\cdot 2^{i-2}(\kappa+1)^{i-1}I(M)$ *for* $2 \le i \le r$.

Proof. We may assume that the residue field $R/\mathfrak{m}$ is infinite. We use induction on r. Let $r = 1$. Then by [MSV11, Proposition 3.1], $e_1(Q,M) = -\lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M))$ for all parameter ideals Q for M. Hence $|e_1(Q,M)| = \lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M)) = I(M)$. Thus the assertion is true in this case.

Let $r \ge 2$. We may assume that $\operatorname{depth} M > 0$. In fact, let $M' = M/\mathrm{H}^0_{\mathfrak{m}}(M)$ and assume that the assertion holds for M'. Set $\kappa' = \operatorname{reg}(G_Q(M'))$. By (2.5),

$$|e_1(Q,M)| \le |e_1(Q,M')| + \lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M)) \le I(M') + \lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M)) = I(M)$$

and for $2 \le i \le r$,

$$\begin{aligned}|e_i(Q,M)| &\le |e_i(Q,M')| + \lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M))\\ &\le (r+1)\cdot 2^{i-2}(\kappa'+1)^{i-1}I(M') + \lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M))\\ &\le (r+1)\cdot 2^{i-2}(\kappa'+1)^{i-1}\left(\sum_{i=1}^{r-1}\binom{r-1}{i}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(M')) + \lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M))\right)\\ &\le (r+1)\cdot 2^{i-2}(\kappa+1)^{i-1}\left(\sum_{i=0}^{r-1}\binom{r-1}{i}\lambda_R(\mathrm{H}^i_{\mathfrak{m}}(M))\right) \text{ [by Lemma 4.2.7(1)]}\\ &= (r+1)\cdot 2^{i-2}(\kappa+1)^{i-1}I(M).\end{aligned}$$

Let $Q = (x_1,\ldots,x_r)$ be such that x_1 is M-superficial in Q. Put $R_1 = R/(x_1)$, $M_1 = M/x_1M$ and $\kappa_1 = \operatorname{reg}(G_{QR_1}(M_1)$. Then using induction hypothesis and Lemmas 4.2.1 and 4.2.7(2), we get

$$|e_1(Q,M)| = |e_1(Q,M_1)| \le I(M_1) \le I(M) \tag{4.13}$$

and for $2 \le i \le r-1$,

$$\begin{aligned}|e_i(Q,M)| &= |e_i(Q,M_1)|\\ &\le r\cdot 2^{i-2}(\kappa_1+1)^{i-1}I(M_1)\end{aligned}$$

$$\leq (r+1)\cdot 2^{i-2}(\kappa+1)^{i-1}I(M). \tag{4.14}$$

Let $i=r$. By [BH98, Theorem 4.4.3], for all $t>\kappa$

$$\lambda_R(Q^tM/Q^{t+1}M) = \sum_{i=0}^{r-1}(-1)^i e_i(Q,M)\binom{t+r-1-i}{r-1-i}$$

and for all $t \geq \kappa_1$,

$$\lambda_R(M_1/Q^{t+1}M_1) = \sum_{i=0}^{r-1}(-1)^i e_i(QR_1,M_1)\binom{t+r-1-i}{r-1-i}. \tag{4.15}$$

Since $\kappa \geq \kappa_1$ by Lemma 4.2.7(2), we have $\lambda_R(Q^tM/Q^{t+1}M) = \lambda_R(M_1/Q^{t+1}M_1)$ for all $t>\kappa$. On the other hand, from (4.6), for all $t \geq 0$, we have

$$\lambda_R(Q^tM/Q^{t+1}M) = \lambda_R(M_1/Q^{t+1}M_1) - \lambda_R((Q^{t+1}M :_M x_1)/Q^tM)).$$

Hence for $t>\kappa$,

$$(Q^{t+1}M :_M x_1) = Q^tM. \tag{4.16}$$

Since $\lambda_R(M/Q^{n+1}M) = \sum\limits_{i=0}^{r}(-1)^i e_i(Q,M)\binom{n+r-i}{r-i}$ for all $n \geq \kappa$ by [BH98, Theorem 4.4.3], we get

$$\begin{aligned}
(-1)^r e_r(Q,M) &= \lambda_R(M/Q^{n+1}M) - \sum_{i=0}^{r-1}(-1)^i e_i(Q,M)\binom{n+r-i}{r-i}\\
&= \sum_{t=0}^{n}\lambda_R(Q^tM/Q^{t+1}M) - \sum_{i=0}^{r-1}(-1)^i e_i(Q,M)\binom{n+r-i}{r-i}\\
&= \sum_{t=0}^{n}\lambda_R(M_1/Q^{t+1}M_1) - \sum_{t=0}^{n}\lambda_R((Q^{t+1}M :_M x_1)/Q^tM)\\
&\quad - \sum_{t=0}^{n}\sum_{i=0}^{r-1}(-1)^i e_i(Q,M)\binom{t+r-1-i}{r-1-i} \qquad \text{[by (4.6)]}\\
&= \sum_{t=0}^{n}\left(\lambda_R(M_1/Q^{t+1}M_1) - \sum_{i=0}^{r-1}(-1)^i e_i(QR_1,M_1)\binom{t+r-1-i}{r-1-i}\right)\\
&\quad - \sum_{t=0}^{\kappa}\lambda_R((Q^{t+1}M :_M x_1)/Q^tM) \quad \text{[by (4.16) and Proposition 2.2.5]}\\
&= \sum_{t=0}^{\kappa_1}\left(\lambda_R(M_1/Q^{t+1}M_1) - \sum_{i=0}^{r-1}(-1)^i e_i(QR_1,M_1)\binom{t+r-1-i}{r-1-i}\right)\\
&\quad - \sum_{t=0}^{\kappa}\lambda_R((Q^{t+1}M :_M x_1)/Q^tM) \qquad \text{[by (4.15)]}
\end{aligned}$$

This implies that

$$
\begin{aligned}
&|e_r(Q,M)| \leq \\
&\sum_{t=0}^{\kappa_1}\left(\left|\lambda_R(M_1/Q^{t+1}M_1) - e_0(QR_1,M_1)\binom{t+r-1}{r-1}\right| + |e_1(Q,M)|\binom{t+r-2}{r-2}\right) \\
&\quad + \sum_{t=0}^{\kappa_1}\sum_{i=2}^{r-1}|e_i(Q,M)|\binom{t+r-1-i}{r-1-i} + \sum_{t=0}^{\kappa}\lambda_R((Q^{t+1}M :_M x_1)/Q^tM) \\
&\leq \sum_{t=0}^{\kappa_1}(r-1)\binom{t+r-2}{r-2}I(M_1) + \sum_{t=0}^{\kappa_1}\binom{t+r-2}{r-2}I(M) \\
&\quad + \sum_{t=0}^{\kappa_1}\sum_{i=2}^{r-1}(r+1)\cdot 2^{i-2}(\kappa+1)^{i-1}\binom{t+r-1-i}{r-1-i}I(M) + \sum_{t=0}^{\kappa}\binom{t+r-2}{r-2}I(M) \\
&\qquad\qquad [\text{by Lemma 4.2.6, (4.13), (4.14) and Corollary 4.2.5 respectively}] \\
&= (r-1)\binom{\kappa_1+r-1}{r-1}I(M_1) + \binom{\kappa_1+r-1}{r-1}I(M) \\
&\quad + \sum_{i=2}^{r-1}(r+1)\cdot 2^{i-2}(\kappa+1)^{i-1}\binom{\kappa_1+r-i}{r-i}I(M) + \binom{\kappa+r-1}{r-1}I(M) \\
&\leq (r+1)\binom{\kappa+r-1}{r-1}I(M) + \sum_{i=2}^{r-1}(r+1)\cdot 2^{i-2}(\kappa+1)^{i-1}\binom{\kappa+r-i}{r-i}I(M) \\
&\qquad\qquad [\text{by Lemmas 4.2.1 and 4.2.7(2)}] \\
&\leq (r+1)(\kappa+1)^{r-1}I(M) + \sum_{i=2}^{r-1}(r+1)\cdot 2^{i-2}(\kappa+1)^{r-1}I(M) \\
&\qquad\qquad \left[\text{since } \binom{m+n}{n} \leq (m+1)^n \text{ for all integers } n \geq 0\right] \\
&= (r+1)(\kappa+1)^{r-1}I(M)\left(1 + \sum_{i=2}^{r-1}2^{i-2}\right) \\
&= (r+1)\cdot 2^{r-2}(\kappa+1)^{r-1}I(M) \qquad \left[\text{since } \sum_{i=2}^{r-1}2^{i-2} = 2^{r-2}-1\right].
\end{aligned}
$$

□

In the following lemma we give a necessary condition for the finiteness of the sets $\Lambda_i(M)$.

Lemma 4.2.9. *Let $(R,\mathfrak{m})$ be a Noetherian local ring and K an $\mathfrak{m}$-primary ideal of R. Let M be a finitely generated R-module of dimension $r \geq 2$. For a fixed $1 \leq k \leq r$,* *assume that*

$$\{e_i(Q, M) : Q \text{ is a parameter ideal for } M \text{ and } Q \subseteq K\}$$

is finite for all $1 \leq i \leq k$*. Then* $\lambda_R(\mathrm{H}^{r-i}_{\mathfrak{m}}(M)) < \infty$ *for all* $1 \leq i \leq k$*.*

In particular, if $\Lambda_i(M)$ *is finite for all* $1 \leq i \leq k$*, then* $\lambda_R(\mathrm{H}^{r-i}_{\mathfrak{m}}(M)) < \infty$ *for all* $1 \leq i \leq k$*.*

Proof. We may assume that R is complete and $k < r$. Let l be an integer such that $\mathfrak{m}^l \subseteq K$. Let $U = U_M(0)$ and $N = M/U$. If $U = 0$ then M is unmixed and the set $\{e_1(Q, M) : Q = (x_1, \ldots, x_r) \subseteq \mathfrak{m}^l$ is a parameter ideal for M with $x_1, \ldots, x_r$ a d-sequence for $M\}$ is finite. Hence by [GGH+15, Lemma 4.1], M is a generalized Cohen-Macaulay module. Thus $\lambda_R(\mathrm{H}^{r-i}_{\mathfrak{m}}(R)) < \infty$ for all $1 \leq i \leq r$.

Assume that $U \neq 0$. By Lemma 4.1.3, $\dim U \leq r - 2$. Hence by [GGH+15, Lemma 3.3], $e_1(Q, M) = e_1(Q, N)$. Thus the set $\{e_1(Q, N) : Q$ is a parameter ideal for M and $Q \subseteq K\}$ is finite. For a parameter ideal q for N, there exists a parameter ideal Q for M such that $QM = qN$, see [GGH+15, Remark 4.4]. So $e_1(q, N) = e_1(Q, M)$ by [GGH+15, Lemma 3.3] which implies that the set $\{e_1(Q, N) : Q$ is a parameter ideal for N and $Q \subseteq K\}$ is also finite. Hence, by $U = 0$ case, N is generalized Cohen-Macaulay. We now show that $t := \dim U \leq r - (k + 1)$. We may assume that $t \geq 1$. Let $x_1, \ldots, x_r$ be a system of parameters for M such that $(x_{t+1}, \ldots, x_r)U = 0$. Since N is a generalized Cohen-Macaulay module, by [Tru86, Lemma 1.5], there exists an integer $l_1 \geq 1$ such that $\mathfrak{m}^{l_1}$ is a standard ideal for N. Let $l_0 = \max\{l_1, l\}$. Then $\mathfrak{m}^{l_0} \subseteq K$ is a standard ideal. Let $n \geq l_0$ and $Q = (x_1^n, \ldots, x_r^n)$. Then by Corollary 2.1.12,

$$e_{r-t}(Q, N) = (-1)^{r-t} \sum_{j=1}^{t} \binom{t-1}{j-1} \lambda_R(\mathrm{H}^j_{\mathfrak{m}}(N)).$$

We have

$$\lambda_R(M/Q^{n+1}M) = \lambda_R(N/Q^{n+1}N) + \lambda_R(U/(Q^{n+1}M \cap U)) \text{ for all } n \geq 0. \tag{4.17}$$

Since the filtration $\{Q^{n+1}M \cap U\}$ is a good Q-filtration,

$$\lambda_R(U/(Q^{n+1}M \cap U)) = \sum_{i=0}^{t} (-1)^i s_i(Q, U) \binom{n+t-i}{t-i} \tag{4.18}$$

for some integers $s_i(Q,U)$ with $s_0(Q,U) = e_0(Q,U)$ and $n \gg 0$. By (4.17) and (4.18),

$$\lambda_R(M/Q^{n+1}M) = \sum_{i=0}^{r}(-1)^i e_i(Q,N)\binom{n+r-i}{r-i} + \sum_{i=0}^{t}(-1)^i s_i(Q,U)\binom{n+t-i}{t-i}$$

for $n \gg 0$. Therefore, for $n \geq l_0$,

$$\begin{aligned}(-1)^{r-t}e_{r-t}(Q,M) &= (-1)^{r-t}e_{r-t}(Q,N) + e_0(Q,U)\\ &= \sum_{j=1}^{t}\binom{t-1}{j-1}\lambda_R(\mathrm{H}^j_{\mathfrak{m}}(N)) + n^t e_0((x_1,\ldots,x_t),U)\\ &\geq n^t.\end{aligned}$$

Thus $\Lambda_{r-t}(M)$ is not finite which implies that $r-t \geq k+1$. Thus $t \leq r-(k+1)$. Consequently, for all $i \geq r-k$,

$$\mathrm{H}^i_{\mathfrak{m}}(U) = 0. \tag{4.19}$$

The exact sequence $0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$ implies that

$$\mathrm{H}^i_{\mathfrak{m}}(M) \simeq \mathrm{H}^i_{\mathfrak{m}}(N) \text{ for all } i \geq t+1. \tag{4.20}$$

Hence, using (4.19), we get that $\mathrm{H}^i_{\mathfrak{m}}(M)$ has finite length for all $r-k \leq i \leq r-1$. □

We recall below a result of Cuong et al. which provides a uniform bound on $\operatorname{reg}(G_Q(M))$.

Theorem 4.2.10. [CLT15, Corollary 4] *Let M be a generalized Cohen-Macaulay module. Then, there exists a constant C such that $\operatorname{reg}(G_Q(M)) \leq C$ for all parameter ideals Q for M.*

It is now easy to see that the sets $\Lambda_i(M)$ are finite for $1 \leq i \leq d$ for a generalized Cohen-Macaulay module M. In the following theorem, we show that it is an if and only if statement. Indeed, finiteness of the sets of first few coefficients implies the finiteness for the rest of the higher coefficients.

Theorem 4.2.11. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension $r \geq 2$. Then the following conditions are equivalent:*

(a) M is a generalized Cohen-Macaulay module;

(b) The set $\Lambda_i(M)$ *is finite for* $1 \leq i \leq r$*;*

(c) The set $\Lambda_i(M)$ *is finite for* $1 \leq i \leq r - \operatorname{depth} M$ *or* $\operatorname{depth} M = r$.

Proof. (a) $\Rightarrow$ (b) Follows from Theorems 4.2.8 and 4.2.10.
(b) $\Rightarrow$ (c) This is clear.
(c) $\Rightarrow$ (a) Let $\operatorname{depth} M \neq r$ and $\Lambda_i(M)$ be finite for $1 \leq i \leq r - \operatorname{depth} M$. By Lemma 4.2.9, $\lambda_R(\mathrm{H}^{r-i}_{\mathfrak{m}}(M)) < \infty$ for $1 \leq i \leq r - \operatorname{depth} M$. Since $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i < \operatorname{depth} M$, M is generalized Cohen-Macaulay. □

We now discuss an example of a local ring for which the sets $\Lambda_i(R)$ are finite except for $i = \dim R - \operatorname{depth} R$ and the ring is not generalized Cohen-Macaulay. This highlights the significance of the finiteness of $\Lambda_i(M)$ for $i = \dim M - \operatorname{depth} M$. This example is motivated by the idea presented in [GO11].

Example 4.2.12. *Let* $(R, \mathfrak{n})$ *be a regular local ring of dimension* $d \geq 2$ *and* $X_1, \ldots, X_d$ *a regular system of parameters of* R*. We put* $\mathfrak{p}_t = (X_1, \ldots, X_{d-t})$ *for some* $1 \leq t \leq d-1$ *and* $D = R/\mathfrak{p}_t$*. Let* $A = R \ltimes D$ *be the idealization of* D *over* R*. Then* A *is a Noetherian local ring with the maximal ideal* $\mathfrak{m} = \mathfrak{n} \times D$ *and* $\dim A = \dim R = d$*. Consider the exact sequence of* A*-modules*

$$0 \longrightarrow D \xrightarrow{j} A \xrightarrow{p} R \longrightarrow 0 \tag{4.21}$$

where $j(x) = (0, x)$ *and* $p(a, x) = a$*. Note that* D *is an* A*-module via* p*. By depth lemma on the exact sequence* (4.21), $\operatorname{depth} A = t$*. Let* Q *be a parameter ideal in* A *and* $q = p(Q) \subseteq R$*. Then we have,*

$$\begin{aligned}\lambda_A(A/Q^{n+1}) &= \lambda_R(R/q^{n+1}) + \lambda_R(D/q^{n+1}D)\\ &= e_0(q, R)\binom{n+d}{d} + \sum_{i=0}^{t}(-1)^i e_i(q, D)\binom{n+t-i}{t-i}\end{aligned}$$

for all $n \gg 0$*. This implies*

$$e_i(Q, A) = \begin{cases} e_0(q, R) & \text{if } i = 0\\ 0 & \text{if } 1 \leq i \leq d-t-1\\ (-1)^{d-t} e_{i-d+t}(q, D) & \text{if } d-t \leq i \leq d.\end{cases} \tag{4.22}$$

In particular, let $t = 1$. Then $e_1(q, D) = 0$ since D is a DVR. Therefore, $e_i(Q, A) = 0$ for $i \neq 0, d-1$ and $e_{d-1}(Q, A) = (-1)^{d-1}e_0(q, D)$. Now let $q = (X_1^{n_1}, \ldots, X_d^{n_d}) \subseteq R$ for some integers $n_i \geq 1$ and $Q = qA$. Then by (4.22) $e_0(Q, A) = e_0(q, R) = n_1 \cdot n_2 \cdots n_d$ and $e_{d-1}(Q, A) = (-1)^{d-1}e_0(q, D) = (-1)^{d-1}n_d$. Hence

$$\Lambda_i(A) = \begin{cases} \{n \mid 0 < n \in \mathbb{Z}\} & \text{if } i = 0 \\ \{0\} & \text{if } 1 \leq i \leq d \text{ and } i \neq d-1 \\ \{(-1)^{d-1}n \mid 0 < n \in \mathbb{Z}\} & \text{if } i = d-1. \end{cases} \tag{4.23}$$

Here, $\operatorname{depth} A = 1$ *and $\mathrm{H}^1_{\mathfrak{m}}(A) \cong \mathrm{H}^1_{\mathfrak{n}}(D)$ is not finitely generated A-module. Hence A is not generalized Cohen-Macaulay.*

As a consequence of Theorem 4.2.10, we obtain a characterization of generalized Cohen-Macaulay rings in terms of the coefficients $g_i^K(Q)$.

