FINITE ELEMENT METHODS FOR ELLIPTIC AND PARABOLIC INTERFACE PROBLEMS

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by

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Dedicated

To my Parents

Certificate

It is certified that the work contained in this thesis entitled **"Finite Element Methods for Elliptic and Parabolic Interface Problems"** by **Bhupen Deka,** a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

April, 2006

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With Regards

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Abstract

The main objective of this thesis is to study the convergence of finite element solutions to the exact solutions of elliptic and parabolic interface problems by means of classical finite element method. Due to low global regularity of the true solution it is difficult to apply the classical finite element analysis to obtain optimal order of convergence for interface problems (cf. [3, 14]). The emphasis is on the theoretical aspects of such methods.

In order to maintain the best possible convergence rate, a finite element discretization is proposed and analyzed for both elliptic and parabolic interface problems. More precisely, we have shown that the finite element solution converges to the exact solution at an optimal rate in L^2 and H^1 norms if the grid lines coincide with the actual interface by allowing interface triangles to be curved triangles. Further, if the grid lines form an approximation to the actual interface, optimal order of convergence in H^1 norm and sub-optimal order in L^2 norm are derived for elliptic problems.

Since the error analysis in finite element method for parabolic equation depends on the error analysis of the corresponding elliptic equation, an attempt has been made to extend the convergence analysis of elliptic interface problems to the parabolic interface problems. Both continuous time Galerkin method and discrete time discontinuous Galerkin methods are discussed. Optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are established for both semidiscrete and fully discrete schemes if the grid lines coincide with the interface. Further, we have shown that semidiscrete and fully discrete solutions converge at optimal rate in $L^2(H^1)$ norm if we used straight triangles instead of curved interface triangles.

As it may be computationally inconvenient to fit the mesh to the interface, a finite element discretization based on a mesh which is independent of the location of the interface is considered and analyzed for both elliptic and parabolic interface problems. For the elliptic case, we establish error estimates of optimal order in H^1 norm and almost optimal order in L^2 norm using an unfitted finite element method. Moreover, we show that the proposed method can be used to derive optimal rate of convergence in $L^2(H^1)$ norm and almost optimal in $L^2(L^2)$ norm in the spatially discrete scheme for parabolic problems. A fully discrete scheme based on backward Euler method is also discussed and related error estimate is derived.

Finally, numerical results for one dimensional test problems are presented to illustrate our theoretical findings.

Contents

Chapter 1

Introduction

The purpose of this thesis is to present some results on finite element Galerkin methods for linear elliptic and parabolic interface problems.

1.1 Problem Description

Interface problems are often referred as differential equations with discontinuous coefficients. The discontinuity of the coefficients corresponds to the fact that the medium consists of two or more physically different materials. To begin with, we first introduce both elliptic and parabolic interface problems.

Elliptic interface problems: Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Further, let $\Omega_1 \subset \Omega$ be an open domain with C^2 smooth boundary Γ and $\Omega_2 = \Omega \backslash \Omega_1$. We now consider the following linear elliptic interface problems of the form

$$\mathcal{L}u = f(x) \quad \text{in } \Omega \tag{1.1.1}$$

subject to the homogeneous Dirichlet boundary condition

$$u(x) = 0 \quad \text{on } \partial\Omega \tag{1.1.2}$$

and interface conditions

$$[u] = 0, \qquad \left[\mathcal{A}\frac{\partial u}{\partial \mathbf{n}}\right] = g(x) \quad \text{along } \Gamma. \tag{1.1.3}$$

The symbol $[v]$ is a jump of a quantity v across the interface Γ, i.e., $[v](x) = v_1(x) - v_2(x),\ x \in \Gamma$, where $v_i(x) = v(x)|_{\Omega_i},\ i = 1, 2$ and $\mathbf{n}$ denotes the unit outward normal to the boundary $\partial\Omega_1$. Here, the operator $\mathcal{L}$ is a second order elliptic partial differential operator of the form

$$\mathcal{L}v = -\nabla \cdot (\mathcal{A}\nabla v) + a(x)v.$$

We assume that the coefficient matrix $\mathcal{A} = (a_{ij}(x))_{i,j=1}^2$ is symmetric and uniformly positive definite in Ω, and $a(x) > 0$ is bounded. Moreover, the matrix $\mathcal{A}$ is assumed to be discontinuous along Γ but piecewise smooth in each subdomain Ω_1 and Ω_2, i.e.,

$$\mathcal{A} = \mathcal{A}_l = (a_{ij}^l(x))_{i,j=1}^2 \quad \text{for } x \in \Omega_l,\ l = 1, 2.$$

Here for each l, $\mathcal{A}_l$ is a uniformly positive definite matrix.

Parabolic interface problems: We shall also consider the following linear parabolic interface problems of the form

$$u_t + \mathcal{L}u = f(x,t) \quad \text{in } \Omega \times (0,T] \tag{1.1.4}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \ \text{ in } \Omega; \ \ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T] \tag{1.1.5}$$

and interface conditions

$$[u] = 0, \quad \left[\mathcal{A}\frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \quad \text{along } \Gamma, \tag{1.1.6}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, $\Omega_1 \subset \Omega$ is an open domain with C^2 boundary Γ and $\Omega_2 = \Omega \backslash \Omega_1$. The operator $\mathcal{L}$, symbols $[v]$ and $\mathbf{n}$ are defined as before, and $T < \infty$.

The equations of the form (1.1.1)-(1.1.3) are often encountered in stationary heat conduction problems, material sciences and fluid dynamics. It is the case when two distinct materials or fluids with different conductivities or densities or diffusions are involved. For the literature relating to applications of elliptic differential equations with discontinuous coefficients, one may refer to Ewing [20], Nielsen [42] or Peacemen [45] for the model of the pressure equation arising in reservoir simulation, Reddy [49] for reactor dynamics, Z. Li *et al.* [37] for the model of the potential in the computation of micromagnetics for the ferromagnetic materials or electrostatics for macromolecules. The model equations of the form (1.1.4)-(1.1.6) involving discontinuous coefficients are sometimes called diffraction problems of parabolic types. Such problems arise in non-stationary heat conduction problems in two dimensions with a conduction coefficient which is discontinuous across a smooth interface. For a detailed discussion on models for heat conduction in materials with discontinuous coefficients, see Dautray and Lions [17], Gilberg and Trudinzer [21], Hackbush [24], Ladyzhenskaya *et al.* [30] and Marti [39].

1.2 Notation and Preliminaries

In this section, we shall introduce some standard notation and preliminaries to be used throughout of this work.

All functions considered here are real valued. Let Ω be a bounded domain in $\mathbb{R}^d$, d-dimensional Euclidean space and $\partial\Omega$ denote the boundary of Ω. Let $x = (x_1, x_2, \ldots, x_d) \in \Omega$, and let $dx = dx_1, \ldots dx_d$. Further, let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a d-tuple with nonnegative integer components and denote order of α as $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$. Then, by $D^\alpha \phi$, we shall mean the αth derivative of ϕ defined by

$$D^\alpha \phi = \frac{\partial^{|\alpha|}\phi}{\partial {x_1}^{\alpha_1}, \ldots, \partial {x_d}^{\alpha_d}}.$$

We shall make frequent reference to the following well-known function spaces. For $1 \leq p < \infty$, $L^p(\Omega)$ denotes the linear space of equivalence classes of measurable functions ϕ in Ω such that $\int_\Omega |\phi(x)|^p dx$ exists and is finite. The norm on $L^p(\Omega)$ is given by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |\phi(x)|^p dx \right)^{\frac{1}{p}}, \ \ 1 \leq p < \infty.$$

For $p = \infty$, $L^\infty(\Omega)$ denotes the space of functions ϕ on Ω such that

$$\|\phi\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |\phi(x)| < \infty.$$

When $p = 2$, $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(\phi, \psi) = \int_\Omega \phi(x)\psi(x)dx.$$

By support of a function ϕ, supp ϕ, we mean the closure of all points x with $\phi(x) \neq 0$, i.e.,

$$\text{supp } \phi = \overline{\{x : \phi(x) \neq 0\}}.$$

For any nonnegative integer m, $C^m(\overline{\Omega})$ denotes the space of functions with continuous derivatives upto and including order m in $\overline{\Omega}$. $C_0^m(\Omega)$ is the space all $C^m(\Omega)$ functions with compact support in Ω. Also, $C_0^\infty(\Omega)$ is the space of all infinitely differentiable functions with compact support in Ω.