Theorem 4.2.13. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and K an $\mathfrak{m}$-primary ideal of R. Then the following conditions are equivalent:*

(a) R is generalized Cohen-Macaulay;

(b) $\Lambda_i^K(R)$ is finite for all $1 \leq i \leq d$;

(c) $\Lambda_i^K(R)$ is finite for all $1 \leq i \leq d-1$;

(d) $\delta_i^K(R)$ is finite for all $1 \leq i \leq d-1$.

Proof. By Remark 2.1.8, (a) is equivalent to the generalized Cohen-Macaulayness of K. By (2.9), we have $|\Lambda_i(K)| = |\Lambda_i^K(R)|$ for $1 \leq i \leq d$. Hence (a) $\Rightarrow$ (b) follows from Theorem 4.2.11. (b) $\Rightarrow$ (c) $\Rightarrow$ (d) is obvious. We show (d) $\Rightarrow$ (a). Since $\delta_i^K(R)$ is finite, by (2.9),

$$\{e_i(Q, K) : Q \text{ is a parameter ideal of } R \text{ and } Q \subseteq K\}$$

is finite for all $1 \leq i \leq d-1$. Therefore by Lemma 4.2.9, K is a generalized Cohen-Macaulay R-module. Thus R is generalized Cohen-Macaulay. □

In the following theorem we give a characterization for $M/\mathrm{H}^0_{\mathfrak{m}}(M)$ to be Buchsbaum in terms of $\Lambda_i(M)$.

Theorem 4.2.14. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension $r \geq 2$. Then the following statements are equivalent:*

(a) $M/\operatorname{H}^0_{\mathfrak{m}}(M)$ is a Buchsbaum R-module;

(b) $|\Lambda_i(M)| = 1$ for all $1 \leq i \leq r$;

(c) $|\Lambda_i(M)| = 1$ for all $1 \leq i \leq r - \operatorname{depth} M$ or $\operatorname{depth} M = r$.

Proof. (a) $\Rightarrow$ (b) Let $M' = M/\operatorname{H}^0_{\mathfrak{m}}(M)$. Since M' is Buchsbaum, every parameter ideal Q for M' is standard. Hence by Corollary 2.1.12, $e_i(Q, M') = (-1)^i \sum\limits_{j=0}^{r-i} \binom{r-i-1}{j-1} \lambda_R(\operatorname{H}^j_{\mathfrak{m}}(M'))$ for all $1 \leq i \leq r$. Thus $|\Lambda_i(M')| = 1$ for all $1 \leq i \leq r$. Hence, using (2.5), $|\Lambda_i(M)| = |\Lambda_i(M')| = 1$ for all $1 \leq i \leq r$.
(b) $\Rightarrow$ (c) This is clear.
(c) $\Rightarrow$ (a) Let $\operatorname{depth} M \neq r$ and $|\Lambda_i(M)| = 1$ for all $1 \leq i \leq r - \operatorname{depth} M$. By (2.5), $|\Lambda_i(M)| = |\Lambda_i(M')| = 1$ for all $1 \leq i \leq r - \operatorname{depth} M$. Hence M' is a generalized Cohen-Macaulay R-module by Theorem 4.2.11. This implies that $\widehat{M'}$ is a generalized Cohen-Macaulay $\widehat{R}$-module. Since $\operatorname{depth}_{\widehat{R}} \widehat{M'} > 0$, using Lemma 2.1.10 we conclude that M' is an unmixed module. Hence M' is a Buchsbaum R-module by Theorem 4.0.16. □

As a consequence we give a sufficient condition for $R/\operatorname{H}^0_{\mathfrak{m}}(R)$ to be Buchsbaum in terms of $\Lambda^K_i(R)$.

Theorem 4.2.15. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and K an $\mathfrak{m}$-primary ideal of R.*

1. *Suppose $|\Lambda^K_i(R)| = 1$ for all $1 \leq i \leq d - 1$. Then $R/\operatorname{H}^0_{\mathfrak{m}}(R)$ is Buchsbaum.*
2. *If $R/\operatorname{H}^0_{\mathfrak{m}}(R)$ is Buchsbaum then $|\Lambda^{\mathfrak{m}}_i(R)| = 1$ for all $1 \leq i \leq d$.*

Proof. (1) By (2.9), $|\Lambda^K_i(R)| = |\Lambda_i(K)|$. Hence putting $M = K$ in Theorem 4.2.14, we get that $K/\operatorname{H}^0_{\mathfrak{m}}(K) \cong (K + \operatorname{H}^0_{\mathfrak{m}}(R))/\operatorname{H}^0_{\mathfrak{m}}(R) \subseteq R/\operatorname{H}^0_{\mathfrak{m}}(R)$ is Buchsbaum. Thus, by Lemma 4.1.5(1), $R/\operatorname{H}^0_{\mathfrak{m}}(R)$ is Buchsbaum.

(2) By Lemma 4.1.5(2), $\mathfrak{m}/\operatorname{H}^0_{\mathfrak{m}}(\mathfrak{m})$ is a Buchsbaum R-module. Since $|\Lambda^{\mathfrak{m}}_i(R)| = |\Lambda_i(\mathfrak{m})|$, the result follows from Theorem 4.2.14. □

4.3 The set $\Delta^K(R)$

We now consider the coefficients $g_1^K(I)$ where I is an $\mathfrak{m}$-primary ideal and discuss the finiteness of the sets

$$\Delta^K(R) := \{g_1^K(I) \mid I \text{ is an } \mathfrak{m}\text{-primary ideal of } R\} \text{ and}$$

$$\Delta_R(M) := \{e_1(I, M) \mid I \text{ is an } \mathfrak{m}\text{-primary ideal of } R\}.$$

Our methods include a treatment to the coefficients $e_1(I, M)$ for an $\mathfrak{m}$-primary ideal I. We obtain necessary and sufficient conditions for the finiteness of $\Delta_R(M)$ and $\Delta^K(R)$. The statements in this section can be seen as generalized versions of results of [KT15] which are in the case of rings. We recall the following results from [KT15] for our use.

Theorem 4.3.1. [KT15, Theorem 1.1] *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. Then the following conditions are equivalent:*

(a) $\Delta_R(R)$ is a finite set;

(b) $d = 1$ and $R/\operatorname{H}^0_{\mathfrak{m}}(R)$ is analytically unramified.

Theorem 4.3.2. [KT15, Theorem 1.2] *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one. Then*

$$\sup \Delta_R(R) = \lambda_R(\overline{R}/R)$$

where $\overline{R}$ denotes the integral closure of R in its total ring of fractions. Hence $\Delta_R(R)$ is finite if and only if R is analytically unramified.

In the following lemma we show that if $\Delta_R(M)$ is finite then $\dim M = 1$.

Lemma 4.3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension $r > 0$. Suppose $\Delta_R(M)$ is a finite set. Then $r = 1$.*

Proof. Let I be an $\mathfrak{m}$-primary ideal of R and $k \geq 1$ an integer. We have

$$\lambda_R(M/(I^k)^{n+1}M) = e_0(I^k, M)\binom{n+r}{r} - e_1(I^k, M)\binom{n+r-1}{r-1} + \ldots + (-1)^r e_r(I^k, M). \tag{4.24}$$

Also

$$\lambda_R(M/I^{kn+k}M) = e_0(I, M)\binom{(kn+k-1)+r}{r} - e_1(I, M)\binom{(kn+k-1)+r-1}{r-1}$$

$$+ \ldots + (-1)^r e_r(I, M). \tag{4.25}$$

Note that

$$\binom{kn+k+r-1}{r} = k^r \binom{n+r}{r} + \frac{r-1}{2}\left(k^{r-1} - k^r\right)\binom{n+r-1}{r-1}$$

$$+ \text{ lower degree terms}$$

$$\binom{kn+k+r-2}{r-1} = k^{r-1}\binom{n+r-1}{r-1} + \text{lower degree terms.}$$

Comparing (4.24) and (4.25), we get

$$e_0(I^k, M) = k^r e_0(I, M) \text{ and}$$

$$e_1(I^k, M) = \frac{r-1}{2} k^r e_0(I, M) + \frac{2e_1(I, M) - (r-1)e_0(I, M)}{2} k^{r-1}. \tag{4.26}$$

Since $\Delta_R(M)$ is a finite set, the set $\{e_1(I^k, M) \mid k \geq 1 \text{ is an integer}\}$ is also finite. Hence (4.26) implies $r = 1$. □

In view of Lemma 4.3.3, we may assume that $\dim M = 1$ while examining the finiteness of the set $\Delta_R(M)$. We obtain bounds on this set in the following proposition. Let $\mu_R(M)$ denote the minimal number of generators of an R-module M.

Proposition 4.3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension one and M a finitely generated R-module of dimension one. Then*

1. $\inf \Delta_R(M) = -\lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M))$.

2. $\sup \Delta_R(M) \leq \lambda_{R'}(\overline{R'}/R')\mu_{R'}(M')$, *where* $R' = R/\mathrm{H}^0_{\mathfrak{m}}(R)$ *and* $M' = M/\mathrm{H}^0_{\mathfrak{m}}(M)$. *Here $\overline{R'}$ denotes the integral closure of R' in its total ring of fractions.*

Proof. (1) Let $c = \inf \Delta_R(M)$. By (2.5), for every $\mathfrak{m}$-primary ideal I in R,

$$e_1(I, M) = e_1(I, M') - \lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M)). \tag{4.27}$$

Since M' is Cohen-Macaulay, $e_1(I, M') \geq 0$ by Northcott's inequality for modules (see [Fil67, p. 218]). Thus $e_1(I, M) \geq -\lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M))$ which implies that $c \geq -\lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M))$. Let $Q = (x)$ be a parameter ideal for M. Then, by (4.27), $e_1(Q, M) = -\lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M))$. Hence $c = -\lambda_R(\mathrm{H}^0_{\mathfrak{m}}(M))$.

(2) Let $C = \sup \Delta_R(M)$. Note that M' is a maximal Cohen-Macaulay R'-module. Hence, for every $\mathfrak{m}$-primary ideal I of R, we have

$$
\begin{aligned}
e_1(I, M) &\le e_1(I, M') && \text{[by (4.27)]} \\
&= e_1(IR', M') \\
&\le e_1(IR', M') + e_1(IR', \mathrm{Syz}_1^{R'}(M')) \\
& \qquad \text{[as } \mathrm{Syz}_1^{R'}(M') \text{ is a Cohen-Macaulay } R'\text{-module]} \\
&\le e_1(IR', R')\mu_{R'}(M') && \text{[by [Put03, Proposition 17]} \\
&\le \lambda_{R'}(\overline{R'}/R')\mu_{R'}(M'). && \text{[by Theorem 4.3.2].}
\end{aligned}
$$

Hence $C \le \lambda_{R'}(\overline{R'}/R')\mu_{R'}(M')$. □

Note that in order to obtain bounds on the set $\Delta_R(M)$, the ring R having dimension one in Proposition 4.3.4 is not a restrictive condition as we may pass to $R/\operatorname{Ann}_R(M)$ if needed and assume that $\dim R = \dim M = 1$.

Proposition 4.3.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a Cohen-Macaulay R-module of dimension one. For nonzero modules N and C, consider the exact sequence*

$$0 \longrightarrow N \longrightarrow M \longrightarrow C \longrightarrow 0. \tag{4.28}$$

For an $\mathfrak{m}$-primary ideal I in R, the following statements hold.

1. *If* $\dim\ C = 0$, *then* $e_1(I, M) \ge e_1(I, N) - \lambda_R(C)$.
2. *If* $\dim\ C = 1$, *then* $e_1(I, M) \ge e_1(I, N) + e_1(I, C) \ge e_1(I, N) - \lambda_R(\mathrm{H}^0_{\mathfrak{m}}(C))$.

Proof. Tensoring (4.28) with R/I^{n+1}, we get an exact sequence

$$0 \longrightarrow K_{I,n+1} \longrightarrow \frac{N}{I^{n+1}N} \longrightarrow \frac{M}{I^{n+1}M} \longrightarrow \frac{C}{I^{n+1}C} \longrightarrow 0,$$

where $K_{I,n+1}$ (depends on I and n) is some R-module of finite length. Therefore,

$$\lambda_R(K_{I,n+1}) - \lambda_R\left(\frac{N}{I^{n+1}N}\right) + \lambda_R\left(\frac{M}{I^{n+1}M}\right) - \lambda_R\left(\frac{C}{I^{n+1}C}\right) = 0. \tag{4.29}$$

This implies that $\lambda_R(K_{I,n+1})$ is a polynomial, for $n \gg 0$, of degree at most one. Let $\lambda_R(K_{I,n+1}) = a_I(n+1) + b_I$, for $n \gg 0$, where a_I and b_I are some integers. Since M

is Cohen-Macaulay, N is a Cohen-Macaulay module of dimension one. Hence, by using [BH98, Corollary 4.7.7], we get $a_I = 0$ and hence $\lambda_R(K_{I,n+1}) = b_I$ for $n \gg 0$.

(1) Suppose that dim $C = 0$. Then $I^n C = 0$ for $n \gg 0$. Hence from (4.29), we get that

$$e_1(I, M) = b_I + e_1(I, N) - \lambda_R(C) \geq e_1(I, N) - \lambda_R(C).$$

(2) Suppose $\dim C = 1$. Again from (4.29), we get that

$$\begin{aligned} e_1(I, M) &= b_I + e_1(I, N) + e_1(I, C) \\ &\geq e_1(I, N) + e_1(I, C) \\ &\geq e_1(I, N) - \lambda_R(\mathrm{H}^0_{\mathfrak{m}}(C)) \qquad \text{[by Proposition 4.3.4(1)].} \end{aligned}$$

□

In the next theorem, we give a necessary condition for the finiteness of the set $\Delta_R(M)$. We first consider the case when M is a cyclic module of dimension one.

Lemma 4.3.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M = Rx$ a Cohen-Macaulay R-module of dimension one. Suppose $\Delta_R(M)$ is finite. Then $R/\operatorname{Ann}_R(M)$ is analytically unramified.*

Proof. Note that $M \cong R/\operatorname{Ann}_R(x)$. Let $B = R/\operatorname{Ann}_R(x)$. Since $\lambda_R(B/I^n B) = \lambda_B(B/I^n B)$ for any $\mathfrak{m}$-primary ideal I in R, $e_1(I, B) = e_1(IB, B)$. Since every $\mathfrak{m}B$-primary ideal in B is of the form IB for some $\mathfrak{m}$-primary ideal I in R, finiteness of the set $\Delta_R(M)$ implies that the set $\Delta_B(B)$ is finite. Therefore, by Theorem 4.3.1, B is analytically unramified. □

Theorem 4.3.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension $r > 0$. Let $R' = R/\mathrm{H}^0_{\mathfrak{m}}(R)$ and $M' = M/\mathrm{H}^0_{\mathfrak{m}}(M)$. Then the following conditions are equivalent:*

(a) $\Delta_R(M)$ is a finite set;

(b) $r = 1$ and $R'/\operatorname{Ann}_{R'}(M')$ is analytically unramified.

Proof. (a) $\Rightarrow$ (b) Since $\Delta_R(M)$ is finite, by Lemma 4.3.3, $r = 1$. Thus M' is a Cohen-Macaulay R'-module of dimension one. From (4.27), $|\Delta_R(M)| = |\Delta_R(M')|$ which implies that $\Delta_R(M')$ is a finite set. Since $e_1(I, M') = e_1(IR', M')$, we get that $\Delta_{R'}(M')$ is a finite set. Let $M' = R'x_1 + R'x_2 + \cdots + R'x_m$, where $0 \neq R'x_i \subseteq M'$ is a submodule of M'. Set $N_i := R'x_i$. Since M' is Cohen-Macaulay, N_i is a Cohen-Macaulay R'-module of dimension one. Hence, by Proposition 4.3.5 and Northcott's inequality on modules, for every $\mathfrak{m}$-primary ideal I in R

$$0 \leq e_1(IR', N_i) \leq e_1(IR', M') + c_i,$$

for some non-negative integer c_i which is independent of I. Thus finiteness of the set $\Delta_{R'}(M')$ implies that the set $\Delta_{R'}(N_i)$ is finite for every i. Hence, by Lemma 4.3.6, $R'/\operatorname{Ann}_{R'}(R'x_i)$ is analytically unramified for each i. Let $I_i = \operatorname{Ann}_{R'}(R'x_i)$. Since $\widehat{R'}/I_i\widehat{R'}$ is reduced for each i, $\widehat{R'}/\left(\bigcap_{i=1}^{m} I_i\widehat{R'}\right)$ is reduced. Also, as $\widehat{R'}$ is a flat R'-module,

$$\widehat{\operatorname{Ann}_{R'}(M')} = \operatorname{Ann}_{R'}(M')\widehat{R'} = \left(\bigcap_{i=1}^{m} I_i\right)\widehat{R'} = \bigcap_{i=1}^{m} I_i\widehat{R'}.$$

Hence $\widehat{R'}/\widehat{\operatorname{Ann}_{R'} M'} \cong \widehat{\left(\frac{R'}{\operatorname{Ann}_{R'} M'}\right)}$ is reduced. Thus $R'/\operatorname{Ann}_{R'}(M')$ is analytically unramified.

(b) $\Rightarrow$ (a) Note that $\dim R'/\operatorname{Ann}_{R'}(M') = \dim M' = 1$. Since $R'/\operatorname{Ann}_{R'}(M')$ is analytically unramified, by Proposition 4.3.4, $\Delta_R(M)$ is a finite set. □

As a consequence we give an equivalent criterion for the finiteness of the set $\Delta^K(R)$.

Theorem 4.3.8. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and K an $\mathfrak{m}$-primary ideal of R. Then the following conditions are equivalent:*

(a) $\Delta^K(R)$ is a finite set;

(b) $d = 1$ and $R/\operatorname{H}^0_{\mathfrak{m}}(R)$ is analytically unramified.

Proof. By (2.9), $|\Delta^K(R)| = |\Delta_R(K)|$. Let $R' = R/\operatorname{H}^0_{\mathfrak{m}}(R)$. Since $\operatorname{Ann}_{R'}(KR') = 0$, using Theorem 4.3.7 we get the result. □

In what follows, we will give a description of the set $\Delta^K(R)$ and determine its supremum when R is Cohen-Macaulay.

Theorem 4.3.9. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one with infinite residue field.*

1. *Let $S = \{x \in R : x \text{ is } R\text{-regular}\}$. For a maximal Cohen-Macaulay R-module M,*
$$\Delta_R(M) \subseteq \{\lambda_R(N/M) : M \subseteq N \subseteq S^{-1}M,\ N \text{ is a finitely generated } R\text{-module}\}.$$
2. *For an $\mathfrak{m}$-primary ideal K in R, $\Delta^K(R)$ is same as*
$$\{\lambda_R(KB/K) - \lambda_R(R/K) : R \subseteq B \subseteq \overline{R},\ B \text{ is a finitely generated } R\text{-module}\}.$$
Further, $\sup \Delta^K(R) = \lambda_R(K\overline{R}/K) - \lambda_R(R/K)$.