We now introduce the notion of Sobolev spaces. Let m be the nonnegative integer and let p be such that $1 \leq p < \infty$. The Sobolev space of order (m, p) on Ω, denoted by $W^{m,p}(\Omega)$, is defined as a linear space of functions (or equivalence class of functions) in $L^p(\Omega)$ whose distributional derivatives upto order m are also in $L^p(\Omega)$, i.e.,

$$W^{m,p}(\Omega) = \{\phi : \ D^\alpha \phi \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}.$$

The space $W^{m,p}(\Omega)$ is endowed with the norm

$$\begin{aligned}\|\phi\|_{m,p} &= \left(\int_\Omega \sum_{0\le|\alpha|\le m} |D^\alpha \phi(x)|^p dx\right)^{\frac{1}{p}} \\ &= \left(\sum_{0\le|\alpha|\le m} \|D^\alpha\phi\|^p\right)^{\frac{1}{p}}, \quad 1\le p<\infty.\end{aligned}$$

When $p=\infty$, the norm on the space $W^{m,\infty}(\Omega)$ is defined by

$$\|\phi\|_{m,\infty} = \max_{0\le|\alpha|\le m} \|D^\alpha\phi(x)\|_{L^\infty(\Omega)}.$$

For $p=2$, these spaces will be denoted by $H^m(\Omega)$. The space $H^m(\Omega)$ is a Hilbert space with natural inner product defined by

$$(\phi,\psi) = \sum_{0\le|\alpha|\le m} \int_\Omega D^\alpha\phi D^\alpha\psi dx, \quad \phi,\psi\in H^m(\Omega).$$

The Sobolev space $H^m(\Omega)$ (respectively, $H_0^m(\Omega)$) is also defined as the closure of $C^m(\Omega)$ (respectively, $C_0^\infty(\Omega)$) with respect to the norm $\|\phi\|_m = \|\phi\|_{m,2}$. This result is true under some smoothness assumption on the boundary $\partial\Omega$. Clearly, $L^2(\Omega)=H^0(\Omega)$ and $H^m(\Omega)=W^{m,2}(\Omega)$. We also need the fractional space $H^{\frac{1}{2}}(\Omega)$ equipped with the norm

$$\|\psi\|_{H^{\frac{1}{2}}(\partial\Omega)} = \inf_{w\in H^1(\Omega)} \{\|w\|_{H^1(\Omega)} : \gamma_0 w = \psi\},$$

where γ_0 is a trace operator. For a more complete discussion on Sobolev spaces, see Adams [1].

We shall also use the following spaces in our error analysis. For a given Banach space $\mathcal{B}$, we define, for $m=0,1,$

$$W^{m,p}(0,T;\mathcal{B}) = \left\{u(t)\in\mathcal{B} \text{ for a.e. } t\in(0,T) \text{ and} \sum_{j=0}^m \int_0^T \left\|\frac{\partial^j u(t)}{\partial t^j}\right\|_{\mathcal{B}}^p dt<\infty\right\}$$

equipped with the norm

$$\|u\|_{W^{m,p}(0,T;\mathcal{B})} = \left(\sum_{j=0}^m \int_0^T \left\|\frac{\partial^j u(t)}{\partial t^j}\right\|_{\mathcal{B}}^p dt\right)^{\frac{1}{p}}.$$

We write $H^m(0,T;\mathcal{B}) = W^{m,2}(0,T;\mathcal{B})$ and $L^2(0,T;\mathcal{B}) = H^0(0,T;\mathcal{B})$. When no risk of confusion exists we shall write $L^2(\mathcal{B})$ for $L^2(0,T;\mathcal{B})$.

Below, we shall discuss some preliminary materials which will be of frequent use in error analysis in the subsequent chapters.

The bilinear form $A(\cdot,\cdot)$ associated with the operator $\mathcal{L}$, given by

$$A(w,v) = \int_\Omega \{\mathcal{A}\nabla w \cdot \nabla v + a(x)wv\}dx, \tag{1.2.1}$$

satisfies the following boundedness and coercive properties: For $\phi, \psi \in H^1(\Omega)$, there exists positive constants C and c such that

$$A(\phi,\psi) \leq C\|\phi\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)}$$

and

$$A(\phi,\phi) \geq c\|\phi\|^2_{H^1(\Omega)}.$$

From time to time we shall also use the following inequalities (see, Hardy *et al.* [25]):

(i) Young's inequality: For $a, b \geq 0$ and $\epsilon > 0$, the following inequality

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$$

holds.
(ii) Cauchy-Schwarz inequality: For all $a, b \geq 0, 1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

In integral form, if ϕ and ψ are both real valued and $\phi \in L^p$ and $\psi \in L^q$, then

$$\int_\Omega \phi\psi \leq \|\phi\|_p\|\psi\|_q.$$

For $p = q = 2$, the above inequality is known as **Schwarz's inequality**. The discrete version of Schwarz's inequality may be stated as:
(iii) Let $\phi_j, \psi_j, j = 1, 2, \ldots, n$ be positive real numbers. Then

$$\sum_{j=1}^n \phi_j\psi_j \leq \left(\sum_{j=1}^n \phi_j^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^n \psi_j^2\right)^{\frac{1}{2}}.$$

Below, we state without proof, the following continuous version of Grownwall's lemma. For a proof, see [47].

Lemma 1.2.1 *(***Grownwall's Lemma***) Let $G(t)$ be a continuous function and $H(t)$ a nonnegative continuous function on its interval $t_0 \leq t \leq t_0 + a$. If a continuous function $F(t)$ has the property*

$$F(t) \leq G(t) + \int_{t_0}^t F(s)H(s)ds \quad \textit{for } \; t \in [t_0, t_0 + a],$$

then

$$F(t) \le G(t) + \int_{t_0}^{t} G(s)H(s)\,exp\left[\int_{s}^{t} H(\tau)d\tau\right]ds \quad for\ t \in [t_0, t_0 + a].$$

In particular, when $G(t) = C$ *a nonnegative constant, we have*

$$F(t) \le C\,exp\left[\int_{t_0}^{t} H(s)ds\right] \quad for\ t \in [t_0, t_0 + a].$$

In addition, we shall also work on the following spaces

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$$

and

$$Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2)$$

equipped with the norms

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}$$

and

$$\|v\|_Y = \|v\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega_1)} + \|v\|_{H^1(\Omega_2)},$$

respectively.