Proof. (1) Let I be an $\mathfrak{m}$-primary ideal in R. Let $J = (x) \subseteq I$ be a minimal reduction of I. Since R (resp. M) is Cohen-Macaulay, x is a non-zero-divisor on R and M. We set
$$\frac{I^n}{x^n} = \left\{\frac{a}{x^n} : a \in I^n\right\} \subseteq S^{-1}R.$$
Let $s = r_J(I)$ and $N = M[\frac{I}{x}] \subseteq S^{-1}M$. Then $M \subseteq N = \bigcup\limits_{n \geq 0} \frac{I^n M}{x^n} = \frac{I^n M}{x^n} \cong I^n M$ for $n \geq s$. Thus N is a finitely generated R-module. We claim that $e_1(I, M) = \lambda_R(N/M)$. We have
$$\begin{aligned}\lambda_R\left(\frac{M}{I^{n+1}M}\right) &= \lambda_R\left(\frac{M}{J^{n+1}M}\right) - \lambda_R\left(\frac{I^{n+1}M}{J^{n+1}M}\right)\\ &= e_0(I, M)(n+1) - \lambda_R\left(\frac{I^{n+1}M}{J^{n+1}M}\right) \qquad \text{for } n \gg 0.\end{aligned}$$
This implies that $e_1(I, M) = \lambda_R\left(\frac{I^{n+1}M}{J^{n+1}M}\right)$ for $n \gg 0$. Since $\frac{I^{n+1}M}{J^{n+1}M} \cong \frac{N}{M}$ for $n \gg 0$, $e_1(I, M) = \lambda_R(N/M)$.

(2) Let $\Gamma(R) := \{\lambda_R(KB/K) : R \subseteq B \subseteq \overline{R},\ B \text{ is a finitely generated } R\text{-module}\}$. First we show that $\Delta_R(K) = \Gamma(R)$. By part (1), $e_1(I, K) = \lambda_R(N/K)$, where $N = K[\frac{I}{x}]$ and (x) is a minimal reduction of I. Put $B = R[\frac{I}{x}]$. Let $s = r_{(x)}(I)$. Then $B = \frac{I^n}{x^n} \cong I^n$ for all $n \geq s$. Thus B is a finitely generated R-module which implies that $B \subseteq \overline{R}$. Also, $KB = K[\frac{I}{x}] = N$. Hence $e_1(I, K) = \lambda_R(KB/K) \in \Gamma(R)$.

Now, let $R \subseteq B \subseteq \overline{R}$ and B is finitely generated R-module. Then there exists a non-zero-divisor $x \in R$ such $xB \subseteq R$. Let $I = xB$. Then I is an $\mathfrak{m}$-primary ideal in R and $I^2 = xI$. Hence $R[\frac{I}{x}] = \frac{I}{x} = B$. A similar argument as above shows that $e_1(I, K) = \lambda_R(KB/K)$. Hence $\Gamma(R) \subseteq \Delta_R(K)$. Thus $\Gamma(R) = \Delta_R(K)$. Now (2) follows from (2.9).

Let $C := \sup \Delta^K(R)$. Then from the description of the set, $C \leq \lambda_R(K\overline{R}/K) - \lambda_R(R/K)$. Hence in order to prove the second assertion we may assume that C is finite. Then, by Theorem 4.3.8, R is analytically unramified and hence $\overline{R}$ is a finite R-module. Again using the set description, we get $C \geq \lambda_R(K\overline{R}/K) - \lambda_R(R/K)$. □

Remark 4.3.10. *1. The containment in Theorem 4.3.9(1) can be strict. Let R be a Cohen-Macaulay local ring of dimension one and I an $\mathfrak{m}$-primary ideal. Choose an integer t such that $e_1(I,R)$ is not divisible by t. Let $M = R^t$. Then $e_1(J,M) = te_1(J,R)$ for every $\mathfrak{m}$-primary ideal J in R. By Theorem 4.3.9(1), $e_1(I,R) = \lambda_R(B/R)$, for a finite R-module B such that $R \subseteq B \subseteq S^{-1}R$. Now, $R^t \subseteq N := B \oplus R \oplus \cdots \oplus R \subseteq (S^{-1}R)^t$ and N is a finite R-module. Also, $\lambda_R(N/M) = \lambda_R(B/R) = e_1(I,R)$. Suppose there exists an $\mathfrak{m}$-primary ideal J in R such that $e_1(J,M) = \lambda_R(N/M) = e_1(I,R)$. Then $te_1(J,R) = e_1(I,R)$ which is a contradiction. This implies that the containment in Theorem 4.3.9(1) can be strict.*

2. Let R be a Cohen-Macaulay local ring of dimension one and $M = R^t$. In this case, $\Delta_R(M) = \{te_1(I,R) : I$ is an $\mathfrak{m}$-primary ideal in $R\}$. Since $\sup \Delta_R(R) = \lambda_R(\overline{R}/R)$ by Theorem 4.3.2. Hence, $\sup \Delta_R(M) = t\lambda_R(\overline{R}/R) = \lambda_R(\overline{R}/R)\mu_R(M)$ which shows that the bound in Proposition 4.3.4(2) can be achieved.

5 Higher coefficients

In this chapter, we focus on the higher coefficients $g_i^K(Q)$ for $i \geq 2$. In Chapter 3, we proved that if R is Cohen-Macaulay and $Q \subseteq K$ is a parameter ideal, then $g_i^K(Q) = (-1)^i \lambda_R(R/K)$ for $1 \leq i \leq d$. We also showed that $g_1^K(Q) \leq -\lambda_R(R/K)$ if depth $R \geq d-1$. It is natural to seek similar bounds for higher coefficients and to obtain the conditions for which the equalities $g_i^K(Q) = (-1)^i \lambda_R(R/K)$ hold true. In this chapter, we characterize the equality $g_2^K(Q) = \lambda_R(R/K)$ under certain assumptions on the form rings. We also examine the difference function $\Delta(f)$, defined below, with $f(n) = P_K(Q,n) - H_K(Q,n)$.

Definition 5.0.11. *Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function. The first difference function $\Delta(f)$ is defined by $\Delta(f(n)) = f(n+1) - f(n)$. We define the i-th difference function $\Delta^i(f)$ by $\Delta^i(f) = \Delta(\Delta^{i-1}(f))$. By convention, $\Delta^0(f) = f$.*

In Section 5.1, we discuss the Hilbert coefficients and the difference function in one dimensional case and show that $P_K(Q,n) - H_K(Q,n) \geq 0$ for all $n \geq 0$. This section builds the base for applying induction in the next section.

In Section 5.2, we characterize the equality $g_2^K(Q) = \lambda_R(R/K)$ for a parameter ideal Q in a ring of depth at least $d-1$ under the assumption that $G(Q)$ has depth at least $d-1$. With the same conditions, we show that the equality $g_2^K(Q) = \lambda_R(R/K)$ enforces high depth on the fiber cone $F_K(Q)$.

The next important result of this chapter is proved in Section 5.3. It restores the behavior, the difference function of $P_K(Q,n) - H_K(Q,n)$ has in one dimension, to higher dimensions subject to some conditions on form rings. Several corollaries follow from this result. Most notably, we get that $g_i^K(Q) \leq (-1)^i \lambda_R(R/K)$ for $1 \leq i \leq d$ and equality for some i forces $g_j((Q) = (-1)^i\lambda_R(R/K)$ for $i \leq j \leq d$.

The methods employed in this chapter are inspired by [Mcc13]. We recall below the so-called Sally machine for fiber cones. This is an an important tool for us.

Lemma 5.0.12. [JV05b, Lemma 2.7] *(Sally-machine for fiber cones) Let $(R, \mathfrak{m})$ be a Noetherian local ring and Q an $\mathfrak{m}$-primary ideal. Let K be an ideal such that $Q \subseteq K$. Let $x \in Q$ be such that x^* (resp. x^o) is superficial in $G(Q)$ (resp. $F_K(Q)$). Let $R_1 = R/(x)$. If* $\operatorname{depth} F_{KR_1}(QR_1) \geq 1$*, then x^o is regular in $F_K(Q)$.*

As an easy application of the above lemma, we state a general version to be able to apply induction efficiently in the results concerning depth of fiber cones.

Lemma 5.0.13. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and Q an $\mathfrak{m}$-primary ideal. Let K be an ideal such that $Q \subseteq K$. Let $x_1, \ldots, x_k \in Q$ be such that $x_1^*, \ldots, x_k^*$ (resp. $x_1^o, \ldots, x_k^o$) is superficial sequence in $G(Q)$ (resp. $F_K(Q)$). Let $R_k = R/(x_1, \ldots, x_k)$.*

1. *Suppose* $\operatorname{depth} G(Q) \geq k - 1$ *and* $\operatorname{depth} F_K(Q) \geq k$*. Then $x_1^o, \ldots, x_k^o$ is a regular sequence in $F_K(Q)$.*
2. *Suppose* $\operatorname{depth} G(Q) \geq k$ *and* $\operatorname{depth} F_{KR_k}(QR_k) \geq 1$ *Then* $\operatorname{depth} F_K(Q) \geq k + 1$.

Proof. We apply induction on k.

(1) Let $k = 1$. Since $\operatorname{depth} F_K(Q) \geq 1$, we can choose a homogeneous regular element $y \in F_K(Q)$. Since x_1^o is superficial in $F_K(Q)$, so $(0 : x^o) \cap F_K(Q)_n = 0$ for $n \geq n_0$ for some integer n_0. If $w \in F_K(Q)$ be a homogeneous element such that $x^o w = 0$, then $x^o w y^t = 0$ where t is chosen such that $p = \deg(w) + t\deg(y) \geq n_0$. Therefore $wy^t \in (0 : F_K(Q))_p = 0$ which implies $w = 0$. So x^o is a regular element. Now let $k > 1$ and the assertion holds true $k-1$. Then $x_1^o, \ldots, x_{k-1}^o$ is a regular sequence in $F_K(Q)$. By Proposition 2.2.4(2), we may assume that $x_1^*, \ldots, x_{k-1}^*$ is a regular sequence in $G(Q)$. So $F_{KR_{k-1}}(QR_{k-1}) \cong F_K(Q)/(x_1^o, \ldots, x_{k-1}^o)F_K(Q)$. This implies $\operatorname{depth} F_{KR_{k-1}}(QR_{k-1}) \geq 1$ where $R_{k-1} = R/(x_1, \ldots, x_{k-1})$. By $k = 1$ case, x_k^o is regular in $F_{KR_{k-1}}(QR_{k-1})$. Hence $x_1^o, \ldots, x_k^o$ is a regular sequence in $F_K(Q)$.

(2) Set $R_1 = R/(x_1)$. Since $x_1^*, \ldots, x_k^*$ is a regular sequence in $G(Q)$ by Proposition 2.2.4(2), we have $F_{KR_1}(QR_1) \cong F_K(Q)/x_1^o F_K(Q)$. So it is enough to see that $\operatorname{depth} F_{KR_1}(QR_1) \geq k$ and x_1^o is regular in $F_K(Q)$. For $k = 1$, it is evident from Lemma 5.0.12. Let $k > 1$. Then $R_k = R_1/(x_2, \ldots, x_k)R_1$. Since $\operatorname{depth} G(QR_1) =$

depth $G(Q)/x_1^*G(Q) \geq k-1$, we get depth $F_{KR_1}(QR_1) \geq k$ by induction hypothesis. Again by Lemma 5.0.12, x_1^o is a regular element in $F_K(Q)$. $\square$

5.1 Dimension one

In this section, we determine the Hilbert polynomial of a parameter ideal in a one dimensional local ring R. In this case, we prove that the difference of Hilbert polynomial and Hilbert function is positive for all non-negative integers. This is a main ingredient for the results in the next section. For an $\mathfrak{m}$-primary ideal K and a parameter ideal containing $Q=(x) \subseteq K$, consider the following ascending chain of ideals.

$$(x)K = ((x)K :_K x^0) \subseteq ((x^2)K :_K x) \subseteq ((x^3)K :_K x^2) \subseteq \dots$$

We set $l := min\ \{i \mid ((x^{n+1})K :_K x^n) = ((x^{i+1})K :_K x^i) \text{ for all } n \geq i\}$ and

$$\tilde{x} = ((x^{l+1})K :_K x^l).$$

Note that $l \geq 0$.

Proposition 5.1.1. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension one and K an $\mathfrak{m}$-primary ideal. Let $Q=(x) \subseteq K$ be a parameter ideal of R. Then*

1. $g_0^K(Q) = \lambda_R(K/\tilde{x})$ *and* $g_1^K(Q) = \sum\limits_{i=0}^{l-1}\Big(\lambda_R(K/\tilde{x}) - \lambda_R(K/((x^{i+1})K :_K x^i))\Big) - \lambda_R(R/K)$.
2. $P_K(Q,n) - H_K(Q,n) = \sum\limits_{i=n}^{\infty}\Big(\lambda_R(K/((x^{i+1})K :_K x^i)) - \lambda_R(K/\tilde{x})\Big)$ *for all* $n \in \mathbb{Z}$.

Consequently, $P_K(Q,n) - H_K(Q,n) \geq 0$ *for all* $n \geq 0$.

Proof. (1) Note that $(x^i)K/(x^{i+1})K \cong K/((x^{i+1})K :_K x^i)$ for $i \geq 0$ as modules. Therefore,

$$\begin{aligned} H_K(Q,n) = \lambda_R(R/KQ^n) &= \lambda_R(R/K) + \sum_{i=0}^{n-1}\lambda_R((x^i)K/(x^{i+1})K) \\ &= \lambda_R(R/K) + \sum_{i=0}^{n-1}\lambda_R(K/((x^{i+1})K :_K x^i)). \end{aligned}$$

for all $n \in \mathbb{Z}$. In particular for $n \geq l$,

$$\lambda_R(R/KQ^n) = \lambda_R(R/K) + \sum_{i=0}^{l-1} \lambda_R(K/((x^{i+1})K :_K x^i)) + (n-l)\lambda_R(K/((x^{l+1})K :_K x^l)).$$

So the Hilbert polynomial is

$$P_K(Q,n) = (n-l)\lambda_R(K/((x^{l+1})K :_K x^l)) + \sum_{i=0}^{l-1} \lambda_R(K/((x^{i+1})K :_K x^i)) + \lambda_R(R/K)$$

which gives $g_0^K(Q) = \lambda_R(K/\tilde{x})$ and $g_1^K(Q) = \sum_{i=0}^{l-1} \Big(\lambda_R(K/\tilde{x}) - \lambda_R(K/((x^{i+1})K :_K x^i))\Big) - \lambda_R(R/K)$.

(2) Clearly for $n \geq l$,

$$P_K(Q,n) = H_K(Q,n). \tag{5.1}$$

For all $n \leq l-1$,

$$\begin{aligned} P_K(Q,n) - H_K(Q,n) &= \sum_{i=0}^{l-1} \lambda_R(K/((x^{i+1})K :_K x^i)) + (n-l)\lambda_R(K/\tilde{x}) \\ &\quad - \sum_{i=0}^{n-1} \lambda_R(K/((x^{i+1})K :_K x^i)) \\ &= \sum_{i=n}^{l-1} \Big(\lambda_R(K/((x^{i+1})K :_K x^i)) - \lambda_R(K/\tilde{x})\Big) \\ &= \sum_{i=n}^{\infty} \Big(\lambda_R(K/((x^{i+1})K :_K x^i)) - \lambda_R(K/\tilde{x})\Big) \end{aligned}$$

where the last equality holds since $((x^{i+1})K :_K x^i) = \tilde{x}$ for $i \geq l$.

For the rest, note that $((x^{i+1})K :_K x^i) \subseteq \tilde{x}$ implies $\lambda_R(K/((x^{i+1})K :_K x^i)) - \lambda_R(K/\tilde{x}) \geq 0$ for $i \geq 0$. Therefore $P_K(Q,n) - H_K(Q,n) \geq 0$ for $n \geq 0$. □

The corollary below gives a formula for postulation number.

Corollary 5.1.2. *Let R, K and Q be as in Proposition 5.1.1. Then*

$$\eta_K(Q) = \min\{i \mid ((x^{n+1})K :_K x^n) = ((x^{i+1})K :_K x^i) \text{ for all } n \geq i\} - 1.$$

Proof. Say $l = \min\{i \mid ((x^{n+1})K :_K x^n) = ((x^{i+1})K :_K x^i)$ for all $n \geq i\}$. By (5.1), $\eta_K(Q) \leq l-1$. Suppose $P_K(Q,l-1) = H_K(Q,l-1)$. Then by Proposition 5.1.1(2),

$$P_K(Q,l-1) - H_K(Q,l-1) = 0 = \sum_{i=l-1}^{\infty} \Big(\lambda_R(K/((x^{i+1})K :_K x^i)) - \lambda_R(K/\tilde{x})\Big).$$

So $\lambda_R(K/((x^l)K :_{_K} x^{l-1})) = \lambda_R(K/\tilde{x})$ which implies $\tilde{x} = ((x^l)K :_{_K} x^{l-1})$. This contradicts the minimality of l. Hence $\eta_{_K}(Q) = l-1$. □

Corollary 5.1.3. *Let R be a Noetherian local ring of dimension $d \geq 2$ and* $\operatorname{depth} R \geq d-1$. *Let K be an $\mathfrak{m}$-primary ideal and $Q \subseteq K$ a parameter ideal of R. Then*

1. $g_0^K(Q) - g_1^K(Q) \geq \lambda_R(R/KQ) - (d-1)\lambda_R(R/K)$.
2. $f_0^K(Q) \geq e_1(Q) - e_0(Q) - (d-1)\lambda_R(R/K) + \lambda_R(R/KQ)$.

Proof. (1) We may assume that $Q = (x_1, \ldots, x_d)$ such that $x_1, \ldots, x_{d-1} \in Q \setminus KQ$ and $x_1^o, \ldots, x_{d-1}^o$ is superficial sequence in $F_K(Q)$. Since $\operatorname{depth} R \geq d-1$, we may choose $x_1, \ldots, x_{d-1}$ to be a regular sequence in R. Let $R_{d-1} = R/(x_1, \ldots, x_{d-1})$ which is a one dimensional local ring and QR_{d-1} is a parameter ideal of R_{d-1}. By Proposition 5.1.1, for all $n \geq 0$,

$$P_{KR_{d-1}}(QR_{d-1}, n) - H_{KR_{d-1}}(QR_{d-1}, n) \geq 0.$$

Putting $n = 1$ and using the fact that $g_i^K(Q) = g_i^{KR_{d-1}}(QR_{d-1})$ for $i = 0, 1$, from Proposition 2.2.9, we get that

$$g_0^K(Q) - g_1^K(Q) \geq \lambda_R(R/(KQ + (x_1, \ldots, x_{d-1}))). \tag{5.2}$$

Claim. $\lambda_R(R/(KQ+(x_1, \ldots, x_{i+1}))) \geq \lambda_R(R/KQ)-(i+1)\lambda_R(R/K))$ for $0 \leq i \leq d-2$.