We now turn to the literature concerning the regularity of elliptic and parabolic problems with discontinuous coefficients. A good number of articles are available in literature describing existence, uniqueness and regularity results for the problems (1.1.1)-(1.1.3) and (1.1.4)-(1.1.6) (see, [14], [17], [21], [24], [30] and [39]). Due to the presence of discontinuous coefficients, the solution u, in general, does not belong to $H^2(\Omega)$ even if the coefficients are very smooth in each individual subdomain. But one can expect higher local regularity of the solution when the coefficients are locally smoother (cf. [30]). Concerning the elliptic interface problems, we have the following regularity result. For a proof, see Chen and Zou [14], and Ladyzhenskaya *et al.* [30].

Theorem 1.2.1 *Let* $f \in L^2(\Omega)$ *and* $g \in H^{\frac{1}{2}}(\Gamma)$. *Then the problem (1.1.1)-(1.1.3) has a unique solution* $u \in X \cap H^1_0(\Omega)$ *and* u *satisfies a priori estimate*

$$\|u\|_X \le C\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)}\right).$$

Regarding the parabolic interface problems (1.1.4)-(1.1.6), we have the following regularity result (cf. [14, 30]).

Theorem 1.2.2 *Let* $f \in H^1(0,T;L^2(\Omega))$, $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$ *and* $u_0 \in H^1_0(\Omega)$. *Then the problem (1.1.4)-(1.1.6) has a unique solution* $u \in L^2(0,T;X) \cap H^1(0,T;Y) \cap H^1_0(\Omega)$.

1.3 A Brief Survey on Numerical Methods

In this section, we shall discuss a brief survey of the relevant literature concerning elliptic and parabolic interface problems. Solving interface problems efficiently and accurately is still a challenge because of many irregularities associated with them. Many numerical methods designed for interface problems do not work or work poorly. Thus, the numerical solution to the interface problem is challenging as well as interesting also.

Finite Difference Method: LeVeque and Li [32] proposed an immersed interface method for elliptic interface problems defined on a regular domain for which a uniform rectangular grid can be used. Then finite difference methods were constructed based on the uniform grid and the jump conditions on the interface. The authors applied their methods also for other interface problems, e.g. Stokes flow [33] and the one dimensional moving interface problems [34]. One major disadvantage of these methods is that the resultant linear systems from these methods are non-symmetric and indefinite even if the original problems are self-adjoint and uniformly elliptic. The convergence proofs of these methods are still open.

Finite Element Methods: The analysis of finite element methods for interface problem has become an active research area over the years. The finite element methods for interface problems may be grouped into two categories: *Fitted finite element method* and *Unfitted finite element method* depending on the choice of the discretization. In fitted finite element method, the discretization is made in such a way that the grid line is either isoparametrically fitted to the interface or an approximation of the smooth interface. In unfitted finite element methods, the discretization is independent of the location of the interface.

In order to put the results of this thesis into proper prospective, we first give a brief account of the development of the finite element methods for such problems. The numerical solutions of interface problems by means of finite element Galerkin procedures have been investigated by several authors. One of the first finite element methods treating interface problem (1.1.1)-(1.1.3) has been studied by Babuška in [3]. In [3], the author has formulated the problem as an equivalent minimization problem and then finite element methods are used to solve the minimization problem. Under some approximation assumptions on finite element spaces, Babuška has obtained sub-optimal order error estimate in H^1 norm. The algorithm in [3] requires the exact evaluation of line integrals on the boundary of the domain and on the interface, and exact integrals on the interface finite elements are also needed. The author of [23] has proposed an infinite element method which may be considered as a certain scheme of mesh refinement for elliptic interface problems with interfaces consisting of straight lines. The optimal energy norm error estimate has been achieved in [23]. The algorithm discussed in [23] is not suitable for curved interfaces. In the absence of variational crimes, finite element approximation of (1.1.1)-(1.1.3) has been studied by Barrett and Elliott in [8]. They have shown that the finite element solution converges to the true solution at optimal rate in L^2 and H^1 norms over any interior subdomain. In [8], it is assumed that the solution and the normal derivatives of the solution are continuous along the interface, and fourth order differentiable on each subdomain.

For the problems (1.1.1)-(1.1.3), Bramble and King [9] have considered a finite element method in which the domains Ω_1 and Ω_2 are replaced by polygonal domains $\Omega_{1,h}$ and $\Omega_{2,h}$, respectively. Then, the Dirichlet data and the interface function are transferred to the polygonal boundaries. Finally, discontinuous Galerkin finite element method is applied to the perturbed problem defined on the polygonal domains. Optimal order error estimates are derived for rough as well as smooth boundary data.

Recently, under practical regularity assumptions on the true solution, the convergence of finite element method is studied in [14] and [42]. In [14], Chen and Zou have considered a practical piecewise linear finite element approximation for solving second order elliptic interface problem with $\mathcal{L}u = -\nabla \cdot (\beta \nabla u)$ in a polygonal domain, where the coefficient β is assumed to be positive and piecewise constant in each subdomains. They have proved almost optimal order of convergence in L^2 and energy norms. More precisely, the error bounds obtained by Chen and Zou [14] are optimal up to the factor $\log h$. Under the assumptions on the source term $f|_{\Omega_1} = 0$ and the interface function $g = 0$, Neilsen [42] has proved optimal order of convergence in H^1 norm in the presence of arbitrarily small ellipticity. The algorithm in [42] requires that the interface triangles follow exactly the actual interface Γ.

More recently, for non-matching grids, the authors of [27] have studied elliptic interface problems by mortar element method and obtained the optimal order estimates in L^2 and H^1 norms. Explicitly realizable mortar conditions are introduced to couple the individual discretizations. In addition, the effect of numerical quadrature on finite element solution has also been discussed in [27] and the related optimal order estimates are derived. In [27], it is assumed that the grid lines coincide with the smooth interface. The authors of [46] have used the spectral theory in a crucial way to derive optimal error estimates in weighted norms. The error bounds in this method are independent of the coefficients in the appropriate weighted norms.

Since it is computationally inconvenience to construct a mesh fitted to the interface Γ, an unfitted finite element method for the elliptic interface problem (1.1.1)-(1.1.3) is proposed by several authors in [5]-[8], [26] and [37]. The first unfitted finite element method for elliptic interface problem is due to Barrett and Elliott [8]. Instead of studying the convergence of the interface problem (1.1.1)-(1.1.3), the authors of [8] have studied the finite element approximation to a penalized problem. They have shown that the finite element solution converges to true solution at optimal rate in the L^2 and energy norms over any interior subdomain for a mesh which is independent of the location of the interface. Recently, while the elliptic interface problem is treated by discontinuous Galerkin methods in [26], based on the cartesian triangulations the authors of [37] have obtained the optimal rate of convergence for energy norm via conforming finite element method. The basis functions in [37] are constructed to satisfy the interface jump conditions either exactly or approximately. For non conforming case, the convergence of finite element solution to the exact solution is still open.

We now turn to the finite element Galerkin approximation to parabolic interface problems (1.1.4)-(1.1.6). Although a good number of articles is devoted to the finite element approximation of elliptic interface problems, the literature seems to lack concerning the convergence of finite element solutions to the true solutions of parabolic interface problems. For the backward Euler time discretization, Chen and Zou [14] have studied the convergence of fully discrete solution to the exact solution using fitted finite element methods. They have proved almost optimal error estimates in L^2 and energy norms when global regularity of the solution is low. Semidiscrete and fully discrete finite element approximations of linear parabolic equations without interface have been studied extensively, [58] offers an excellent review of the main results and mathematical techniques of this subject and contains a comprehensive list of references.