Proof of the Claim. Let $R_i = R/(x_1, \ldots, x_i)$ for $0 \leq i \leq d-2$. Note that $\lambda_R(R_i/KR_i) = \lambda_R(R/K)$ and $\lambda_R(R_i/(KQR_i)) = \lambda_R(R/(KQ+(x_1, \ldots, x_i)))$. Using the following exact sequence recursively

$$0 \longrightarrow \frac{(KQR_i : x_{i+1})}{KR_i} \longrightarrow R_i/KR_i \xrightarrow{x_{i+1}} R_i/(KQR_i) \longrightarrow R_i/(KQR_i, x_{i+1}) \longrightarrow 0, \tag{5.3}$$

we get for $0 \leq i \leq d-2$,

$$\begin{aligned}
\lambda_R(R/(KQ + (x_1, \ldots, x_{i+1}))) &\geq \lambda_R(R/(KQ + (x_1, \ldots, x_i))) - \lambda_R(R/K) \\
&\geq \lambda_R(R/(KQ + (x_1, \ldots, x_{i-1}))) - 2\lambda_R(R/K) \\
&\vdots \\
&\geq \lambda_R(R/KQ) - (i+1)\lambda_R(R/K).
\end{aligned} \tag{5.4}$$

□

In particular,

$$\lambda_R(R/(KQ + (x_1, \ldots, x_{d-1})) \geq \lambda_R(R/KQ) - (d-1)\lambda_R(R/K). \tag{5.5}$$

Now (1) follows from (5.2) and (5.5).

(2) Use $f_0^K(Q) = e_1(Q) - g_1^K(Q)$ from (2.11). $\square$

Note that if x_{i+1}^o is a non-zero-divisor on $F_{KR_i}(QR_i)$ for all $0 \leq i \leq d-2$ in above proof, then $(KQR_i : x_{i+1}) = KR_i$ in (5.3). Therefore, we get equality at each step of (5.4) and consequently in (5.5). More precisely, if $x_1^o, \ldots, x_{d-1}^o$ is a regular sequence in $F_K(Q)$ then

$$\lambda_R(R/(KQ + (x_1, \ldots, x_{d-1}))) = \lambda_R(R/KQ) - (d-1)\lambda_R(R/K). \tag{5.6}$$

This relation will be used in the proof of Remark 5.1.5. Observe that the assumption on the depth of R in Corollary 5.1.3 is necessary as evidenced by the following example.

Example 5.1.4. *Let $S = k[[X, Y, Z]]$ be a power series ring over a field k and $I = (XZ, YZ, Z^2)$. Consider the ring $R = S/I = k[[x, y, z]]$. Then* $\dim R = 2$ *and* $\operatorname{depth} R = 0$. *Let $Q = (x, y)$ and $K = (x, y, z)$. Then*

$$P_K(Q, n) = \binom{n+1}{2} + n + 2.$$

So $g_0^K(Q) - g_1^K(Q) = 2$ whereas $\lambda_R(R/KQ) - \lambda_R(R/K) = 4 - 1 = 3$.

Remark 5.1.5. *We highlight the following inequalities for the coefficients of $\mathfrak{m}$-primary ideals $Q \subseteq K$ in a Cohen-Macaulay local ring R with* $\dim R = d$. *These can be seen as a generalization of the Northcott's inequality ($e_1(Q) \geq e_0(Q) - \lambda_R(R/Q)$). We have*

1. *$g_0^K(Q) - g_1^K(Q) \leq \lambda_R(R/KQ)$.*
2. *Suppose* $\operatorname{depth} G(Q) \geq d-1$ *and* $\operatorname{depth} F_K(Q) \geq d-1$, *then $g_0^K(Q) - g_1^K(Q) \leq \lambda_R(R/KQ) - (d-1)\lambda_R(R/K)$.*

Proof. (1) Let $x_1, \ldots, x_{d-1} \in Q \setminus KQ$ be such that $x_1^*, \ldots, x_{d-1}^*$ (resp. $x_1^o, \ldots, x_{d-1}^o$) is a superficial sequence in $G(Q)$ (resp. $F_K(Q)$). Let $R_{d-1} = R/(x_1, \ldots, x_{d-1})$. Since R is

Cohen-Macaulay, we may assume that $x_1, \ldots, x_{d-1}$ is a regular sequence in R. Therefore, by Proposition 2.2.9, $g_i^K(Q) = g_i^{KR_{d-1}}(QR_{d-1})$ for $i = 0, 1$. Applying Northcott's inequality on KR_{d-1} as an R_{d-1}-module and using (2.9), we get

$$\begin{aligned} g_0^K(Q) - g_1^K(Q) &= g_0^{KR_{d-1}}(QR_{d-1}) - g_1^{KR_{d-1}}(QR_{d-1}) \\ &= g_0^{KR_{d-1}}(QR_{d-1}) + \lambda_R(R_{d-1}/KR_{d-1}) - e_1(QR_{d-1}, KR_{d-1}) \\ &= e_0(QR_{d-1}, KR_{d-1}) + \lambda_R(R_{d-1}/KR_{d-1}) - e_1(QR_{d-1}, KR_{d-1}) \\ &\leq \lambda_R(R_{d-1}/KR_{d-1}) + \lambda_R(KR_{d-1}/QKR_{d-1}) \\ &= \lambda_R(R/KQ + (x_1, \ldots, x_{d-1})) \qquad (5.7) \\ &\leq \lambda_R(R/KQ) \end{aligned}$$

(2) Since $\operatorname{depth} G(Q) \geq d-1$ and $\operatorname{depth} F_K(Q) \geq d-1$, we may assume that $x_1^*, \ldots, x_{d-1}^*$ is a regular sequence in $G(Q)$ and $x_1^o, \ldots, x_{d-1}^o$ is a regular sequence in $F_K(Q)$. By (5.6) and (5.7),

$$g_0^K(Q) - g_1^K(Q) \leq \lambda_R(R/KQ) - (d-1)\lambda_R(R/K).$$

□

5.2 Higher coefficients

In this section, we obtain a condition on the coefficient $g_2^K(Q)$ of a parameter ideal Q which ensures almost maximal depth of the fiber cone $F_K(Q)$ provided the associated graded rings have high depth. In order to prove our main results, we first derive a formula for $g_d^K(Q)$ in the following lemma.

Lemma 5.2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and K an $\mathfrak{m}$-primary ideal of R. Let $Q \subseteq K$ be an $\mathfrak{m}$-primary ideal and $x \in Q \setminus KQ$ be such that x^* (resp. x^o) is superficial in $G(Q)$ (resp. $F_K(Q)$). Then for all $t \gg 0$,*

$$\begin{aligned} (-1)^d g_d^K(Q) = \sum_{n=1}^{t} \Big(H_{KR_1}(QR_1, n) - P_{KR_1}(QR_1, n) \Big) - \sum_{n=1}^{t} \lambda_R((KQ^n : x)/KQ^{n-1}) \\ + t\lambda_R(0 : x) + \lambda_R(R/K) \end{aligned}$$

where $R_1 = R/(x)$.

Proof. From the exact sequence

$$0 \longrightarrow \frac{(KQ^n : x)}{KQ^{n-1}} \longrightarrow R/KQ^{n-1} \xrightarrow{x} R/KQ^n \longrightarrow R/(KQ^n, x) \longrightarrow 0,$$

$$\lambda_R(R/(KQ^n, x)) = \lambda_R(R/KQ^n) - \lambda_R(R/KQ^{n-1}) + \lambda_R((KQ^n : x)/KQ^{n-1})$$

for $n \in \mathbb{Z}$. So for all $t \gg 0$,

$$\begin{aligned}
&\sum_{n=1}^{t}\Big(H_{KR_1}(QR_1, n) - P_{KR_1}(QR_1, n)\Big) \\
&= \sum_{n=1}^{t}\Big(\lambda_R(R/KQ^n) - \lambda_R(R/KQ^{n-1})\Big) + \sum_{n=1}^{t}\Big(\lambda_R((KQ^n : x)/KQ^{n-1}) - P_{KR_1}(QR_1, n)\Big) \\
&= \lambda_R(R/KQ^t) - \lambda_R(R/K) + \sum_{n=1}^{t}\lambda_R((KQ^n : x)/KQ^{n-1}) \\
&\quad - \sum_{n=1}^{t}\sum_{i=0}^{d-1}(-1)^i\binom{n+d-2-i}{d-1-i}g_i^{KR_1}(QR_1) \\
&= \sum_{i=0}^{d}(-1)^i\binom{t+d-1-i}{d-i}g_i^K(Q) - \lambda_R(R/K) + \sum_{n=1}^{t}\lambda_R((KQ^n : x)/KQ^{n-1}) \\
&\quad - \sum_{i=0}^{d-1}(-1)^i\binom{t+d-1-i}{d-i}g_i^{KR_1}(QR_1) \\
&= -t\lambda_R(0 : x) + (-1)^d g_d^K(Q) - \lambda_R(R/K) + \sum_{n=1}^{t}\lambda_R((KQ^n : x)/KQ^{n-1})
\end{aligned}$$

where last equality holds since $g_i^{KR_1}(QR_1) = g_i^K(Q)$ for $0 \le i \le d-2$ and $g_{d-1}^{KR_1}(QR_1) = g_{d-1}(Q) + (-1)^{d-1}\lambda_R(0 : x)$ by Proposition 2.2.9. By rearranging the terms, we get

$$\begin{aligned}
(-1)^d g_d^K(Q) = &\sum_{n=1}^{t}\Big(H_{KR_1}(QR_1, n) - P_{KR_1}(QR_1, n)\Big) - \sum_{n=1}^{t}\Big(\lambda_R((KQ^n : x)/KQ^{n-1})\Big) \\
&+ t\lambda_R(0 : x) + \lambda_R(R/K).
\end{aligned}$$

□

Remark 5.2.2. *If x is a non-zero-divisor in above, then $(KQ^n : x) = KQ^{n-1}$ for $n \gg 0$. So,*

$$\begin{aligned}
(-1)^d g_d^K(Q) = &\sum_{n=1}^{\infty}\Big(H_{KR_1}(QR_1, n) - P_{KR_1}(QR_1, n)\Big) - \sum_{n=1}^{\infty}\Big(\lambda_R((KQ^n : x)/KQ^{n-1})\Big) \\
&+ \lambda_R(R/K). \qquad (5.8)
\end{aligned}$$

The following theorem provides a necessary and sufficient condition for the second Hilbert coefficient of a parameter ideal Q with $\operatorname{depth} G(Q) \geq d-2$ to achieve its maximum value in a ring of depth at least $d-1$. This condition also determines the other coefficients.

Theorem 5.2.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and $\operatorname{depth} R \geq d-1$. Let K be an $\mathfrak{m}$-primary ideal and $Q \subseteq K$ a parameter ideal of R. Then*

1. *(a) $g_2^K(Q) \leq \lambda_R(R/K)$.*

   (b) $f_1^K(Q) \geq e_2(Q) + e_1(Q) - e_0(Q) - d\lambda_R(R/K) + \lambda_R(R/KQ)$.

2. *Suppose $\operatorname{depth} G(Q) \geq d-2$. Then $g_2^K(Q) = \lambda_R(R/K)$ if and only if $\eta_K^*(Q) < 2-d$ and $\operatorname{depth} F_K(Q) \geq d-1$.*

3. *$g_2^K(Q) = \lambda_R(R/K)$ implies $g_i^K(Q) = (-1)^i \lambda_R(R/K)$ for $2 \leq i \leq d$.*

Proof. 1(a) Suppose $d = 2$. Let $x_1 \in Q \setminus KQ$ such that x_1^* (resp. x_1^o) is superficial in $G(Q)$ (resp. $F_K(Q)$) and $R_1 = R/(x_1)$. Since $\operatorname{depth} R \geq 1$, x_1 is a non-zero-divisor on R. Then by Remark 5.2.2,

$$\begin{aligned} g_2^K(Q) &= \sum_{n=1}^{\infty} \Big(H_{KR_1}(QR_1, n) - P_{KR_1}(QR_1, n) \Big) - \sum_{n=1}^{\infty} \Big(\lambda_R((KQ^n : x_1)/KQ^{n-1}) \Big) \\ &\quad + \lambda_R(R/K) \qquad (5.9) \\ &\leq \lambda_R(R/K) \end{aligned}$$

where the last inequality holds since $H_{KR_1}(QR_1, n) - P_{KR_1}(QR_1, n) \leq 0$ for all $n \geq 1$ by Proposition 5.1.1.

For $d > 2$, let $Q = (x_1, \ldots, x_d)$ such that $x_1^*, \ldots, x_{d-2}^*$ (resp. $x_1^o, \ldots, x_{d-2}^o$) is a superficial sequence in $G(Q)$ (resp. $K_K(Q)$). Since $\operatorname{depth} R \geq d-1$, we may assume that $x_1, \ldots, x_{d-2}$ is a regular sequence in R. We put $R_{d-2} = R/(x_1, \ldots, x_{d-2})$. Then $\dim R_{d-2} = 2$ and QR_{d-2} is a parameter ideal. So by Proposition 2.2.9 and induction hypothesis, $g_2^K(Q) = g_2^K(QR_{d-2}) \leq \lambda_R(R_{d-2}/KR_{d-2}) = \lambda_R(R/K)$.

1(b) From (2.11),

$$f_1^K(Q) = e_2(Q) - g_2^K(Q) + e_1(Q) - g_1^K(Q)$$

$$\geq e_2(Q) - \lambda_R(R/K) + e_1(Q) - g_1^K(Q) \qquad \text{[by part 1(a)]}$$

$$\geq e_2(Q) + e_1(Q) - e_0(Q) - d\lambda_R(R/K) + \lambda_R(R/KQ) \quad \text{[by Corollary 5.1.3(1)].}$$

(2) Suppose $d = 2$ and $g_2^K(Q) = \lambda_R(R/K)$. Then from (5.9), we get $H_{KR_1}(QR_1, n) = P_{KR_1}(QR_1, n)$ and $(KQ^n : x_1) = KQ^{n-1}$ for $n \geq 1$. This implies $\eta^*_{KR_1}(QR_1) < 1$ and x_1^o is regular in $F_K(Q)$. Now by Lemma 2.1.4, $\eta^*_K(Q) = \eta^*_{KR_1}(QR_1) - 1 < 0$.

Let $d > 2$ and $g_2^K(Q) = \lambda_R(R/K)$. Let R_{d-2} be as in part 1(a). Since depth $R \geq d-1$, so $g_2^{KR_{d-2}}(QR_{d-2}) = g_2^K(Q) = \lambda_R(R/K)$ by Proposition 2.2.9. By induction hypothesis, $\eta^*_{KR_{d-2}}(QR_{d-2}) < 0$ and depth $F_{KR_{d-2}}(QR_{d-2}) \geq 1$. Since depth $G(Q) \geq d-2$, we get depth $F_K(Q) \geq d-1$ with $x_1^o, \ldots, x_{d-1}^o$ a regular sequence in $F_K(Q)$ by Lemma 5.0.13 and $\eta^*_K(Q) = \eta^*_{KR_{d-2}}(QR_{d-2}) - (d-2) < 2-d$.

For the backward implication of (2), suppose $\eta^*_K(Q) < 2-d$. Then

$$P_K(Q, n) = H^*_K(Q, n)$$

for all $n \geq 2-d$. This gives

$$P_K(Q, 0) = (-1)^d g_d^K(Q) = \lambda_R(R/K) \text{ and}$$

$$P_K(Q, n) = \sum_{i=0}^{d} (-1)^i g_i^K(Q) \binom{n+d-1-i}{d-i} = 0 \text{ for } 2-d \leq n \leq -1.$$

On solving the above set of equations, we get $g_i^K(Q) = (-1)^i \lambda_R(R/K)$ for $2 \leq i \leq d$.

(3) $g_2^K(Q) = \lambda_R(R/K)$ implies $\eta^*_K(Q) < 2-d$ by part (2). Therefore it follows from the proof of part (2) that $g_i^K(Q) = (-1)^i \lambda_R(R/K)$ for $3 \leq i \leq d$. □

In the following examples, we see that the condition on depth R in Theorem 5.2.3 can not be relaxed. At the same time, the upper bound for $g_2^K(Q)$ can be achieved even if depth $R < d-1$. In both examples, we use the method of Example 3.2.12 for computing $P_K(Q, n)$.

Example 5.2.4. *Let $S = k[[X, Y, Z]]$ be a power series ring over a field k and $I = (YZ, X^2Y, Y^3)$. Consider the ring $R = S/I = k[[x, y, z]]$. Then $\dim R = 2$ and depth $R = 0$. Let $Q = (x, z^2)$ and $K = (x, y, z^2)$. Then*

$$P_K(Q, n) = 2\binom{n+1}{2} + 2n + 6.$$

So $g_2^K(Q) = 6 > \lambda_R(R/K) = 2$.

Example 5.2.5. *Let* $S = k[[X,Y,Z,W]]$ *be a power series ring and* $I = (YZ, X^2Y, YW, Y^3)$. *Consider the ring* $R = S/I = k[[x,y,z,w]]$. *Then* $\dim R = 3$ *and* $\operatorname{depth} R = 0$. *Let* $Q = (x, z^2, w^2)$ *and* $K = (x, y, z^2, w^2)$. *Then*

$$P_K(Q,n) = 4\binom{n+2}{3} + 4\binom{n+1}{2}n + 4n + 8.$$

We see that $g_2^K(Q) = \lambda_R(R/K) = 4$ *but* $g_3^K(Q) = -8 \neq -\lambda_R(R/K)$.

5.3 The difference function

We now prove the other main result of this chapter concerning the difference function. In Section 5.1 ,we discussed the difference function of $P_K(Q,n) - H_K(Q,n)$ in case of one dimensional local rings. We extend our discussion to dimension d in this section.