1.4 Motivation and Objectives

This section elucidates our contributions and motivation for the present study. The physical world is replete with examples of free surfaces, material interface and moving boundaries that interact with a surrounding fluid. There are interfaces that separate air and water (in the case of bubbles or free surface flows) and boundaries between two materials of different physical properties (in porous media flow or mixing layers). While the mathematical modelling of the interaction is a difficult problem in itself, another formidable task is developing a numerical method that solves these problems effectively and efficiently.

Solving the elliptic or parabolic equations with discontinuous coefficients by means of classical finite element methods usually leads to the loss in accuracy (cf. [3, 14]).

One major difficulty is that the solution has low global regularity and the elements do not fit with the interface of general shape. For non-interface problems, one can assume full regularities of the solutions (at least $H^2(\Omega)$) on whole physical domain. But, for the interface problems, the global regularity of the solutions is low. So the classical analysis is difficult to apply for the convergence analysis of the interface problems.

In the present work, we have used three types of different meshes to study the convergence of finite element solutions to the exact solutions of elliptic and parabolic interface problems. Optimal order error estimates in L^2 and H^1 norms are shown to hold for a finite element discretization where grid lines coincide with the actual interface. The results presented in this thesis not only generalize the work of [42] but also establish the optimal rate of convergence in L^2 norm for elliptic interface problems (cf. [52]). Further, for the purpose of numerical computations we discuss the effect of numerical quadrature on finite element solution and the related optimal order estimates are also established.

It is costly to generate the mesh whose grid lines coincide with the actual interface of general shape. Therefore, a modification of the method is proposed and analyzed in this thesis by assuming interface triangles to be straight triangles instead of curved triangles. The proposed method yields optimal order convergence in H^1 norm and sub-optimal in L^2 norm for the elliptic interface problems.

Since the finite element analysis of parabolic problem depends on the analysis of elliptic problem, therefore, an attempt has been made to extend the convergence analysis of elliptic interface problems to the parabolic problems with discontinuous coefficients. The goal of this work is to study the convergence of parabolic interface problems for both types of triangulation in the case of fitted finite element methods. While the continuous time Galerkin method is discussed for the spatially discrete scheme, the discontinuous Galerkin method is analyzed for the fully discrete scheme. If the grid lines coincide with the interface, optimal order of convergence in $L^2(L^2)$ and $L^2(H^1)$ norms are established (cf. [53]). When the finite element discretization is based on a triangulation consisting of straight triangles, the semidiscrete and fully discrete solutions converge at optimal rate in $L^2(H^1)$ norm.

If the triangulation is aligned with the interface (body fitting grid), a second order accurate approximation to the solution of interface problem can be generated by the Galerkin finite element method with standard linear basis functions (cf. [8], [14], [52]-[53]). However, it is difficult and time consuming to generate body fitting grid for an interface problem in which the interface changes its topology during the iteration. Such a difficulty becomes even more severe for moving interface problems because a new grid has to be generated at each time step. In that case, it may be advantageous to use the same mesh on the domain for the moving problems. This gives the motivation to work on unfitted mesh for interface problems.

In this thesis, we have analyzed an unfitted finite element method for both elliptic and time dependent parabolic interface problems. Optimal error estimate is derived in H^1 norm and almost optimal in L^2 norm for elliptic interface problems. Moreover, for the time dependent parabolic problems, we have shown that the proposed method can be used to obtain optimal rate of convergence in $L^2(H^1)$ norm for both semidiscrete and fully discrete solutions. Further, the error estimate for the spatially discrete scheme is shown to be almost optimal in $L^2(L^2)$ norm. To the best of our knowledge, the analysis of an unfitted finite element method for the time dependent problems has not been studied before.

1.5 Organization of the Thesis

The organization of the thesis is as follows: Chapter 2 deals with the error analysis for elliptic interface problems using curved interface triangles. In addition, for the purpose of practical implementation the effect of numerical quadrature on finite element solution is also analyzed.

Chapter 3 is devoted to the convergence of finite element Galerkin method for elliptic interface problems with straight triangles. Optimal H^1 norm and sub-optimal L^2 norm error estimates are derived for arbitrary shape but smooth interfaces.

In Chapter 4, we analyze the continuous time Galerkin method for the spatially discrete scheme for parabolic interface problems. Optimal error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are established for a finite element discretization where the interface triangles are assumed to be curved triangles instead of straight triangles like classical finite element methods. When the triangulation is based on straight triangles, we have shown that the semidiscrete solution converges to the exact solution at optimal rate in $L^2(H^1)$ norm for the fitted finite element method.

In Chapter 5, we discuss the fully discrete schemes based on backward Euler type time discretization for the fitted finite element methods. Optimal error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are derived if the grid lines coincide with the actual interface. Further, optimal error estimate in $L^2(H^1)$ norm is also achieved when the triangulation is based on straight triangles.

Chapter 6 is concerned with the *a priori* error estimates of an unfitted finite element method for both linear elliptic and time dependent parabolic interface problems. The proposed method yields optimal order error estimates in H^1 norm and almost optimal in L^2 norm for elliptic interface problems. Moreover, the proposed method can be used to obtain optimal rates of convergence in $L^2(H^1)$ norm for both semidiscrete and fully discrete schemes. Further, $L^2(L^2)$ norm error estimate is shown to be optimal upto a factor $\log h$ for the semidiscrete problem.

Chapter 7 provides the information about the performance of our numerical algorithms for one dimensional test problems.

Finally, Chapter 8 concerns with the critical evaluation of results highlighting the contributions made by this thesis. It also provides information for the scope of future investigations.

For clarity of presentation we have repeatedly given equations (1.1.1)-(1.1.3) or (1.1.4)-(1.1.6) at the beginning of subsequent chapters.

Chapter 2

Finite Element Method for Elliptic Interface Problems: Part-I

In this chapter, we analyze the finite element method for elliptic interface problems. An isoparametric type of discretization is used to prove optimal order error estimates in L^2 and H^1 norms when the global regularity of the solution is low. In addition, for the purpose of practical implementation, the effect of numerical quadrature on finite element solution is also analyzed and related optimal error estimates are obtained. [1]

2.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ is an open domain with C^2 boundary Γ. Let $\Omega_2 = \Omega \backslash \Omega_1$. We recall the following linear elliptic interface problems of the form

$$\mathcal{L}u = f(x) \quad \text{in } \Omega \tag{2.1.1}$$

subject to the boundary condition

$$u(x) = 0 \quad \text{on } \partial\Omega \tag{2.1.2}$$

and jump conditions on the interface

$$[u] = 0, \qquad \left[\mathcal{A}\frac{\partial u}{\partial \mathbf{n}}\right] = g(x) \quad \text{along } \Gamma. \tag{2.1.3}$$

Here, $f = f(x)$ and $g = g(x)$ are real valued functions in Ω and Γ, respectively. The operator $\mathcal{L}$, symbols $[v]$ and $\mathbf{n}$ are defined as in Chapter 1.

[1] *Numer. Funct. Anal. Optim., 27 (2006), pp. 99-115*

As a first step towards finite element approximation for the elliptic interface problem (2.1.1)-(2.1.3), let us define $H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$. Further, we recall the bilinear form $A(.,.) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ corresponding to the operator $\mathcal{L}$, i.e.,

$$A(w, v) = \int_\Omega \{\mathcal{A}\nabla w \cdot \nabla v + a(x)wv\}dx.$$

Then the weak formulation of the problem (2.1.1)-(2.1.3) may be stated as: Find $u \in H_0^1(\Omega)$ such that u satisfies

$$A(u, v) = (f, v) + \langle g, v\rangle_\Gamma \quad \forall\, v \in H_0^1(\Omega). \tag{2.1.4}$$

Here, $(\cdot, \cdot)$ and $\langle\cdot, \cdot\rangle_\Gamma$ are used to denote the inner products of the $L^2(\Omega)$ and $L^2(\Gamma)$ spaces, respectively.

The purpose of the present chapter is to establish optimal *a priori* error estimates for elliptic interface problems. An attempt has been made to study the convergence of elliptic interface problem (2.1.1)-(2.1.3) by using a finite element discretization where interface triangles are assumed to be curved triangles instead of straight triangles as in classical finite element methods. Optimal order error estimates in L^2 and H^1 norms are derived when the global regularity of the solution is low. In practice, it may cause some technical difficulties for the evaluation of the integrals over those curved elements near the interface Γ. It would make the numerical implementation much easier if we can replace these integrals over the curved elements by some well-known quadrature rule. Therefore, for the purpose of numerical computations, we discuss the effect of numerical quadrature on finite element solution and the related optimal order error estimates are also established. Based on the fact that the curved triangles follow exactly the actual interface Γ, the convergence of finite element solution is studied by several authors, see [3], [8] and [42]. The main crucial technical tools used in our analysis are some Sobolev embedding inequality, extension theorem and Nitsche's trick.

The organization of this chapter is as follows. In Section 2.2, we describe the finite element discretization and the related approximation properties of the finite element spaces. Section 2.3 is concerned with the error analysis and finally, the effect of numerical quadrature is discussed in Section 2.4.

2.2 Finite Element Discretization

In this section, we shall describe an isoparametric type finite element discretization for the problems (2.1.1)-(2.1.3) and recall some basic approximation properties associated with the finite element spaces.

For the purpose of finite element approximation we now describe the triangulation of Ω as follows: Let $\mathcal{T}_h$ be a triangulation of Ω with mesh parameter h, $0 < h < 1$. We first approximate the domain Ω_1 by a domain Ω_1^h with the polygonal boundary Γ_h whose vertices all lie on the interface Γ. Let Ω_2^h be the approximation for the domain Ω_2 with polygonal exterior $\partial\Omega_2^h$ and interior boundary Γ_h. Further, let $\{P_j\}_{j=1}^{m_h}$ be the set of all nodes of the triangulation $\mathcal{T}_h$ lying on the interface Γ, and let $\{e_j\}(j = 1, \ldots, m_h)$ be the edge connecting the two neighboring points P_j and P_{j+1} such that $P_{m_h+1} = P_1$. Let $\mathcal{T}_h^*$ be a triangulation obtained with a modification of $\mathcal{T}_h$. $\mathcal{T}_h^*$ is obtained by changing those triangles of $\mathcal{T}_h$ having one edge e_j (for some $1 \leq j \leq m_h$) into curved triangles having two original edges unchanged but having their third edge e_j replaced with the curved segment. The element $K \in \mathcal{T}_h^*$ with one curved edge along interface Γ is called interface curved triangle. The set of all curved triangles in Ω is denoted by $\mathcal{T}_C^*$.

Triangulation $\mathcal{T}_h^*$ of the domain Ω be such that it satisfies the following conditions:

($\mathcal{A}1$) $\overline{\Omega} = \cup_{K\in\mathcal{T}_h^*} K$.

($\mathcal{A}2$) If K_1, $K_2 \in \mathcal{T}_h^*$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex or edge or one curved edge of both triangles.

($\mathcal{A}3$) Each interface triangle K intersects Γ(interface) in at most two vertices and has at most one curved edge.

($\mathcal{A}4$) For each triangle $K \in \mathcal{T}_h^*$, let $r_K, \overline{r}_K$ be the radii of its inscribed and circumscribed circles, respectively. Let $h = \max\{\overline{r}_K : K \in \mathcal{T}_h^*\}$. We assume that, for some fixed $h_0 > 0$, there exists two positive constants C_0 and C_1 independent of h such that

$$C_0 h \leq \mathrm{diam(K)} \leq C_1 h \quad \forall K \in \mathcal{T}_h^*, \quad \forall h \in (0, h_0).$$

Assumption ($\mathcal{A}4$) allows us to relate L^2 and H^1 norms of the polynomials in each element of $\mathcal{T}_h^*$ by

$$\|v\|_{H^1(K)} \leq Ch^{-1}\|v\|_{L^2(K)} \quad \forall K \in \mathcal{T}_h^* \tag{2.2.1}$$

for any polynomial $v \in P_1(K)$ (cf. Lemma 4.5.3, [12]), where $P_1(K)$ denotes the set of all polynomials defined over K of degree less than or equal to one.

Let V_h be a family of finite element subspaces of $H_0^1(\Omega)$ defined on $\mathcal{T}_h^*$ consisting of piecewise linear polynomials vanishing on the boundary $\partial\Omega$. For the existence of such type of finite element space based on a triangulation where the grid lines coincide with the actual interface, we refer to [42] (see, pages 372, 373, 384, and 385).

The finite element approximation is then defined to be the function $u_h \in V_h$ such that

$$A(u_h, v_h) = (f, v_h) + \langle g, v_h\rangle_\Gamma \quad \forall\, v_h \in V_h. \tag{2.2.2}$$

From (2.1.4) and (2.2.2), we notice that

$$A(u-u_h, v_h) = 0 \quad \forall\ v_h \in V_h. \tag{2.2.3}$$

Since u_h is the best approximation of u with respect to the H^1 norm, we have

$$\|u-u_h\|_{H^1(\Omega)} \le C \inf_{w_h \in V_h} \|u-w_h\|_{H^1(\Omega)}.$$

Define the inner product $[\cdot,\cdot]_E$ on $H^1(\Omega)\times H^1(\Omega)$ by

$$[\phi,\psi]_E = [\phi,\psi]_{1,\Omega_1} + [\phi,\psi]_{1,\Omega_2},$$

where $[\phi,\psi]_{1,\Omega_k}\ :\ H^1(\Omega_k)\times H^1(\Omega_k) \to \mathbb{R}$ is defined by

$$[\phi,\psi]_{1,\Omega_k} = \int_{\Omega_k} \{\mathcal{A}_k \nabla\phi\cdot\nabla\psi + a(x)\phi\psi\}dx, \quad k=1,2, \tag{2.2.4}$$

and the associated energy norms are

$$\|\phi\|_E = ([\phi,\phi]_E)^{\frac{1}{2}}, \quad \|\phi\|_{1,\Omega_k} = ([\phi,\phi]_{1,\Omega_k})^{1/2}, \quad k=1,2.$$

Then

$$\|\phi\|_E \le C\left(\sum_{k=1}^{2}\|\phi\|^2_{1,\Omega_k}\right)^{1/2},$$

for some $C>0$. By the poincáre's inequality and the assumptions on the coefficients, there exists positive constants C_1, C_2, C_3 and C_4 such that

$$C_1\|\phi\|_{H^1(\Omega)} \le \|\phi\|_E \le C_2\|\phi\|_{H^1(\Omega)} \quad \forall \phi \in H^1_0(\Omega) \tag{2.2.5}$$

and

$$C_3\|\phi\|_{1,\Omega_k} \le \|\phi\|_{H^1(\Omega_k)} \le C_4\|\phi\|_{1,\Omega_k} \quad \forall \phi \in H^1(\Omega_k) \tag{2.2.6}$$

hold true for $k=1,2$. For optimal error estimates it is not convenient to work on general finite element space V_h. We, therefore, need the following specific assumptions on V_h.