Theorem 5.3.1. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d > 0$ *and* K *an* $\mathfrak{m}$*-primary ideal of* R. *Let* $Q \subseteq K$ *be a parameter ideal with* $\operatorname{depth} G(Q) \geq d-1$ *and* $\operatorname{depth} F_K(Q) \geq d-1$. *Then, for* $0 \leq i \leq d+1$ *and* $n \geq 0$,

$$(-1)^i \Delta^{d+1-i}(P_K(Q,n) - H_K(Q,n)) \geq 0.$$

Proof. Suppose $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfies $f(n) = 0$ for $n \gg 0$ and $\Delta(f(n)) \geq 0$ for $n \geq 0$. Then $f(n+1) - f(n) \geq 0$ for $n \geq 0$. This implies $f(n) \leq 0$ for $n \geq 0$. Therefore if $(-1)^i\Delta^{d+1-i}(f(n)) \geq 0$ for $n \geq 0$, then $(-1)^i\Delta^{d+1-i-1}(f(n)) \leq 0$ for $n \geq 0$. So, it is enough to prove the result for $i = 0$ i.e. for all $n \geq 0$,

$$\Delta^{d+1}(P_K(Q,n) - H_K(Q,n)) \geq 0.$$

We use induction on d. Suppose $d = 1$. Then by Proposition 5.1.1,

$$\begin{aligned}
\Delta^2(P_K(Q,n) - H_K(Q,n)) &= \Delta^2 \sum_{i=n}^{\infty} \Big(\lambda_R(K/((x^{i+1})K :_K x^i)) - \lambda_R(K/\tilde{x})\Big) \\
&= \Delta\left(\lambda_R(K/\tilde{x}) - \lambda_R(K/((x^{n+1})K :_K x^n))\right) \\
&= -\lambda_R\left(K/((x^{n+2})K :_K x^{n+1})) + \lambda_R(K/((x^{n+1})K :_K x^n)\right) \\
&\geq 0.
\end{aligned}$$

Let $d > 1$. Let $x \in Q \setminus KQ$ be a part of the generating set of Q such that x^* (resp. x^o) is superficial in $G(Q)$ (resp. $F_K(Q)$). Since $\operatorname{depth} G(Q) \geq d-1$ and $\operatorname{depth} F_K(Q) \geq$

$d-1$, we may assume that x^o is is regular in $F_K(Q)$ and x^* is regular in $G(Q)$. Hence $(KQ^n : x) = KQ^{n-1}$ for all $n \geq 1$. Let $R_1 = R/(x)$. By induction,

$$\Delta^d(P_{KR_1}(QR_1, n) - H_{KR_1}(QR_1, n)) \geq 0$$

for all $n \geq 0$. From the exact sequence

$$0 \longrightarrow \frac{(KQ^n : x)}{KQ^{n-1}} \longrightarrow R/KQ^{n-1} \xrightarrow{x} R/KQ^n \longrightarrow R/(KQ^n, x) \longrightarrow 0,$$

we get

$$H_{KR_1}(QR_1, n) = H_K(Q, n) - H_K(Q, n-1) \text{ for } n \geq 1.$$

This implies $P_{KR_1}(QR_1, n) = P_K(Q, n) - P_K(Q, n-1)$ for all n. Therefore, for all $n \geq 0$,

$$\begin{aligned}\Delta^{d+1}(P_K(Q, n) - H_K(Q, n)) &= \Delta^d(\Delta(P_K(Q, n) - H_K(Q, n)))\\ &= \Delta^d(P_{KR_1}(QR_1, n+1) - H_{KR_1}(QR_1, n+1))\\ &\geq 0\end{aligned}$$

by induction hypothesis. □

Corollary 5.3.2. *Let R. K and Q be as in Theorem 5.3.1. Suppose* $P_K(Q, k) = H_K(Q, k)$ *for some* $k \geq 0$. *Then* $P_K(Q, n) = H_K(Q, n)$ *for all* $n \geq k$.

Proof. Let $i = d$ in Theorem 5.3.1. Then $(-1)^d\Delta(P_K(Q, n) - H_K(Q, n)) \geq 0$ which gives

$$(-1)^d(P_K(Q, n+1) - H_K(Q, n+1)) \geq (-1)^d(P_K(Q, n) - H_K(Q, n))$$

for all $n \geq 0$. Since $P_K(Q, n) - H_K(Q, n) = 0$ for $n \gg 0$, we get

$$0 \geq (-1)^d(P_K(Q, n) - H_K(Q, n)) \geq (-1)^d(P_K(Q, k) - H_K(Q, k)) = 0 \quad \text{for} \quad n \geq k.$$

This implies $P_K(Q, n) = H_K(Q, n)$ for all $n \geq k$. □

In the following corollaries, we obtain some inequalities for higher coefficients $g_i^K(Q)$ provided the form rings have high depths. With this condition, we also see that if $g_i^K(Q)$ attains the boundary value $(-1)^i\lambda_R(R/K)$ for some i, then $g_j^K(Q)$ for $j \geq i$ also attains the value $(-1)^j\lambda_R(R/K)$.

Corollary 5.3.3. *Let R. K and Q be as in Theorem 5.3.1. Then*

1. (a) $g_i^K(Q) \leq (-1)^i \lambda_R(R/K)$ *for* $1 \leq i \leq d$.

   (b) $(-1)^{i+1}\Big(g_0^K(Q) - g_1^K(Q) + \ldots + (-1)^i g_i^K(Q) - (\lambda_R(R/KQ) - (d - i)\lambda_R(R/K))\Big) \geq 0$ *for* $1 \leq i \leq d$.

2. $f_i^K(Q) \geq e_{i+1}(Q) + e_i(Q)$ *for* $1 \leq i \leq d-1$.

Proof. 1(a) It is enough to prove the result for $i = d$. We can then use reduction modulo superficial sequence and Proposition 2.2.9 to get $g_i^K(Q) \leq (-1)^i \lambda_R(R/K)$. Putting $i = d+1$ in Theorem 5.3.1, we get

$$(-1)^{d+1}(P_K(Q,n) - H_K(Q,n)) \geq 0 \text{ for all } n \geq 0. \tag{5.10}$$

For $n = 0$, this gives $(-1)^{d+1}((-1)^d g_d^K(Q) - \lambda_R(R/K)) \geq 0$ which implies $g_d^K(Q) \leq (-1)^d \lambda_R(R/K)$.

1(b) Putting $n = 1$ in (5.10), we get

$$(-1)^{d+1}(g_0^K(Q) - g_1^K(Q) + \ldots + (-1)^d g_d^K(Q) - \lambda_R(R/KQ)) \geq 0$$

which gives the result for $i = d$. Suppose $1 \leq i \leq d-1$. We may assume that $Q = (x_1, \ldots, x_d)$ such that $x_1^o, \ldots, x_{d-i}^o$ is a superficial sequence in $F_K(Q)$ and $x_1^*, \ldots, x_{d-i}^*$ is a superficial sequence in $G(Q)$. We may choose $x_1, \ldots, x_{d-i}$ such that $x_1^*, \ldots, x_{d-i}^*$ (resp. $x_1^o, \ldots, x_{d-i}^o$) is a regular sequence in $G(Q)$ (resp. $F_K(Q)$). Let $R_{d-i} = R/(x_1, \ldots, x_{d-i})$. Then $\dim R_{d-i} = i$ and QR_{d-i} is a parameter ideal of R_{d-i}. By Proposition 2.2.9, $g_j(QR_{d-i}) = g_j(Q)$ for $0 \leq j \leq i$. Therefore,

$$(-1)^{i+1}\Big(g_0^K(Q) - g_1^K(Q) + \ldots + (-1)^i g_i^K(Q) - \lambda_R(R_{d-i}/KQR_{d-i})\Big) \geq 0.$$

By (5.6), we have

$$\lambda_R(R_{d-i}/KQR_{d-i}) = \lambda_R(R/(KQ + (x_1, \ldots, x_{d-i}))) = \lambda_R(R/KQ) - (d-i)\lambda_R(R/K)$$

This completes the proof of part 1(b).

(2) From (2.11) and part 1(a),

$$\begin{aligned} f_i^K(Q) &= e_{i+1}(Q) - g_{i+1}^K(Q) + e_i(Q) - g_i^K(Q) \\ &\geq e_{i+1}(Q) + (-1)^{i+2}\lambda_R(R/K) + e_i(Q) + (-1)^{i+1}\lambda_R(R/K) \\ &\geq e_{i+1}(Q) + e_i(Q). \end{aligned}$$

$\square$

Corollary 5.3.4. *Let R. K and Q be as in Theorem 5.3.1. Suppose* $g_i^K(Q) = (-1)^i\lambda_R(R/K)$ *for some* $1 \le i \le d-1$. *Then* $g_j(Q) = (-1)^j\lambda_R(R/K)$ *for* $i \le j \le d$.

Proof. It is enough to prove the result for $i = d-1$. Suppose $g_{d-1}(Q) = (-1)^{d-1}\lambda_R(R/K)$. Let $x \in Q \setminus KQ$ be such that x^o is superficial in $F_K(Q)$ and x^* is superficial in $G(Q)$. We may choose x such that x^* (resp. x^o) is regular in $G(Q)$ (resp. $F_K(Q)$). Let $R_1 = R/(x)$. Then $\operatorname{depth} G(QR_1) \ge d-2$ and $\operatorname{depth} F_{KR_1}(QR_1) \ge d-2$. Also,

$$\begin{aligned} P_{KR_1}(QR_1,0) &= (-1)^{d-1}g_{d-1}(QR_1) \\ &= (-1)^{d-1}g_{d-1}(Q) = \lambda_R(R/K) \\ &= H_{KR_1}(QR_1,0). \end{aligned}$$

Hence by Corollary 5.3.2, $P_{KR_1}(QR_1,n) = H_{KR_1}(QR_1,n)$ for all $n \ge 0$. This gives $P_{KR_1}(QR_1,n) = H^*_{KR_1}(QR_1,n)$ for all $n \ge 0$, i.e., $\eta^*_{KR_1}(QR_1) \le -1$ which implies $\eta^*_K(Q) \le -2$ by Lemma 2.1.4. In particular, $(-1)^d g_d^K(Q) = P_K(Q,0) = H^*_K(Q,0) = \lambda_R(R/K)$. □

Remark 5.3.5. *In particular, by putting* $K = Q$ *and using* (2.12), *we get that for* $1 \le i \le d$

$$e_i(Q) = g_i^K(Q) + g_{i-1}^K(Q) \le 0$$

for a parameter ideal Q provided $\operatorname{depth} G(Q) \ge d-1$. *Furthermore, if* $e_i(Q) = 0$ *for some* $i \ge 1$, *then* $(-1)^i\lambda_R(R/K) \le -g_{i-1}^K(Q) = g_i^K(Q) \le (-1)^i\lambda_R(R/K)$. *This implies* $g_i^K(Q) = (-1)^i\lambda_R(R/K)$. *By Corollary 5.3.4,* $g_j^K(Q) = (-1)^j\lambda_R(R/K)$ *for* $j \ge i$. *Thus* $e_j(Q) = g_j^K(Q) + g_{j-1}^K(Q) = 0$ *for all* $j \ge i$.

6

Uniform bounds for Hilbert coefficients

The main theme of this chapter is to find upper and lower bounds for the Hilbert coefficients $e_i(Q)$ for a parameter ideal Q. In case of an $\mathfrak{m}$-primary ideal Q in a Cohen-Macaulay local ring, various relations among Hilbert coefficients have been explored by several authors and bounds are also given. The case when R is not Cohen-Macaulay is quite different. Throughout the chapter, we consider the rings with depth at least $d-1$ and the coefficients $e_i(Q)$ for parameter ideals Q. When the associated graded rings have high depth, we obtain some nice uniform bounds for $e_i(Q)$. A key ingredient of our proofs in this chapter is the following. Given a parameter ideal Q of a local ring of dimension d with $[\mathrm{H}^i_{\mathcal{M}}(G(Q))]_n = 0$ for all $n \leq -1$ and $0 \leq i \leq d-3$, we can obtain an integer $l \gg 0$ such that $\operatorname{depth} G(Q^l) \geq d-2$. To see an immediate advantage of this idea, let $d = 3$. Then there exists an integer $l \gg 0$ such that $\operatorname{depth} G(Q^l) \geq 1$.

In Section 6.1, we discuss some lemmas concerning the local cohomology modules $\mathrm{H}^i_{\mathcal{M}}(G(I))$ for an ideal I. We also recall the statements of a number of results from [Bla97] and [Hoa93]. These results mainly develop the setting for the proofs of our main results in subsequent sections. The first three lemmas of this section are very basic but we include them exclusively for clarity.

In Section 6.2, we prove that $e_d(Q) \leq 0$ if $[\mathrm{H}^i_{\mathcal{M}}(G(Q))]_n = 0$ for all $n \leq -1$ and $0 \leq i \leq d-3$. Note that this condition holds true if $\operatorname{depth} G(Q) \geq d-2$. As a consequence of this result, we show that $e_3(Q) \leq 0$ for a parameter ideal Q if $\operatorname{depth} R \geq d-1$. For the higher coefficients $e_i(Q)$ for $2 \leq i \leq d$, we are able to prove that $e_i(Q) \leq 0$ if $\operatorname{depth} G(Q) \geq d-2$ and $e_i(Q) \geq -\lambda_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R))$ if $\operatorname{depth} G(Q) \geq d-1$.

We discuss the vanishing of the last coefficient $e_d(Q)$ in Section 6.3. If $\operatorname{depth} G(Q) \geq d-2$ and $Q = (x_1, \ldots, x_d)$ with $x_1^*, \ldots, x_{d-1}^*$ a superficial sequence in $G(Q)$, then $e_d(Q) =$

0 implies that $x_1^l, \ldots, x_{d-1}^l, x_d^{(d-1)l}$ is a d-sequence. In particular for the vanishing of $e_2(Q)$, we obtain equivalent conditions. Our results generalize similar results of [GO11] and [Mcc13] . Our proofs are inspired by the methods of [GO11].

6.1 Preliminary results

In this section, we discuss few lemmas which are the key steps for the results of subsequent sections. The main motivation for framing these lemmas is to be able to avoid the higher local cohomology modules wherever we can. For this section, let $(R, \mathfrak{m})$ be a local ring and $I \subseteq \mathfrak{m}$ an arbitrary ideal. It is well known that

$$\operatorname{depth}(G_+, G(I)) = \inf\{i | H^i_{G_+}(G(I)) \neq 0\} \text{ and}$$

$$H^i_{G_+}(G(I)) = 0 \text{ for } i > \dim G(I) = d.$$

Indeed for an $\mathfrak{m}$-primary ideal I, $d = \sup\{i | H^i_{G_+}(G(I)) \neq 0\}$. We frequently use the fact that for all $i \in \mathbb{Z}$,

$$H^i_{\mathcal{M}}(G(I)) = H^i_{\mathcal{R}_+}(G(I)) = H^i_{G_+}(G(I)).$$

It is known that there exists $m_I \in \mathbb{Z}$ such that

$$[\mathrm{H}^i_{\mathcal{M}}(G(I))]_{n+1} = 0 \text{ for all } n \geq m_I \text{ and for all } i \in \mathbb{Z}. \tag{6.1}$$

Lemma 6.1.1. *Let $x \in I \setminus I^2$ such that x^* is a regular element in $G(I)$. Then for all $n \geq m_I$ and for all $i \in \mathbb{Z}$,*

$$[\mathrm{H}^{i+1}_{\mathcal{M}}(G(I))]_n \cong [\mathrm{H}^i_{\mathcal{M}}(G(IR_1))]_{n+1}$$

where $R_1 = R/(x)$. In particular, $[\mathrm{H}^i_{\mathcal{M}}(G(IR_1))]_{n+1} = 0$ for all $n \geq m_I + 1$.

Proof. Since x^* is regular in $G(I)$, we have $G(IR_1) \cong G(I)/x^*G(I)$. Consider the following exact sequence and the induced long exact sequence of local cohomology modules.

$$0 \longrightarrow G(I)(-1) \xrightarrow{x^*} G(I) \longrightarrow G(IR_1) \longrightarrow 0$$

$$\ldots \longrightarrow \mathrm{H}^i_{\mathcal{M}}(G(I)) \longrightarrow \mathrm{H}^i_{\mathcal{M}}(G(IR_1)) \longrightarrow \mathrm{H}^{i+1}_{\mathcal{M}}(G(I))(-1) \longrightarrow \mathrm{H}^{i+1}_{\mathcal{M}}(G(I)) \longrightarrow \ldots$$

The conclusion follows easily. □

Lemma 6.1.2. *Let $x_1, \ldots, x_t \in I \setminus I^2$ such that $x_1^*, \ldots, x_t^*$ is a regular sequence in $G(I)$. Then, for $1 \leq j \leq t$,*

$$[\mathrm{H}^j_{\mathcal{M}}(G(I))]_{m_I} \cong [\mathrm{H}^{j-1}_{\mathcal{M}}(G(IR_1))]_{m_I+1} \cong \ldots \cong [\mathrm{H}^0_{\mathcal{M}}(G(IR_j))]_{m_I+j}$$

where $R_j = R/(x_1, \ldots, x_j)$. Moreover, for all $n \geq m_I + j$ and for all $i \in \mathbb{Z}$,

$$[\mathrm{H}^i_{\mathcal{M}}(G(IR_j))]_{n+1} = 0.$$

Proof. We apply induction on t. Since $[\mathrm{H}^i_{\mathcal{M}}(G(I))]_{n+1} = 0$ for all $n \geq m_I$, we get the result for $t = 1$ from Lemma 6.1.1. Now suppose $t > 1$ and the assertion holds for $t-1$. By Lemma 6.1.1, for all $i \in \mathbb{Z}$,

$$[\mathrm{H}^i_{\mathcal{M}}(G(I))]_{m_I} \cong [\mathrm{H}^{i-1}_{\mathcal{M}}(G(IR_1))]_{m_I+1} \text{ and} \tag{6.2}$$

$$[\mathrm{H}^i_{\mathcal{M}}(G(IR_1))]_{n+1} = 0 \text{ for } n \geq m_I + 1.$$

Since $x_1^*, \ldots, x_t^*$ is a regular sequence in $G(I)$, we have $G(IR_1) \cong G(I)/x_1^*G(I)$ and $x_2^*, \ldots, x_t^*$ is a regular sequence in $G(IR_1)$. By induction hypothesis, for $1 \leq k \leq t-1$,

$$[\mathrm{H}^k_{\mathcal{M}}(G(IR_1))]_{m_I+1} \cong [\mathrm{H}^{k-1}_{\mathcal{M}}(G(IR_2))]_{m_I+2} \cong \ldots \cong [\mathrm{H}^0_{\mathcal{M}}(G(IR_{k+1}))]_{m_I+1+k} \text{ and} \tag{6.3}$$

$$[\mathrm{H}^i_{\mathcal{M}}(G(IR_{k+1}))]_{n+1} = 0 \text{ for all } n \geq m_I + 1 + k \text{ and for all } i \in \mathbb{Z}. \tag{6.4}$$

Now let $1 \leq j \leq t$. If $j = 1$, then the results follows from Lemma 6.1.1. Assume that $j > 1$. Then (6.2), (6.3) and (6.4) with $k = j-1$ give

$$[\mathrm{H}^j_{\mathcal{M}}(G(I))]_{m_I} \cong [\mathrm{H}^{j-1}_{\mathcal{M}}(G(IR_1))]_{m_I+1} \cong \ldots \cong [\mathrm{H}^0_{\mathcal{M}}(G(IR_j))]_{m_I+j} \text{ and}$$

$$[\mathrm{H}^i_{\mathcal{M}}(G(IR_j))]_{n+1} = 0 \text{ for all } n \geq m_I + j \text{ and for all } i \in \mathbb{Z}.$$

□

The next lemma relates the local cohomology of Rees algebra and the associated graded ring.

Lemma 6.1.3. $[\mathrm{H}^i_{\mathcal{R}_+}(\mathcal{R})]_n \cong [\mathrm{H}^i_{\mathcal{R}_+}(G(I))]_n$ *for all $n > m_I - 1$ and for all $i \in \mathbb{Z}$.*

Proof. Consider the following exact sequences with the canonical maps.

$$0 \longrightarrow \mathcal{R}_+ \longrightarrow \mathcal{R} \longrightarrow R \longrightarrow 0 \text{ and } 0 \longrightarrow \mathcal{R}_+(1) \longrightarrow \mathcal{R} \longrightarrow G(I) \longrightarrow 0$$

and apply the functor $H^i_{\mathcal{R}_+}(*)$ to get

$$\ldots \longrightarrow H^{i-1}_{\mathcal{R}_+}(R) \longrightarrow H^i_{\mathcal{R}_+}(\mathcal{R}_+) \longrightarrow H^i_{\mathcal{R}_+}(\mathcal{R}) \longrightarrow H^i_{\mathcal{R}_+}(R) \longrightarrow \ldots \tag{6.5}$$

$$\longrightarrow H^{i-1}_{\mathcal{R}_+}(G(I)) \longrightarrow H^i_{\mathcal{R}_+}(\mathcal{R}_+)(1) \longrightarrow H^i_{\mathcal{R}_+}(\mathcal{R}) \longrightarrow H^i_{\mathcal{R}_+}(G(I)) \longrightarrow H^{i+1}_{\mathcal{R}_+}(\mathcal{R}_+)(1) \longrightarrow \tag{6.6}$$

Since $[H^i_{\mathcal{R}_+}(G(I))]_n = 0$ for all $i \in \mathbb{Z}$ and for all $n > m_I$, we get

$$[H^i_{\mathcal{R}_+}(\mathcal{R}_+)]_{n+1} \cong [H^i_{\mathcal{R}_+}(\mathcal{R})]_n \tag{6.7}$$

for all $n > m_I$ and for all $i \in \mathbb{Z}$ from exact sequence (6.6). Further by exact sequence (6.5), we have

$$[H^i_{\mathcal{R}_+}(\mathcal{R}_+)]_{n+1} \cong [H^i_{\mathcal{R}_+}(\mathcal{R})]_{n+1}$$

for all $n \geq 1$ and $i \in \mathbb{Z}$. This gives $[H^i_{\mathcal{R}_+}(\mathcal{R})]_n \cong [H^i_{\mathcal{R}_+}(\mathcal{R})]_{n+1}$ for all $n > m_I$. Since $[H^i_{\mathcal{R}_+}(\mathcal{R})]_n = 0$ for all $n \gg 0$, see [Bla97, Theorem 3.7], $[H^i_{\mathcal{R}_+}(\mathcal{R})]_n = 0$ for all $n > m_I$. Therefore $[H^i_{\mathcal{R}_+}(\mathcal{R}_+)(1)]_n = 0$ for all $n > m_I - 1$ and for all $i \in \mathbb{Z}$ from (6.7). Now by exact sequence (6.6), we get that

$$[\mathrm{H}^i_{\mathcal{R}_+}(\mathcal{R})]_n \cong [\mathrm{H}^i_{\mathcal{R}_+}(G(I))]_n$$

for all $n > m_I - 1$ and for all $i \in \mathbb{Z}$. □

To be able to use the method of induction, we need the existence of superficial elements. As discussed earlier, we may pass to the extension ring $R(X)$ if needed to have superficial elements. We remark below that the properties of R which we are interested in are preserved under this passage. We set

$$a_i\,(G(I)) := \sup\{n \in \mathbb{Z} : [\mathrm{H}^i_{\mathcal{R}_+}(G(I))]_n \neq 0\} \text{ and}$$

$$b_i\,(G(I)) := \inf\{n \in \mathbb{Z} : [\mathrm{H}^i_{\mathcal{R}_+}(G(I))]_n \neq 0\}.$$

By convention, if $\mathrm{H}^i_{\mathcal{R}_+}(G(I)) = 0$ then we set $a_i(G(I)) = -\infty$ and $b_i(G(I)) = \infty$.

Remark 6.1.4. *1. Let $S = R(X)$ and $\mathfrak{n} = \mathfrak{m}S$. Since S is faithfully flat R-algebra, the Rees ring $\mathcal{S} = S(IS) = R(I) \otimes_R S$ is flat over the Rees ring $\mathcal{R} = R(I)$ and $\mathcal{S}_+ = \mathcal{R}_+\mathcal{S}$ is the extension of $\mathcal{R}_+$ in $\mathcal{S}$. Therefore $\mathrm{H}^i_{\mathcal{S}_+}(G(IS)) = \mathrm{H}^i_{\mathcal{S}_+}(S(IS)/IS(I)) = \mathrm{H}^i_{\mathcal{S}_+}(S(I) \otimes_{R(I)} R(I)/IR(I)) \cong S(I) \otimes_{R(I)} (\mathrm{H}^i_{\mathcal{R}_+}(R(I)/IR(I))))$, see [ILL+07, Proposition 7.15] for the isomorphism. It follows that $[\mathrm{H}^i_{\mathcal{R}_+}(G(I))]_n = 0$ for all $n \leq -1$ if and only if $[\mathrm{H}^i_{\mathcal{S}_+}(G(IS))]_n = 0$ for all $n \leq -1$.*

2. *Note that* $b_i(G(I) \geq 0$ *for some* i *is equivalent to* $[\mathrm{H}^i_{\mathcal{R}_+}(G(I))]_n = 0$ *for all* $n \leq -1$. *In particular, if* $\operatorname{depth} G(I) \geq t+1$ *for some integer* t *then* $b_i(G(I) \geq 0$ *for* $0 \leq i \leq t$.

We recall the following results from [Hoa93] in order to prove our main results. Let $s(I) = \dim F_{\mathfrak{m}}(I)$ denote the analytic spread of I. It is well known that $s(I) = \dim R$ for an $\mathfrak{m}$-primary ideal I.

Theorem 6.1.5. [Hoa93, Theorem 2.1] *Let* $(R, \mathfrak{m})$ *be a local ring and* $I \subseteq \mathfrak{m}$ *an arbitrary ideal of* R. *Then* $r_J(I^n)$ *is independent of* J *and stable for all* $n \gg 0$. *Namely, for all* $n > \max\{|a_i(G(I))| \ : \ a_i(G(I)) \neq -\infty\}$,

$$r_J(I^n) = \begin{cases} s(I) & \text{if } a_{s(I)}(G(I)) \geq 0, \\ s(I)-1 & \text{if } a_{s(I)}(G(I)) < 0 \end{cases}$$

where J *is any minimal reduction of* I^n.

Lemma 6.1.6. [Hoa93, Lemma 2.4] *Assume that* $s(I) \geq 1$. *Then for all* $i \leq s(I)$ *and* $n \geq 1$,

1. $a_i(G(I^n)) \leq [a_i(G(I))/n]$,
2. $b_i(G(I^n)) \geq [b_i(G(I))/n]$ *and*
3. $a_{s(I)}(G(I^n)) = [a_{s(I)}G(I)/n]$

where $[x] = \max\{m \in \mathbb{Z} \ : \ m \leq x\}$.

We recall the following results from [Bla97] for our use.

Theorem 6.1.7. [Bla97, Theorem 4.1] *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d > 0$ *and* I *an* $\mathfrak{m}$*-primary ideal of* R. *Then for all* $n \in \mathbb{Z}$,

$$P(I,n) - H(I,n) = \sum_{i=0}^{d} (-1)^i \lambda_R([\mathrm{H}^i_{\mathcal{R}_+}(\mathcal{R}^*)]_n).$$

Theorem 6.1.8. [Bla97, Theorem 3.8] *Let* R *be a Noetherian ring and* $I \subseteq R$ *be an ideal. Then for all* $i \geq 2$, *there are isomorphisms of graded* $\mathcal{R}$*-modules*

$$\mathrm{H}^i_{\mathcal{R}_+}(\mathcal{R}) \cong \mathrm{H}^i_{\mathcal{R}_+}(\mathcal{R}^*)$$

and there is an exact sequence of graded $\mathcal{R}$-modules

$$0 \longrightarrow \mathrm{H}^0_{\mathcal{R}_+}(\mathcal{R}) \longrightarrow \mathrm{H}^0_{\mathcal{R}_+}(\mathcal{R}^*) \longrightarrow \mathcal{R}^*/\mathcal{R} \longrightarrow \mathrm{H}^1_{\mathcal{R}_+}(\mathcal{R}) \longrightarrow \mathrm{H}^1_{\mathcal{R}_+}(\mathcal{R}^*) \longrightarrow 0.$$

In particular,

$$[\mathrm{H}^0_{\mathcal{R}_+}(\mathcal{R})]_n \cong [\mathrm{H}^0_{\mathcal{R}_+}(\mathcal{R}^*)]_n \text{ and } [\mathrm{H}^1_{\mathcal{R}_+}(\mathcal{R})]_n \cong [\mathrm{H}^1_{\mathcal{R}_+}(\mathcal{R}^*)]_n$$

for all $n \geq 0$.

The following lemma plays a crucial role in most of our results. We show that given a parameter ideal Q with $b_i(G(Q)) \geq 0$ for $0 \leq i \leq d-3$, there exists an integer l such that $G(Q^l)$ has high depth.

Lemma 6.1.9. *Suppose* depth $R \geq d-1$. *Let Q be parameter ideal such that* $[\mathrm{H}^i_{\mathcal{M}}(G(Q))]_n = 0$ *for all* $n \leq -1$ *and* $0 \leq i \leq d-3$. *Then for* $l \gg 0$,

1. *We have*

$$[\mathrm{H}^i_{\mathcal{M}}(G(Q^l))]_n = \begin{cases} 0 & \text{for all } n \geq 1 \text{ and } i \in \mathbb{Z}, \\ 0 & \text{for all } n \geq 0 \text{ and } i = d, d-2. \end{cases} \tag{6.8}$$

   Furthermore, $\mathrm{H}^i_{\mathcal{M}}(G(Q^l)) = 0$ *for* $0 \leq i \leq d-3$. *In particular,* depth $G(Q^l) \geq d-2$.

2. *Suppose* $\mathrm{H}^{d-2}_{\mathcal{M}}(G(Q))$ *is finitely graded and* $[\mathrm{H}^{d-1}_{\mathcal{M}}(G(Q^l))]_0 = 0$ *for some integer* l. *Then* depth $G(Q^l) \geq d-1$.

Proof. In view of Remark 6.1.4(1), we may assume that the residue field $R/\mathfrak{m}$ is infinite. Let $Q = (x_1, \ldots, x_d)$ such that $x_1^*, \ldots, x_d^*$ is a superficial sequence in $G(Q)$. For $l \geq 1$, we put $I = Q^l$. Given that $[\mathrm{H}^i_{\mathcal{M}}(G(Q))]_n = 0$ for all $n \leq -1$ and $0 \leq i \leq d-3$, equivalently $b_i(G(Q)) \geq 0$ for $0 \leq i \leq d-3$. Thus $b_i(G(I)) \geq [b_i(G(Q))/l] \geq 0$ for $0 \leq i \leq d-3$ by Lemma 6.1.6. Choose $l > max\{|a_i(G(Q))| : a_i(G(Q)) \neq -\infty\}$ and $y_i = x_i^l$ for $1 \leq i \leq d$. Then $y_1^*, \ldots, y_d^*$ is a superficial sequence in $G(I)$ and

$$I^d = (y_1, \ldots, y_d)I^{d-1}. \tag{6.9}$$

Since $J = (y_1, \ldots, y_d)$ is a reduction of I, $\mu(J) = d$. Hence J is a minimal reduction of I with $r_J(Q^l) \leq d-1$. So by [Tru87, Proposition 3.2], $a_d(G(I)) \leq r_J(I) - d$. This gives

$a_d(G(I)) < 0$ and $a_i(G(I)) \leq [a_i(G(Q))/l] \leq 0$ for $i \leq d-1$ by choice of l and Lemma 6.1.6.

(1) It follows that

$$[\mathrm{H}^i_{\mathcal{M}}(G(I))]_n = \begin{cases} 0 & \text{for all } n \geq 1 \text{ and } i \in \mathbb{Z}, \\ 0 & \text{for all } n \neq 0 \text{ and } 0 \leq i \leq d-3, \\ 0 & \text{for all } n \geq 0 \text{ and } i = d. \end{cases} \tag{6.10}$$

We now show that $[\mathrm{H}^i_{\mathcal{M}}(G(I))]_0 = 0$ for $0 \leq i \leq d-2$ by induction on d. For $d = 2$, $[\mathrm{H}^0_{\mathcal{M}}(G(I))]_0 \cong [\mathrm{H}^0_{\mathcal{R}_+}(R(I))]_0 = 0$ by Lemma 6.1.3 as $\operatorname{depth} R \geq 1$. Let $d \geq 3$ and $[\mathrm{H}^i_{\mathcal{M}}(G(I))]_0 = 0$ for $0 \leq i \leq s$ for some $s \leq d-3$. Using (6.10), we get that $\mathrm{H}^i_{\mathcal{M}}(G(I)) = 0$ for $0 \leq i \leq s$. So $\operatorname{depth}(G(I)) \geq s+1$ and $y_1^*, \ldots, y_{s+1}^*$ is a regular sequence in $G(I)$. By Lemma 6.1.2,

$$[\mathrm{H}^{s+1}_{\mathcal{M}}(G(I))]_0 \cong [\mathrm{H}^0_{\mathcal{M}}(G(IR_{s+1}))]_{s+1} \text{ and} \tag{6.11}$$

$$[\,\mathrm{H}^i_{\mathcal{M}}(G(IR_{s+1}))]_n = 0 \text{ for all } n \geq s+2 \text{ and } i \in \mathbb{Z}$$

where $R_{s+1} = R/(y_1, \ldots, y_{s+1})$. Thus by Lemma 6.1.3, $[\mathrm{H}^0_{\mathcal{M}}(G(IR_{s+1}))]_{s+1} \cong [\mathrm{H}^0_{\mathcal{R}_+}(R(IR_{s+1}))]_{s+1} = 0$ since $\operatorname{depth} R_{s+1} \geq 1$. Therefore $[\mathrm{H}^{s+1}_{\mathcal{M}}(G(I))]_0 = 0$ by (6.11).

(2) By part (1), $\operatorname{depth} G(I) \geq d-2$. We show that $\mathrm{H}^{d-2}_{\mathcal{M}}(G(I)) = 0$ for all $l \gg 0$ which implies $\operatorname{depth} G(I) \geq d-1$. By Lemma 6.1.6, $b_{d-2}(G(I)) \geq [b_{d-2}(G(Q))/l] \geq -1$ for all $l \gg 0$. Thus, by part (1) and (6.10),

$$[\mathrm{H}^{d-2}_{\mathcal{M}}(G(I))]_n = 0 \text{ for } n \neq -1.$$

Now it is enough to show that $[\mathrm{H}^{d-2}_{\mathcal{M}}(G(I))]_{-1} = 0$. Given that $[\mathrm{H}^{d-1}_{\mathcal{M}}(G(I))]_0 = 0$. By part (1) and (6.10), we get that $[\mathrm{H}^i_{\mathcal{M}}(G(I))]_n = 0$ for all $n \geq 0$ and $i \in \mathbb{Z}$. Also $\operatorname{depth} G(I) \geq d-2$ implies that $y_1^*, \ldots, y_{d-2}^*$ is a regular sequence in $G(I)$. Therefore by Lemma 6.1.2,

$$[\mathrm{H}^{d-2}_{\mathcal{M}}(G(I))]_{-1} \cong [\mathrm{H}^0_{\mathcal{M}}(G(IR_{d-2}))]_{d-3} \text{ and}$$

$$[\,\mathrm{H}^i_{\mathcal{M}}(G(IR_{d-2}))]_{n+1} = 0 \text{ for } n \geq d-3.$$

So by Lemma 6.1.3, $[\mathrm{H}^0_{\mathcal{M}}(G(IR_{d-2}))]_{d-3} \cong [\mathrm{H}^0_{\mathcal{M}}(R(IR_{d-2}))]_{d-3} = 0$ as $\operatorname{depth} R_{d-2} \geq 1$. Hence $[\mathrm{H}^{d-2}_{\mathcal{M}}(G(I))]_{-1} = 0$. □

6.2 Bounding the Hilbert coefficients

We now prove our main results in which we obtain bounds on the coefficients $e_i(Q)$ with certain conditions on the local cohomology modules of $G(Q)$. In particular, if $\operatorname{depth} G(Q) \geq d-2$, then these conditions are satisfied. Our bound is uniform in the sense that it is independent of Q. The next theorem and corollary provide an upper bound on $e_i(Q)$. In Theorem 6.2.7, we obtain lower bound.

Theorem 6.2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and* $\operatorname{depth} R \geq d-1$. *Let Q be a parameter ideal of R with $b_i(G(Q)) \geq 0$ for $0 \leq i \leq d-3$. Then*

$$e_d(Q) \leq 0.$$

Proof. We may assume that the residue field is infinite by Remark 6.1.4(1). Let $Q = (x_1, \ldots, x_d)$ such that $x_1^*, \ldots, x_d^*$ is a superficial sequence in $G(Q)$. For an integer $l \gg 0$, we put $I = Q^l$. By Lemma 6.1.9(1),

$$[\mathrm{H}^i_{\mathcal{M}}(G(I))]_0 = 0 \text{ for } 0 \leq i \leq d-2. \tag{6.12}$$

Since $[\mathrm{H}^i_{\mathcal{R}_+}(R(I)^*)]_0 \cong [\mathrm{H}^i_{\mathcal{R}_+}(R(I))]_0$ for all $i \geq 0$ due to Theorem 6.1.8, Theorem 6.1.7 yields that

$$\begin{aligned}(-1)^d e_d(I) &= P(I,0) - H(I,0)) \\ &= \sum_{i=0}^{d} (-1)^i \lambda_R([\mathrm{H}^i_{\mathcal{R}_+}(R(I)^*)]_0) \\ &= \sum_{i=0}^{d} (-1)^i \lambda_R([\mathrm{H}^i_{\mathcal{R}_+}(R(I))]_0).\end{aligned} \tag{6.13}$$

By Lemma 6.1.9(1), $[\mathrm{H}^i_{\mathcal{M}}(G(I))]_n = 0$ for all $n \geq 1$ and $i \in \mathbb{Z}$. Therefore by Lemma 6.1.3 and (6.13), we get

$$(-1)^d e_d(I) = \sum_{i=0}^{d} (-1)^i \lambda_R([\mathrm{H}^i_{\mathcal{M}}(G(I))]_0) = (-1)^{d-1} \lambda_R([\mathrm{H}^{d-1}_{\mathcal{M}}(G(I))]_0)$$

where the last equality holds due to (6.12) and the fact that $[\mathrm{H}^d_{\mathcal{M}}(G(I))]_0 = 0$ from (6.8). This implies

$$e_d(Q) = e_d(I) = -\lambda_R([\mathrm{H}^{d-1}_{\mathcal{M}}(G(I))]_0) \leq 0. \tag{6.14}$$

□

Indeed, we can show that the coefficients $e_i(Q)$, for $2 \leq i \leq d-1$, are all non-positive when $\operatorname{depth} G(Q) \geq d-2$.

Corollary 6.2.2. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 2$ *and* $\operatorname{depth} R \geq d-1$. *Let* Q *be a parameter ideal of* R *such that* $\operatorname{depth}(G(Q)) \geq d-2$. *Then, for* $2 \leq i \leq d$,

$$e_i(Q) \leq 0.$$

Proof. We may assume that the residue field $R/\mathfrak{m}$ is infinite. Let $Q = (x_1, \ldots, x_d)$ such that $x_1^*, \ldots, x_d^*$ is a superficial sequence in $G(Q)$. By Theorem 6.2.1 $e_d(Q) \leq 0$. Note that $\operatorname{depth} R \geq d-1$ implies that $x_1, \ldots, x_{d-1}$ is a regular sequence. Set $R_i = R/(x_1, \ldots, x_i)$ for $1 \leq i \leq d-2$. Since $x_1^*, \ldots, x_{d-2}^*$ is a regular sequence in $G(Q)$, $G(QR_i) \cong G(Q)/(x_1^*, \ldots, x_i^*)G(Q)$ and $\operatorname{depth} G(QR_i) \geq d-i-2$ for $i \leq d-2$. Now using Proposition 2.2.5, we get $e_i(Q) = e_i(QR_{d-i}) \leq 0$ for $2 \leq i \leq d$. □

We refer to Example 4.2.12 here to emphasize that the depth condition on the ring is necessary in the above corollary.