(i) For all $\phi \in H^2(\Omega_1)$ and for all $\chi \in H^2(\Omega_2)\cap\{\psi : \psi = 0 \text{ on } \partial\Omega\}$

$$\inf_{w_h^{(1)} \in V_{h,\Omega_1}} \|\phi - w_h^{(1)}\|_{H^1(\Omega_1)} \le Ch\|\phi\|_{H^2(\Omega_1)} \tag{2.2.7}$$

and

$$\inf_{w_h^{(2)} \in V_{h,\Omega_2}} \|\chi - w_h^{(2)}\|_{H^1(\Omega_2)} \le Ch\|\chi\|_{H^2(\Omega_2)}, \tag{2.2.8}$$

where $V_{h,\Omega_k} = \{v_h|_{\Omega_k} \; : \; v_h \in V_h\}$. Note that the assumptions (2.2.7)-(2.2.8) are satisfied as the triangulation $\mathcal{T}_h^*$ is constructed such that the interface Γ coincides with grid lines of $\mathcal{T}_h^*$ (cf. Brenner and Scott [12] or Hackbusch [24]). In addition, we shall need the following assumptions on V_h which are now stated in terms of sets. These assumptions can be verified for various types of finite element spaces (cf. [10] and [11]).

(ii) Let $T_{\Omega_k} : H^1(\Omega_k) \to H^{\frac{1}{2}}(\Gamma)$, for k=1,2, be the trace operators. Set G_{h,Ω_k} to be the restriction of all $\psi \in V_{h,\Omega_k}$ on the interface Γ, i.e.,

$$G_{h,\Omega_k} = \{T_{\Omega_k}(\psi) : \psi \in V_{h,\Omega_k}\}.$$

We now introduce an important operator $F_{h,k} : G_{h,\Omega_k} \to V_{h,\Omega_s}$ with $s = 1(2)$ whenever $k = 2(1)$. For any $\alpha_h \in G_{h,\Omega_k}$, define $F_{h,k}\alpha_h|_\Gamma = \alpha_h$. Then the discrete system

$$[F_{h,k}\alpha_h, \psi]_{1,\Omega_s} = 0 \quad \forall \; \psi \in V_{h,\Omega_s} \cap H_0^1(\Omega_s)$$

with the boundary condition $F_{h,k}\alpha_h|_\Gamma = \alpha_h$ has a unique solution.

For $\alpha_h \in G_{h,\Omega_k}$, it now follows from Nielsen ([42], p. 372) that

$$C\|F_{h,k}\alpha_h\|_{H^{\frac{1}{2}}(\Gamma)} = C\|\alpha_h\|_{H^{\frac{1}{2}}(\Gamma)} \leq \|F_{h,k}\alpha_h\|_{H^1(\Omega_s)} \leq C\|\alpha_h\|_{H^{\frac{1}{2}}(\Gamma)}. \tag{2.2.9}$$

2.3 Convergence Results

This section is devoted to estimate the error between the finite element solution u_h and the exact solution u. We shall prove optimal order error bounds in H^1 and L^2 norms.

Let $v_h^{(k)}$, $k = 1, 2$, be the best approximation of u on V_{h,Ω_k} with respect to the inner product $[\cdot, \cdot]_{1,\Omega_k}$, i.e.,

$$[u - v_h^{(k)}, \psi]_{1,\Omega_k} = 0 \quad \forall \;\; \psi \in V_{h,\Omega_k}. \tag{2.3.1}$$

This implies

$$[u - v_h^{(k)}, \psi]_{1,\Omega_k} = 0 \quad \forall \;\; \psi \in V_{h,\Omega_k} \cap H_0^1(\Omega_k). \tag{2.3.2}$$

Now, consider the following subspaces of V_h

$$S_{h,\Omega_k} = \left\{\psi^{(k)} \in V_h \; : \; \text{supp}\,(\psi^{(k)}) \subset \overline{\Omega}_k \quad \text{and} \quad \psi^{(k)}|_{\Omega_k} \in H_0^1(\Omega_k)\right\},$$
$$S_{h,\Omega_k}^{\perp} = \left\{q_h^{(k)} \in V_h \; : \; [q_h^{(k)}, \psi]_E = 0 \quad \forall\, \psi \in S_{h,\Omega_k}\right\}.$$

Let

$$\tilde{\psi}^{(k)} = \begin{cases} \psi^{(k)} & \text{in} \quad \Omega_k, \\ 0 & \text{in} \quad \Omega\backslash\Omega_k \end{cases} \tag{2.3.3}$$

be an extension of $\psi^{(k)} \in V_{h,\Omega_k} \cap H_0^1(\Omega_k)$. Clearly, $\tilde{\psi}^{(k)} \in V_h$. Thus, the space $S_{h,\Omega_k}^{\perp}$ defined above is equivalent to: For every $q_h^{(k)} \in S_{h,\Omega_k}^{\perp}$,

$$[q_h^{(k)}, \psi]_{1,\Omega_k} = 0 \quad \forall \psi \in V_{h,\Omega_k} \cap H_0^1(\Omega_k) \tag{2.3.4}$$

and hence, subtracting from (2.3.2), we have

$$[v_h^{(k)} - q_h^{(k)}, \psi^{(k)}]_{1,\Omega_k} = [u, \psi^{(k)}]_{1,\Omega_k} \quad \forall\, \psi^{(k)} \in V_{h,\Omega_k} \cap H_0^1(\Omega_k),\ \forall q_h^{(k)} \in S_{h,\Omega_k}^{\perp}. \tag{2.3.5}$$

Below, we shall prove a lemma which plays a crucial role in the subsequent analysis.

Lemma 2.3.1 *Assume that either $f|_{\Omega_1} = 0$ or $f|_{\Omega_2} = 0$. Then, for $k = 1, 2$, there exists a positive constant C independent of h such that the following estimates*

$$\|v_h^{(k)} - q_h^{(k)}\|_{H^1(\Omega_k)} \le C \|v_h^{(k)} - q_h^{(k)}\|_{H^{\frac{1}{2}}(\Gamma)} \quad \forall q_h^{(k)} \in S_{h,\Omega_k}^{\perp}$$

hold true, where $f|_{\Omega_k}$ is the restriction of f over Ω_k.

Proof. Take $k = 1$. For $\alpha_h \in V_{h,\Omega_1}$, let $\phi \in X \cap H_0^1(\Omega)$ be the solution of the following interface problem:

$$\begin{aligned} \mathcal{L}\phi &= 0 \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega, \\ [\phi] = 0,\ \left[\mathcal{A}\frac{\partial \phi}{\partial \mathbf{n}}\right] &= \alpha_h \quad \text{along} \quad \Gamma. \end{aligned}$$

By Theorem 1.2.1, ϕ satisfies the regularity estimate

$$\|\phi\|_X \le C \|\alpha_h\|_{H^{\frac{1}{2}}(\Gamma)}. \tag{2.3.6}$$

Note that, for the extension $\tilde{\psi}^{(1)} \in V_h$ defined by (2.3.3) of $\psi^{(1)} \in V_{h,\Omega_1} \cap H_0^1(\Omega_1)$, we have

$$[\phi, \tilde{\psi}^{(1)}]_E = [\phi, \psi]_{1,\Omega_1} = 0. \tag{2.3.7}$$

Since $f|_{\Omega_1} = 0$, using (2.1.4), we obtain

$$[u, \psi^{(1)}]_{1,\Omega_1} = 0 \quad \forall\, \psi^{(1)} \in V_{h,\Omega_1} \cap H_0^1(\Omega_1),$$

and hence, with an aid of (2.3.5), yields

$$[v_h^{(1)} - q_h^{(1)}, \psi^{(1)}]_{1,\Omega_1} = 0 \quad \forall\, \psi^{(1)} \in V_{h,\Omega_1} \cap H_0^1(\Omega_1),\ \forall q_h^{(1)} \in S_{h,\Omega_1}^{\perp}. \tag{2.3.8}$$