Example 6.2.3. *In Example 4.2.12, let* R *be a regular local ring of dimension* $d = 4$ *and* $q = (X_1, \ldots, X_d) = \mathfrak{m}$. *We put* $t = 2$ *so that* $D = R/(X_1, X_2)$ *and* $A = R \ltimes D$. *Then as already discussed* $\dim A = 4$, $\operatorname{depth} A = 2$. *Since* $G(qA) = G(qR) \ltimes G(qD)(-1)$ *see [Put03, Remark 2], we have* $\operatorname{depth} G(Q) = 2$ *but* $e_2(Q, A) > 0$ *by* (4.22) *where* $Q = qA$.

A noteworthy consequence of Theorem 6.2.1 is the following result.

Theorem 6.2.4. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 3$ *and* $\operatorname{depth} R \geq d-1$. *Let* Q *be a parameter ideal. Then* $e_3(Q) \leq 0$.

Proof. We may assume that $R/\mathfrak{m}$ is infinite. Then using reduction modulo superficial elements and Proposition 2.2.5, it is enough to assume that $d = 3$. The result now follows from Theorem 6.2.1. □

The following example shows that the assumption on the depth of the ring can not be relaxed from Theorem 6.2.4.

Example 6.2.5. [GO15, Example 4.7] *Let* $(S, \mathfrak{n})$ *be a regular local ring of dimension* $d = 4$ *and infinite residue field* $S/\mathfrak{n}$. *Let* X, Y, Z, W *be a regular system of parameters of* S *and* $R = S/((X) \cap (Y^3, Z, W))$. *Let* x, y, z, w *be the images of* X, Y, Z, W *in*

R respectively and $\mathfrak{m} = (x, y, z, w)R$ be the maximal ideal of R. Then $\dim R = 3$, $\operatorname{depth} R = 1$ *and $U = (x)$ is the unmixed component of R. Let $Q = (x-y, x-z, x-w)R$ and $T = R/(x)$. Then T is a regular local ring with* $\dim T = 3$ *and $QT = \mathfrak{m}T$. The following exact sequence*

$$0 \longrightarrow (x) \longrightarrow R \longrightarrow R/(x) \longrightarrow 0$$

gives that

$$\begin{aligned}\lambda_R(R/Q^{n+1}R) &= \lambda_R(T/\mathfrak{m}^{n+1}T) + \lambda_R(U/Q^{n+1}U)\\ &= \binom{n+3}{3} + e_0(Q,U)\binom{n+1}{1} - e_1(Q,U)\end{aligned} \tag{6.15}$$

for all $n \gg 0$. We have $(x) \cong R/I$ where $I = (y^3, z, w)R$ and $Q(R/I) = \mathfrak{m}(R/I)$, so $e_0(Q,U) = e_0(\mathfrak{m}, R/I)$ and $e_1(Q,U) = e_1(\mathfrak{m}, R/I)$. To evaluate these values, we look at the Hilbert series of the associated graded ring $G(\mathfrak{m}(R/I))$ which is

$$\frac{1 + t + \ldots + t^2}{1-t}$$

Hence $e_0(Q,U) = e_0(\mathfrak{m}, R/I) = 3$ and $e_1(Q,U) = e_1(\mathfrak{m}, R/I) = 3$. By (6.15)*, we get $e_3(Q,R) = 3 > 0$.*

The following lemma is crucial for obtaining lower bound on $e_i(Q)$. We also obtain a necessary condition for the vanishing of $e_d(Q)$ in Theorem 6.3.1 as an application of this lemma. We treat the associated graded ring $G(I)$ as the quotient of the Rees algebra $R(I)/IR(I)$.

Lemma 6.2.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and* $\operatorname{depth} R \geq d-1$*. Let I be an $\mathfrak{m}$-primary ideal such that* $\operatorname{depth} G(I) \geq d-1$ *and*

$$[\mathrm{H}^i_{\mathcal{M}}(G(I))]_n = \begin{cases} 0 & \text{for all } n \geq 1 \text{ and } i \in \mathbb{Z}, \\ 0 & \text{for all } n \geq 0 \text{ and } i = d. \end{cases} \tag{6.16}$$

Let $J = (y_1, \ldots, y_d)$ be a reduction of I with $I^d = JI^{d-1}$ and $y_1^*, \ldots, y_{d-1}^*$ is a superficial sequence in $G(I)$. Then

$$e_d(I) = -\lambda_R\left(\frac{((y_1, \ldots, y_{d-1}) : y_d) \cap (I^{d-1} + (y_1, \ldots, y_{d-1}))}{(y_1, \ldots, y_{d-1})}\right). \tag{6.17}$$

Proof. Since $\operatorname{depth} G(I) \geq d-1$, $y_1^*, \ldots, y_{d-1}^*$ is a regular sequence in $G(I)$ and $\mathrm{H}^i_{\mathcal{M}}(G(I)) = 0$ for $0 \leq i \leq d-2$. By Lemma 6.1.3, $[\mathrm{H}^i_{\mathcal{M}}(G(I))]_0 \cong [\mathrm{H}^i_{\mathcal{R}_+}(R(I))]_0$ for all i. Therefore using (6.13),

$$\begin{aligned}(-1)^d e_d(I) &= \sum_{i=0}^{d}(-1)^i \lambda_R([\mathrm{H}^i_{\mathcal{R}_+}(R(I))]_0)\\ &= \sum_{i=0}^{d}(-1)^i \lambda_R([\mathrm{H}^i_{\mathcal{M}}(G(I))]_0)\end{aligned}$$

Using (6.16) and Lemma 6.1.2 respectively, we get that

$$e_d(I) = -\lambda_R([\mathrm{H}^{d-1}_{\mathcal{M}}(G(I))]_0) = -\lambda_R([\mathrm{H}^0_{\mathcal{M}}(G(IR_{d-1}))]_{d-1})$$

where $R_{d-1} = R/(y_1, \ldots, y_{d-1})$. Now consider the map

$$\rho : \frac{((y_1, \ldots, y_{d-1}) : y_d) \cap (I^{d-1} + (y_1, \ldots, y_{d-1}))}{(y_1, \ldots, y_{d-1})} \longrightarrow [\mathrm{H}^0_{\mathcal{M}}(G(IR_{d-1}))]_{d-1}$$

defined as $\rho(\bar{x}) = \overline{\bar{x}t^{d-1}}$ where $\bar{x}$ and $\overline{\bar{x}t^{d-1}}$ are the images of $x \in R$ in $R/(y_1, \ldots, y_{d-1})$ and $\bar{x}t^{d-1} \in R(IR_{d-1})$ in $G(IR_{d-1})$ respectively. It is now enough to show that ρ is an isomorphism. To show surjectivity, let $\alpha = \overline{\bar{x}t^{d-1}} \in [\mathrm{H}^0_{\mathcal{M}}(G(IR_{d-1}))]_{d-1}$ with $x \in I^{d-1}$. Then

$$\bar{y}_d t \ \cdot \ \overline{\bar{x}t^{d-1}} = \overline{\overline{y_d x} t^d} \in [\mathrm{H}^0_{\mathcal{M}}(G(IR_{d-1}))]_d. \tag{6.18}$$

Now recall that $y_1^*, \ldots, y_{d-1}^*$ is a regular sequence in $G(I)$, so $(y_1, \ldots, y_{d-1}) \cap I^j = (y_1, \ldots, y_{d-1})I^{j-1}$ for all $j \geq 1$. Since $[\mathrm{H}^0_{\mathcal{M}}(G(IR_{d-1}))]_d = 0$ by Lemma 6.1.2 and $I^d = JI^{d-1}$, (6.18) yields that

$$y_d x \in (I^{d+1} + (y_1, \ldots, y_{d-1})) \cap I^d = ((y_1, \ldots, y_{d-1}) \cap I^d) + I^{d+1} = (y_1, \ldots, y_{d-1})I^{d-1} + y_d I^d.$$

Let $y_d x = \sum_{i=1}^{d-1} r_i y_i + s y_d$ where $r_i \in I^{d-1}$ and $s \in I^d$. This implies $y_d(x-s) \in (y_1, \ldots, y_{d-1})$. So $x - s \in ((y_1, \ldots, y_{d-1}) : y_d) \cap I^{d-1}$ and $\rho(\overline{x-s}) = \alpha$. Hence ρ is surjective.

Now let $x \in ((y_1, \ldots, y_{d-1}) : y_d) \cap I^{d-1}$ such that $\rho(\bar{x}) = \overline{\bar{x}t^{d-1}} = 0$ in $[G(IR_{d-1})]_{d-1}$. Then

$$x \in ((y_1, \ldots, y_{d-1}) : y_d) \cap (I^d + (y_1, \ldots, y_{d-1})) = (y_1, \ldots, y_{d-1}) + \left(((y_1, \ldots, y_{d-1}) : y_d) \cap I^d\right).$$

Claim: Let $n \geq d$ be an integer. Then

$$((y_1, \ldots, y_{d-1}) : y_d) \cap I^n \subseteq (y_1, \ldots, y_{d-1}) + ((y_1, \ldots, y_{d-1}) : y_d) \cap I^{n+1}).$$

Proof of Claim. Let $x \in ((y_1, \ldots, y_{d-1}) : y_d) \cap I^n$, then $y_d x \in (y_1, \ldots, y_{d-1})$. So

$$\bar{y}_d t \cdot \overline{\bar{x}t^n} = \overline{\overline{y_d x} t^{n+1}} = 0 \text{ in } [G(IR_{d-1})]_{n+1}$$

which implies $\overline{\bar{x}t^n} \in [G(IR_{d-1})]_n$ is annihilated by some power of $\mathcal{M}$. Thus $\overline{\bar{x}t^n} \in [\mathrm{H}^0_{\mathcal{M}}(G(IR_{d-1}))]_n = 0$. Recall that $[\mathrm{H}^0_{\mathcal{M}}(G(IR_{d-1}))]_n = 0$ for all $n \geq d$ by Lemma 6.1.2. This gives that $x \in (y_1, \ldots, y_{d-1}) + I^{n+1}$. So $x \in (y_1, \ldots, y_{d-1}) + ((y_1, \ldots, y_{d-1}) : y_d) \cap I^{n+1})$. □

By above claim, $x \in (y_1, \ldots, y_{d-1}) + ((y_1, \ldots, y_{d-1}) : y_d) \cap I^n) \subseteq (y_1, \ldots, y_{d-1}) + I^n$ for all $n \geq d$. This implies $x \in (y_1, \ldots, y_{d-1})$ and ρ is injective. Thus

$$e_d(Q) = -\lambda_R \left(\frac{((y_1, \ldots, y_{d-1}) : y_d) \cap (I^{d-1} + (y_1, \ldots, y_{d-1})}{(y_1, \ldots, y_{d-1})} \right).$$

This completes the proof. □

Theorem 6.2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and* $\operatorname{depth} R \geq d-1$. *Let Q be a parameter ideal of R such that $b_i(G(Q)) \geq 0$ for $0 \leq i \leq d-2$. Then*

$$-\lambda_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R)) \leq e_d(Q).$$

Proof. We may assume that $R/\mathfrak{m}$ is infinite. Let $Q = (x_1, \ldots, x_d)$ such that $x_1^*, \ldots, x_d^*$ is a superficial sequence in $G(Q)$. For an integer $l \gg 0$, let $I = Q^l$ and $y_i = x_i^l$ for $1 \leq i \leq d$. Then $y_1^*, \ldots, y_d^*$ is a superficial sequence in $G(I)$ and

$$I^d = (y_1, \ldots, y_d) I^{d-1}. \tag{6.19}$$

By Lemma 6.1.9, for $l \gg 0$, $\operatorname{depth} G(I) \geq d-2$ and

$$[\mathrm{H}^i_{\mathcal{M}}(G(I))]_n = \begin{cases} 0 & \text{for all } n \geq 1 \text{ and } i \in \mathbb{Z}, \\ 0 & \text{for all } n \geq 0, \text{ and } i = d, d-2. \end{cases} \tag{6.20}$$

Since $b_{d-2}(G(Q)) \geq 0$, we get $b_{d-2}(G(I)) \geq [b_{d-2}(G(Q))/l] \geq 0$ by Lemma 6.1.6. Using (6.20), we get that $\mathrm{H}^{d-2}_{\mathcal{M}}(G(I)) = 0$ for $l \gg 0$. Hence $\operatorname{depth} G(I) \geq d-1$. By Lemma 6.2.6,

$$e_d(Q) = e_d(I) = -\lambda_R \left(\frac{((y_1, \ldots, y_{d-1}) : y_d) \cap (I^{d-1} + (y_1, \ldots, y_{d-1}))}{(y_1, \ldots, y_{d-1})} \right)$$

$$\geq -\lambda_R\left(\frac{((y_1,\ldots,y_{d-1}):y_d)}{(y_1,\ldots,y_{d-1})}\right)$$

$$\geq -\lambda_R(\mathrm{H}^0_{\mathfrak{m}}(R/(y_1,\ldots,y_{d-1}))) \tag{6.21}$$

where the last inequality holds since

$$\frac{((y_1,\ldots,y_{d-1}):y_d)}{(y_1,\ldots,y_{d-1})} \subseteq \mathrm{H}^0_{\mathfrak{m}}(R/(y_1,\ldots,y_{d-1}))$$

Now let $R_i = R/(y_1,\ldots,y_i)$ for $i \leq d-1$ and $R_0 = R$. Note that $y_1,\ldots,y_{d-1}$ is a regular sequence in R and $\operatorname{depth} R_i \geq d-i-1$. For $0 \leq i \leq d-2$, the exact sequence

$$0 \longrightarrow R_i \xrightarrow{y_{i+1}} R_i \longrightarrow R_{i+1} \longrightarrow 0$$

gives the long exact sequence of local cohomology modules

$$0 \longrightarrow \mathrm{H}^{d-i-2}_{\mathfrak{m}}(R_{i+1}) \longrightarrow \mathrm{H}^{d-i-1}_{\mathfrak{m}}(R_i) \xrightarrow{y_{i+1}} \mathrm{H}^{d-i-1}_{\mathfrak{m}}(R_i) \longrightarrow \ldots.$$

Thus for $0 \leq i \leq d-2$,

$$\lambda_R(\mathrm{H}^{d-i-2}_{\mathfrak{m}}(R_{i+1}) \leq \lambda_R(\mathrm{H}^{d-i-1}_{\mathfrak{m}}(R_i)$$

Putting the values of i successively, we get

$$\lambda_R(\mathrm{H}^0_{\mathfrak{m}}(R_{d-1})) \leq \lambda_R(\mathrm{H}^1_{\mathfrak{m}}(R_{d-2})) \leq \ldots \leq \lambda_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R)).$$

Hence $e_d(Q) \geq -\lambda_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R))$ by (6.21). □

Corollary 6.2.8. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. Let Q be a parameter ideal of R such that* $\operatorname{depth} G(Q) \geq d-1$. *Then for $2 \leq i \leq d$,*

$$-\lambda_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R)) \leq e_i(Q). \tag{6.22}$$

Proof. As discussed earlier, we may assume that the residue field $R/\mathfrak{m}$ is infinite. Let $Q = (x_1,\ldots,x_d)$ such that $x_1^*,\ldots,x_d^*$ is a superficial sequence in $G(Q)$. By Theorem 6.2.7, $-\lambda_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R)) \leq e_d(Q)$.

Let $R_i = R/(x_1,\ldots,x_i)$ for $1 \leq i \leq d-1$ and $R_0 = R$. Since $x_1^*,\ldots,x_{d-1}^*$ is a regular sequence in $G(Q)$, $G(QR_i) \cong G(Q)/(x_1^*,\ldots,x_i^*)G(Q)$ and $\operatorname{depth} G(QR_i) \geq d-i-1$. Hence by Theorem 6.2.7,

$$-\lambda_R(\mathrm{H}^{i-1}_{\mathfrak{m}}(R_{d-i})) \leq e_i(Q) \quad \text{for} \quad 2 \leq i \leq d-1. \tag{6.23}$$

Since $x_1, \ldots, x_{d-1}$ is a regular sequence in R, we have the following exact sequence for $0 \le i \le d-2$,

$$0 \longrightarrow R_i \xrightarrow{x_{i+1}} R_i \longrightarrow R_{i+1} \longrightarrow 0$$

which gives the long exact sequence

$$0 \longrightarrow \mathrm{H}^{d-i-2}_{\mathfrak{m}}(R_{i+1}) \longrightarrow \mathrm{H}^{d-i-1}_{\mathfrak{m}}(R_i) \longrightarrow \ldots.$$

Thus for all $0 \le i \le d-2$, we get

$$\lambda_R(\mathrm{H}^{d-i-2}_{\mathfrak{m}}(R_{i+1})) \le \lambda_R(\mathrm{H}^{d-i-1}_{\mathfrak{m}}(R_i)). \tag{6.24}$$

Hence

$$\lambda_R(\mathrm{H}^{i-1}_{\mathfrak{m}}(R_{d-i})) \le \lambda_R(\mathrm{H}^{i}_{\mathfrak{m}}(R_{d-i-1})) \le \ldots \le \lambda_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R)).$$

Therefore by (6.23), we get

$$-\lambda_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R)) \le e_i(Q).$$

□

We include an example where (6.22) does not hold.

Example 6.2.9. *We recall Example 4.2.12 again with* $\dim R = d = 5$ *and* $t = 2$*. Then* $\dim A = 5$ *and* $\operatorname{depth} A = 2$*. By* (4.22)*,* $e_3(Q, A) < 0$ *for any parameter ideal* Q *of* A *whereas* $-\lambda_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(A)) = 0$.

6.3 Vanishing of coefficients

We now give a necessary condition for the vanishing of $e_d(Q)$. We highlight that the hypothesis of the following theorem is satisfied if $\operatorname{depth} G(Q) \ge d-2$.

Theorem 6.3.1. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \ge 2$ *and* $\operatorname{depth} R \ge d-1$*. Let* $Q = (x_1, \ldots, x_d)$ *be a parameter ideal of* R *such that* $x_1^*, \ldots, x_{d-1}^*$ *is a superficial sequence in* $G(Q)$*. Let* $b_i(G(Q)) \ge 0$ *for* $0 \le i \le d-3$ *and* $[\mathrm{H}^{d-2}_{\mathcal{M}}(G(Q))]_n = 0$ *for all* $n \ll 0$ *i.e.* $\mathrm{H}^{d-2}_{\mathcal{M}}\, G(Q)$ *is finitely graded. Suppose*

$$e_d(Q) = 0.$$

Then $x_1^l, \ldots, x_{d-1}^l, x_d^{(d-1)l}$ *is a d-sequence in* R *for all integers* $l \ge 1$.