From (2.3.7) and (2.3.8), we obtain

$$[\phi - (v_h^{(1)} - q_h^{(1)}), \psi^{(1)}]_{1,\Omega_1} = 0 \quad \forall\, \psi^{(1)} \in V_{h,\Omega_1} \cap H_0^1(\Omega_1),\ \forall q_h^{(1)} \in S_{h,\Omega_1}^{\perp}. \tag{2.3.9}$$

Note that, for every $\alpha_h \in V_{h,\Omega_1}$, we find $\tilde{\alpha_h} \in V_{h,\Omega_2}$ such that $\alpha_h|_\Gamma = \tilde{\alpha_h}|_\Gamma$. Clearly, $\alpha_h - F_{h,2}\tilde{\alpha_h} \in V_{h,\Omega_1} \cap H_0^1(\Omega_1)$ and

$$\|F_{h,2}\tilde{\alpha_h}\|_{H^1(\Omega_1)} \leq C\|\tilde{\alpha_h}\|_{H^{\frac{1}{2}}(\Gamma)} = C\|\alpha_h\|_{H^{\frac{1}{2}}(\Gamma)}. \tag{2.3.10}$$

By setting $\alpha_h = q_h^{(1)} - v_h^{(1)} \in V_{h,\Omega_1}$ and applying (2.3.9), we have

$$\begin{aligned}
&\|\phi - (q_h^{(1)} - v_h^{(1)})\|^2_{H^1(\Omega_1)} \\
&\leq C\{[\phi, \phi - (q_h^{(1)} - v_h^{(1)})]_{1,\Omega_1} - [F_{h,2}(\tilde{\alpha_h}), \phi - (q_h^{(1)} - v_h^{(1)})]_{1,\Omega_1}\} \\
&\leq C\{\|\phi\|_{H^1(\Omega_1)}\|\phi - (q_h^{(1)} - v_h^{(1)})\|_{H^1(\Omega_1)} \\
&\quad + \|F_{h,2}(\tilde{\alpha_h})\|_{H^1(\Omega_1)}\|\phi - (q_h^{(1)} - v_h^{(1)})\|_{H^1(\Omega_1)}\}.
\end{aligned}$$

Using Young's inequality and estimate (2.3.10), we obtain

$$\begin{aligned}
\|\phi - (q_h^{(1)} - v_h^{(1)})\|^2_{H^1(\Omega_1)} &\leq C\left(\|\phi\|^2_{H^1(\Omega_1)} + \|F_{h,2}(\tilde{\alpha_h})\|^2_{H^1(\Omega_1)}\right) \\
&\leq C\left(\|\phi\|^2_{H^1(\Omega_1)} + \|\alpha_h\|^2_{H^{\frac{1}{2}}(\Gamma)}\right).
\end{aligned}$$

This, together with (2.3.6) and $\alpha_h = q_h^{(1)} - v_h^{(1)}$, yields

$$\begin{aligned}
\|q_h^{(1)} - v_h^{(1)}\|_{H^1(\Omega_1)} &\leq C\left(\|\phi - (q_h^{(1)} - v_h^{(1)})\|_{H^1(\Omega_1)} + \|\phi\|_{H^1(\Omega_1)}\right) \\
&\leq C\|q_h^{(1)} - v_h^{(1)}\|_{H^{\frac{1}{2}}(\Gamma)}.
\end{aligned}$$

The case $k = 2$ can be covered in a similar fashion and hence, we omit the details. This completes the rest of the proof. □

Let $q_h^{(k)} \in S^\perp_{h,\Omega_k}$ be arbitrary. Then the triangle inequality and the approximation properties (2.2.7)-(2.2.8) implies

$$\begin{aligned}
\|u - q_h^{(k)}\|_{1,\Omega_k} &\leq \|u - v_h^{(k)}\|_{1,\Omega_k} + \|v_h^{(k)} - q_h^{(k)}\|_{1,\Omega_k} \\
&\leq \inf_{w_h \in V_{h,\Omega_k}} \|u - w_h\|_{H^1(\Omega_k)} + \|q_h^{(k)} - v_h^{(k)}\|_{H^1(\Omega_k)} \\
&\leq C\left(h\|u\|_{H^2(\Omega_k)} + \|q_h^{(k)} - v_h^{(k)}\|_{H^1(\Omega_k)}\right).
\end{aligned} \tag{2.3.11}$$

Here, we have used the fact that $v_h^{(k)}$, $k = 1, 2$, is the best approximation of u on V_{h,Ω_k} with respect to the inner product $[\cdot, \cdot]_{1,\Omega_k}$.

Now, using Lemma 2.3.1 and trace result, we obtain

$$\begin{aligned}
\|u - q_h^{(k)}\|_{1,\Omega_k} &\leq C\left(h\|u\|_{H^2(\Omega_k)} + \|v_h^{(k)} - q_h^{(k)}\|_{H^{1/2}(\Gamma)}\right) \\
&\leq C\left(h\|u\|_{H^2(\Omega_k)} + \|u - v_h^{(k)}\|_{H^{1/2}(\Gamma)} + \|u - q_h^{(k)}\|_{H^{1/2}(\Gamma)}\right) \\
&\leq C\left(h\|u\|_{H^2(\Omega_k)} + \|u - v_h^{(k)}\|_{H^1(\Omega_k)} + \|u - q_h^{(k)}\|_{H^1(\Omega_s)}\right) \\
&\leq C\left(h\|u\|_{H^2(\Omega_k)} + \inf_{w_h \in V_{h,\Omega_k}} \|u - w_h\|_{H^1(\Omega_k)} + \|u - q_h^{(k)}\|_{H^1(\Omega_s)}\right) \\
&\leq C\left(h\|u\|_{H^2(\Omega_k)} + \|u - q_h^{(k)}\|_{H^1(\Omega_s)}\right),
\end{aligned}$$

where $s = 2(1)$ when $k = 1(2)$.
The above estimates lead to

$$\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} &\leq \inf_{q_h^{(k)} \in V_h} \|u - q_h^{(k)}\|_{H^1(\Omega)} \\
&\leq C \inf_{q_h^{(k)} \in S^{\perp}_{h,\Omega_k}} \|u - q_h^{(k)}\|_E \\
&\leq C \inf_{q_h^{(k)} \in S^{\perp}_{h,\Omega_k}} \left\{ \|u - q_h^{(k)}\|_{1,\Omega_k} + \|u - q_h^{(k)}\|_{1,\Omega_s} \right\} \\
&\leq C \inf_{q_h^{(k)} \in S^{\perp}_{h,\Omega_k}} \left\{ h\|u\|_{H^2(\Omega_k)} + \|u - q_h^{(k)}\|_{1,\Omega_s} \right\}.
\end{aligned}$$

We now construct an extension operator $E_{h,s} : G_{h,\Omega_s} \to V_{h,\Omega_k}$. For any $\alpha_h^{(s)} \in G_{h,\Omega_s}$, we define $E_{h,s}\alpha_h^{(s)} \in V_{h,\Omega_k}$ by $\alpha_h^{(s)} = E_{h,s}\alpha_h^{(s)}|_\Gamma$ and solve the discrete system

$$[E_{h,s}\alpha_h^{(s)}, \psi]_{1,\Omega_k} = 0 \quad \forall \psi \in V_{h,\Omega_k} \cap H^1_0(\Omega_k). \tag{2.3.12}$$

Thus, $E_{h,s}$ enable us to extend every $w_h^{(s)} \in V_{h,\Omega_s}$ to a function $q_h^{(k)} \in S^{\perp}_{h,\Omega_k}$ via (2.3.12). Now, using the approximation properties (2.2.7)-(2.2.8), we conclude that

$$\begin{aligned}
\|u - u_h\|_{H^1(\Omega)} &\leq C \inf_{q_h^{(s)} \in V_{h,\Omega_s}} \left\{ h\|u\|_{H^2(\Omega_k)} + \|u - q_h^{(s)}\|_{1,\Omega_s} \right\} \\
&\leq C \left\{ h\|u\|_{H^2(\Omega_k)} + h\|u\|_{H^2(\Omega_s)} \right\}.
\end{aligned}$$

Thus, we have proved the following theorem.