Proof. For $l \gg 0$, let $I = Q^l$ and $J = (x_1^l, \ldots, x_d^l)$. Then $(x_1^l)^*, \ldots, (x_d^l)^*$ is a superficial sequence in $G(I)$ and $I^d = JI^{d-1}$. Suppose $e_d(Q) = 0$, then (6.14) implies $[\mathrm{H}_{\mathcal{M}}^{d-1}(G(I))]_0 = 0$. Hence by Lemma 6.1.9, $\operatorname{depth} G(I) \geq d-1$.

By Lemma 6.2.6, we get that

$$e_d(Q) = -\lambda_R\left(\frac{((x_1^l, \ldots, x_{d-1}^l) : x_d^l) \cap (I^{d-1} + (x_1^l, \ldots, x_{d-1}^l)}{(x_1^l, \ldots, x_{d-1}^l)}\right).$$

Suppose $e_d(Q) = 0$. Then $((x_1^l, \ldots, x_{d-1}^l) : x_d^l) \cap (I^{d-1} + (x_1^l, \ldots, x_{d-1}^l)) = (x_1^l, \ldots, x_{d-1}^l)$ for all $l \gg 0$. Let $N \geq 1$ be an integer such that for all $l \geq N$,

$$((x_1^l, \ldots, x_{d-1}^l) : x_d^l) \cap I^{d-1} \subseteq (x_1^l, \ldots, x_{d-1}^l). \tag{6.25}$$

Claim: For all $l \geq 1$

$$((x_1^l, \ldots, x_{d-1}^l) : x_d^l) \cap I^{d-1} \subseteq (x_1^l, \ldots, x_{d-1}^l). \tag{6.26}$$

Proof of Claim. Let $1 \leq l < N$ and $y \in ((x_1^l, \ldots, x_{d-1}^l) : x_d^l) \cap I^{d-1}$. Then

$$\begin{aligned} x_d^N \cdot x_1^{N-l} \cdot x_2^{N-l} \cdots x_{d-1}^{N-l} \cdot y &= x_d^{N-l} \cdot x_1^{N-l} \cdot x_2^{N-l} \cdots x_{d-1}^{N-l} \cdot x_d^l y \\ &\in x_d^{N-l} \cdot x_1^{N-l} \cdot x_2^{N-l} \cdots x_{d-1}^{N-l}(x_1^l, \ldots, x_{d-1}^l) \\ &\subseteq (x_1^N, \ldots, x_{d-1}^N). \end{aligned}$$

This implies

$$\begin{aligned} x_1^{N-l} \cdot x_2^{N-l} \cdots x_{d-1}^{N-l} \cdot y &\in ((x_1^N, \ldots, x_{d-1}^N) : x_d^N) \cap Q^{(N-l)(d-1)} I^{d-1} \\ &= ((x_1^N, \ldots, x_{d-1}^N) : x_d^N) \cap Q^{N(d-1)} \\ &\subseteq (x_1^N, \ldots, x_{d-1}^N) \end{aligned} \tag{6.27}$$

where the last containment is due to (6.25). Now we show by induction on d that if y is such that (6.27) holds then $y \in (x_1^l, \ldots, x_{d-1}^l)$. Note that $x_1, \ldots, x_{d-1}$ is a regular sequence. For $d = 2$, $x_1^{N-l} y \in (x_1^N) \implies y \in (x_1^l)$. Let $d > 2$ and (6.27) holds. Set $y' = x_1^{N-l} y$ and $R_1 = R/(x_1^N)$. Let $\overline{\alpha}$ denote the image of an element $\alpha \in R$ in R_1. Then

$$\begin{aligned} & x_2^{N-l} \cdot x_3^{N-l} \cdots x_{d-1}^{N-l} \cdot y' \in (x_1^N, \ldots, x_{d-1}^N) \\ \implies & \overline{x}_2^{N-l} \cdot \overline{x}_3^{N-l} \cdots \overline{x}_{d-1}^{N-l} \cdot \overline{y'} \in (\overline{x}_2^N, \ldots, \overline{x}_{d-1}^N) R_1 \end{aligned}$$

$$\implies \overline{y'} \in (\overline{x}_2^l, \ldots, \overline{x}_{d-1}^l)R_1 \qquad \text{[by induction hypothesis]}$$

$$\implies x_1^{N-l}y = y' \in (x_2^l, \ldots, x_{d-1}^l) + (x_1^N)$$

$$\implies y \in (x_1^l, \ldots, x_{d-1}^l)$$

where the last statement holds since x_1^{N-l} is regular in $R/(x_2^l, \ldots, x_{d-1}^l)$. □

To see that $x_1^l, \ldots, x_{d-1}^l, x_d^{(d-1)l}$ is a d-sequence in R, we use (6.26) repeatedly. For this purpose, let $l \geq 1$ and

$$r \in ((x_1^l, \ldots, x_{d-1}^l) : x_d^{(d-1)l}) \cap (x_d^{(d-1)l})$$

$$\implies rx_d^{(d-2)l} \in ((x_1^l, \ldots, x_{d-1}^l) : x_d^l) \cap I^{d-1} \subseteq (x_1^l, \ldots, x_{d-1}^l)$$

$$\implies rx_d^{(d-3)l} \in ((x_1^l, \ldots, x_{d-1}^l) : x_d^l) \cap I^{d-1} \subseteq (x_1^l, \ldots, x_{d-1}^l)$$

$$\vdots$$

$$\implies r \in ((x_1^l, \ldots, x_{d-1}^l) : x_d^l) \cap I^{d-1} \subseteq (x_1^l, \ldots, x_{d-1}^l).$$

This implies that for $l \geq 1$,

$$((x_1^l, \ldots, x_{d-1}^l) : x_d^{(d-1)l}) \cap (x_1^l, \ldots, x_{d-1}^l, x_d^{(d-1)l}) = (x_1^l, \ldots, x_{d-1}^l).$$

Since $x_1^l, \ldots, x_{d-1}^l$ is a regular sequence, it follows that $x_1^l, \ldots, x_{d-1}^l, x_d^{(d-1)l}$ is a d-sequence for $l \geq 1$. □

For $d = 2$, we give the following statement separately which immediately follows from Lemma 6.2.6 and Theorem 6.3.1. This can also be found in [GO11].

Theorem 6.3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d = 2$ and* $\operatorname{depth} R \geq 1$. *Let $Q = (x_1, x_2)$ be a parameter ideal of R such that x_1^*, x_2^* is a superficial sequence in $G(Q)$. Then for all $l \gg 0$,*

$$e_2(Q) = -\lambda_R \left(\frac{((x_1^l) : x_2^l) \cap Q^l}{(x_1^l)} \right) \leq 0.$$

Further, $e_2(Q) = 0$ implies x_1^l, x_2^l is a d-sequence in R for all $l \geq 1$.

Next, we will see that the converse of the last part of above theorem holds true, namely if x_1, x_2 is a d-sequence in R then $e_2(Q) = 0$. Indeed in Theorem 6.3.4, we extend this characterization for vanishing of $e_2(Q)$ in local rings with dimension $d \geq 2$. In [GO11], Goto and Ozeki gave uniform bounds for $e_2(Q)$ and equivalent conditions for the equality $e_2(Q) = 0$ in two dimensional local rings. Theorem 6.3.4 generalize their result in arbitrary dimension. It also unifies the necessary and sufficient conditions given by Mccune [Mcc13] for the vanishing of $e_2(Q)$. The proof of Theorem 6.3.4 is based on the properties of d-sequences. To recall them, we state the following result which is a part of [GO11, Proposition 3.4].

Proposition 6.3.3. [GO11, Proposition 3.4] *Suppose $d > 0$ and let $Q = (x_1, \dots, x_d)$ be a parameter ideal in R such that $x_1, \dots, x_d$ forms a d-sequence in R. Then*

1. $\lambda_R(R/Q^{n+1}) = \sum_{i=0}^{d} (-1)^i e_i(Q) \binom{n+d-i}{d-i}$ *for all $n \geq 0$. In particular, $\eta(Q) \leq 0$.*
2. $\mathrm{H}^0_{\mathcal{M}}(G(Q)) = [\mathrm{H}^0_{\mathcal{M}}(G(Q))]_0 \cong \mathrm{H}^0_{\mathfrak{m}}(R)$.

Theorem 6.3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$ and* depth $R \geq d-1$. *Let $Q = (x_1, \dots, x_d)$ be a parameter ideal of R such that $x_1^*, \dots, x_d^*$ is a superficial sequence in $G(Q)$. Then the following assertions hold true.*

1. $-\lambda_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R)) \leq e_2(Q) \leq 0$.
2. *The following statements are equivalent:*
   (a) $e_2(Q) = 0$;
   (b) *$x_1^l, \dots, x_d^l$ is d-sequence in R for all integers $l \geq 1$;*
   (c) *$x_1, \dots, x_d$ is d-sequence in R;*
   (d) depth $G(Q) \geq d-1$ *and* $\eta(Q) < 2-d$.
3. $e_2(Q) = 0 \implies e_i(Q) = 0$ *for* $2 \leq i \leq d$.

Proof. (1) Since depth $R \geq d-1$, $x_1, \dots, x_{d-1}$ is a regular sequence. Set $R_i = R/(x_1, \dots, x_i)$ for $1 \leq i \leq d-1$ and $R_0 = R$. By Proposition 2.2.5, $e_2(Q) = e_2(QR_{d-2})$. Hence by Theorems 6.2.1 and 6.2.7, $-\lambda_R(\mathrm{H}^1_{\mathfrak{m}}(R_{d-2})) \leq e_2(Q) \leq 0$. From (6.24), we have that

$$\lambda_R(\mathrm{H}^1_{\mathfrak{m}}(R_{d-2})) \leq \lambda_R(\mathrm{H}^2_{\mathfrak{m}}(R_{d-3})) \leq \ldots \leq \lambda_R(\mathrm{H}^{d-1}_{\mathfrak{m}}(R)).$$

(2) (a) $\implies$ (b) $e_2(Q) = 0 \implies e_2(QR_{d-2}) = 0$. Therefore for all $l \geq 1$, the images of x_{d-1}^l, x_d^l in R_{d-2} is a d-sequence by Theorem 6.3.2. Since $x_1, \dots, x_{d-1}$ is a regular sequence in R, it follows that $x_1^l, \dots, x_d^l$ is a d-sequence in R.

(b) $\Longrightarrow$ (c) It is obvious.

(c) $\Longrightarrow$ (d) Since the images of x_{d-1}, x_d in R_{d-2} is a d-sequence, Proposition 6.3.3 yields that $\mathrm{H}^0_{\mathcal{M}}(G(QR_{d-2})) \cong \mathrm{H}^0_{\mathfrak{m}}(R_{d-2})$. Since $\operatorname{depth} R_{d-2} \geq 1$, we get $\mathrm{H}^0_{\mathcal{M}}(G(QR_{d-2})) = 0$ which implies $\operatorname{depth} G(QR_{d-2}) \geq 1$. Thus by Proposition 2.2.4, $\operatorname{depth} G(Q) \geq d-1$. Since the image of x_d in R_{d-1} is a d-sequence, again by Proposition 6.3.3 we have that $\eta(QR_{d-1}) \leq 0$. Then $\eta(Q) = \eta(QR_{d-1}) - (d-1) \leq 1-d$.

(d) $\Longrightarrow$ (a) $\eta(Q) \leq 1-d$, then $P(Q,n) = H(Q,n) = 0$ for $n = 0, -1, \ldots, 2-d$. By putting the values of n into $P(Q,n)$ successively, we easily get that $e_i(Q) = 0$ for $2 \leq i \leq d$.

(3) It follows from part (2). □

The depth condition on the ring is necessary as evidenced by the following example.

Example 6.3.5. *In Example 4.2.12, let* $\dim R = d \geq 4$ *and* $t = d-3$ *so that* $D = R/(X_1, X_2, X_3)$ *and* $A = R \ltimes D$. *Let* $q = (X_1, \ldots, X_d)$ *and* $Q = qA$. *Then* $e_2(Q, A) = 0$ *but* $\operatorname{depth} G(Q) = d-3$ *and* $e_3(Q, A) \neq 0$ *by* (4.22). *In this case,* $\operatorname{depth} A = d-3$.

7 Summary and future scope

This chapter presents a brief overview of the important results obtained in the thesis. Some highlights are also made to indicate the scope of future investigations.

Hilbert coefficients are important invariants associated to an $\mathfrak{m}$-primary ideal in a Noetherian local ring $(R, \mathfrak{m})$ of dimension d. We investigated the Hilbert coefficients of an $\mathfrak{m}$-primary ideal Q and their relation with certain properties of the ring and the ideal Q. Most of the thesis is devoted to the study of Hilbert coefficients of a parameter ideal Q with respect to an $\mathfrak{m}$-primary ideal K, namely $g_i^K(Q)$, already introduced by Jayanthan and Verma [JV05a]. We developed the necessary technical background needed to deal with $g_i^K(Q)$ in a manner analogous to dealing with $e_i(Q)$.

It is known that $g_i^K(Q)$ for a parameter ideal $Q \subseteq K$ equals $(-1)^i \lambda_R(R/K)$ for $1 \leq i \leq d$ for a Cohen-Macaulay local ring. We provided an alternative proof. Ghezzi et al. solved the Vasconcelos' negativity conjecture and characterized the Cohen-Macaulayness of an unmixed local ring in terms of the vanishing of $e_1(Q)$ for a parameter ideal Q. We generalized their result and obtained a necessary and sufficient condition for the ring to be Cohen-Macaulay in terms of the first two Hilbert coefficients $g_0^K(Q)$ (which is the usual multiplicity) and $g_1^K(Q)$.

The finiteness properties of the set of Hilbert coefficients, where Q varies among the parameter ideals of R, carries useful information about the ring. Ghezzi et al. proved that an unmixed local ring is generalized Cohen-Macaulay (resp. Buchsbaum) if and only if the set of the first Hilbert coefficient $e_1(Q)$ for parameter ideals Q is finite (resp. singleton). We generalize their result for $g_1^K(Q)$ and prove that an unmixed local ring is generalized Cohen-Macaulay if and only if the set of $g_1^K(Q)$ for all parameter ideals Q is finite. If $K = \mathfrak{m}$, then R is Buchsbaum if and only if the above set is singleton. Bringing higher coefficients into consideration, Goto and Ozeki proved that R is generalized Cohen-Macaulay if and only if the sets of the coefficients $e_i(Q)$ for parameter ideals Q are finite for all $1 \leq i \leq d$. We improve upon their result and extended it to modules. We showed that the finiteness of the set of first $\dim M - \operatorname{depth} M$ coefficients imply that M is a generalized Cohen-Macaulay module. In particular, if the set of the coefficients $e_i(Q, M)$ for $1 \leq i \leq \dim M - \operatorname{depth} M$ are finite then so are the sets of remaining higher coefficients. We discussed an example of a local ring for which the sets of the coefficients $e_i(Q)$ for $1 \leq i \leq d$ are finite except for $i = d - \operatorname{depth} R$ and R is not generalized Cohen-Macaulay. This emphasizes the significance of the finiteness of the set of $e_i(Q)$ for $i = d - \operatorname{depth} R$. We have also extended the results of Goto and Ozeki to the coefficients $g_i^K(Q)$. Further, we considered the set of $g_1^K(Q)$ where Q varies among the $\mathfrak{m}$-primary ideals of R and proved that it is finite if and only if $d = 1$ and $R/\operatorname{H}_{\mathfrak{m}}^0(R)$ is analytically unramified.

We further investigated the higher coefficients $g_i^K(Q)$ for $i \geq 2$. We proved that $g_2^K(Q) \leq \lambda_R(R/K)$ and equality implies $g_i^K(Q) = (-1)^i \lambda_R(R/K)$ for all $2 \leq i \leq d$ provided $\operatorname{depth} R \geq d - 1$. In addition, if the associated graded ring has depth at least $d-2$, then $g_2^K(Q) = \lambda_R(R/K)$ implies almost maximal depth of the corresponding fiber cone.

We also examined the difference between the Hilbert polynomial and the Hilbert function with respect to K, namely $P_K(Q, n) - H_K(Q, n)$. In dimension one, we showed that $P_K(Q, n) - H_K(Q, n) \geq 0$ for all $n \geq 0$. In higher dimensions, it is proved that $(-1)^i \Delta^{d+1-i}(P_K(Q, n) - H_K(Q, n)) \geq 0$ for all $0 \leq i \leq d + 1$ and $n \geq 0$ provided $\operatorname{depth} G(Q) \geq d - 1$ and $\operatorname{depth} F_K(Q) \geq d - 1$. Under the same conditions on the form rings, we proved that $g_i^K(Q) \leq (-1)^i \lambda_R(R/K)$ for $1 \leq i \leq d$ and if equality holds for some i then it holds for all $j \geq i$.

In the last part of the thesis, we derived some uniform lower and upper bounds for the coefficients $e_i(Q)$ for a parameter ideal Q. It is proved that $e_3(Q) \leq 0$ for a parameter ideal Q if $\operatorname{depth} R \geq d-1$. In addition, if depth of the associated graded ring is at least $d - 2$ (resp. $d - 1$) then $e_i(Q) \leq 0$ (resp. $e_i(Q) \geq -\lambda_R(\operatorname{H}_{\mathfrak{m}}^{d-1}(R))$ for $2 \leq i \leq d$. We also discussed the vanishing of the last Hilbert coefficient $e_d(Q)$. Suppose $Q = (x_1, \ldots, x_d)$ with $x_1^*, \ldots, x_{d-1}^*$ is a superficial sequence in $G(Q)$. Let $[\operatorname{H}_{\mathfrak{m}}^i(G(Q))]_n = 0$ for all $n \leq -1$ and $0 \leq i \leq d - 3$ and $[\operatorname{H}_{\mathfrak{m}}^{d-2}(G(Q))]_n = 0$ for $n \ll 0$. Then $e_d(Q) = 0$ implies that $x_1^l, \ldots, x_{d-1}^l, x_d^{(d-1)l}$ is a d-sequence for all $l \geq 1$. Consequently, vanishing of $e_2(Q)$ is characterized in a ring of depth at least $d - 1$.

We now propose some problems emerging from the work carried out in the thesis.

- The problem mentioned in Chapter 3 remains unsolved in general i.e., whether $g_1^K(Q) \geq -\lambda_R(R/K)$ for some parameter ideal Q characterizes unixed Cohen-Macaulay local rings.
- To examine whether $g_1^K(Q)$ is constant for all parameter ideals Q in a Buchsbaum local ring R where K is an $\mathfrak{m}$-primary ideal of R.
- To examine the finiteness of the sets of the coefficients $e_i(Q)$ and $g_i^K(Q)$ for $i \geq 2$ when Q varies among the $\mathfrak{m}$-primary ideals and to obtain necessary and sufficient conditions for the finiteness.
- To obtain better bounds for $e_i(Q)$ with weaker assumptions on the depth of the form rings.
- To obtain sufficient conditions for the vanishing of $e_d(Q)$.

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List of publications

1. Kumari Saloni, *On Hilbert coefficients of parameter ideals and Cohen-Macaulayness*, To appear in J. Commut. Algebra.
2. Shreedevi K. Masuti and Kumari Saloni, *On the finiteness of the set of Hilbert coefficients*, To appear in J. Commut. Algebra.
3. Anupam Saikia and Kumari Saloni, *Bounding Hilbert coefficients of parameter ideals*, To be submitted.