Theorem 2.3.1 *Let u and u_h be the solutions of (2.1.1)-(2.1.3) and (2.2.2), respectively. Then, for $f|_{\Omega_1} = 0$ or $f|_{\Omega_2} = 0$ and $g \in H^{\frac{1}{2}}(\Gamma)$, there exists a positive constant C independent of h such that*

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch \left(\|u\|_{H^2(\Omega_1)} + \|u\|_{H^2(\Omega_2)} \right).$$

We are now in a position to prove the following optimal H^1 norm estimate.

Theorem 2.3.2 *Let u and u_h be the solutions of (2.1.1)-(2.1.3) and (2.2.2), respectively. Then, for $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Omega)$, there exists a positive constant C independent of h such that*

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)} \right).$$

Proof. We write $f_k = f|_{\Omega_k}$ be the restriction of f over Ω_k. Define $\tilde{f}_k : \Omega \to \mathbb{R}$ by

$$\tilde{f}_k = \begin{cases} f_k & \text{in } \Omega_k, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{f}_k \in L^2(\Omega)$ and $f = \tilde{f}_1 + \tilde{f}_2$ a.e. in Ω. We now consider the following interface problems: Let w_k be the solution of

$$\begin{aligned} \mathcal{L}w_k &= \tilde{f}_k \quad \text{in } \Omega, \\ w_k &= 0 \quad \text{on } \partial\Omega, \\ [w_k] = 0, \quad \left[\mathcal{A}\frac{\partial w_k}{\partial \mathbf{n}}\right] &= \frac{g}{2} \quad \text{along} \quad \Gamma. \end{aligned}$$

Now, applying Theorem 1.2.1 for the above interface problem, we have

$$\|w_k\|_X \leq C\left\{\|f\|_{L^2(\Omega_k)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)}\right\}. \tag{2.3.13}$$

Let $w_h^k \in V_h$ be the finite element approximation to w_k defined by (2.2.2) with $f = \tilde{f}_k$ and replacing interface function g by $\frac{g}{2}$. Then it is easy to verify that $u = w_1 + w_2$ and $u_h = w_h^1 + w_h^2$. Therefore, it follows from regularity estimate (2.3.13) and Theorem 2.3.1 that

$$\begin{aligned} \|u - u_h\|_{H^1(\Omega)} &\leq \|w_1 - w_h^1\|_{H^1(\Omega)} + \|w_2 - w_h^2\|_{H^1(\Omega)} \\ &\leq C\left(h\|w_1\|_X + h\|w_2\|_X\right) \\ &\leq Ch\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)}\right), \end{aligned}$$

and this completes the proof. □

For the L^2 norm estimate, let us consider the following problem: Find $w \in H_0^1(\Omega)$ such that

$$A(w, v) = (u - u_h, v) \quad \forall v \in H_0^1(\Omega), \tag{2.3.14}$$

with finite element approximation: Find $w_h \in V_h$ such that

$$A(w_h, v_h) = (u - u_h, v_h) \quad \forall v_h \in V_h. \tag{2.3.15}$$

Note that, here $g = 0$ and $f = u - u_h$. Now, we choose $v = u - u_h \in H_0^1(\Omega)$. Then, from (2.3.14) and orthogonality property (2.2.3), we have

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega)}^2 &= A(w, u - u_h) \\ &= A(w - w_h, u - u_h) + A(w_h, u - u_h) \\ &= A(w - w_h, u - u_h) \\ &\leq C\|w - w_h\|_{H^1(\Omega)}\|u - u_h\|_{H^1(\Omega)}. \end{aligned}$$

It now follows from Theorem 2.3.2 that

$$\|u - u_h\|_{L^2(\Omega)}^2 \leq Ch\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)}\right)\|w - w_h\|_{H^1(\Omega)}.$$

Again, applying Theorem 2.3.2 for the problems (2.3.14)-(2.3.15), we get

$$\|u-u_h\|_{L^2(\Omega)}^2 \le Ch^2\left(\|f\|_{L^2(\Omega)}+\|g\|_{H^{\frac{1}{2}}(\Gamma)}\right)\|u-u_h\|_{L^2(\Omega)}, \tag{2.3.16}$$

and this proves the following optimal L^2 norm estimate.

Theorem 2.3.3 *Let u and u_h be the solutions of (2.1.1)-(2.1.3) and (2.2.2), respectively. Then, for $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma)$, there exists a positive constant C independent of h such that*

$$\|u-u_h\|_{L^2(\Omega)} \le Ch^2\left(\|f\|_{L^2(\Omega)}+\|g\|_{H^{\frac{1}{2}}(\Gamma)}\right).$$

Remark 2.3.1 *Note that Theorem 2.3.2 and Theorem 2.3.3 yield optimal error bounds even if the global regularity of the solution u is low on the whole domain Ω.*

2.4 Effect of Numerical Quadrature

In this section, we shall study the effect of numerical quadrature on finite element solutions of the interface problems (2.1.1)-(2.1.3). Optimal order error estimates are derived in H^1 and L^2 norms.

Define an approximation of the original bilinear form $A(u_h, v_h)$ in V_h and induced norm by

$$\overline{A}_h(u_h,v_h)=\sum_{K\in\mathcal{T}_h}\left[\text{meas}(K)\left\{\sum_{i,j=1}^{2}a_{ij}^K(b_K)\frac{\partial u_h}{\partial x_i}\frac{\partial v_h}{\partial x_j}+a(b_K)u_h(b_K)v_h(b_K)\right\}\right],$$

and

$$\|\phi\|_{1,h}=\overline{A}_h(\phi,\phi)^{\frac{1}{2}}.$$

Here, $\mathcal{T}_h$ is the set of straight triangles and b_K is the barycentre of a triangle $K \in \mathcal{T}_h$. a_{ij}^K denote the approximation of a_{ij} which is defined as follows: For each triangle $K \in \mathcal{T}_h$, $a_{ij}^K = a_{ij}^s$ if $K \subset \Omega_s^h, s = 1, 2$. To treat the interface integral $\langle g, v_h\rangle_\Gamma$, for $g \in H^2(\Gamma)$, we define $g_h \in V_h$ by

$$g_h=\sum_{j=1}^{m_h}g(P_j)\Phi_j^h,$$

where $\{\Phi_j^h\}_{j=1}^{m_h}$ is the set of standard nodal basis functions corresponding to the nodes $\{P_j\}_{j=1}^{m_h}$ on the interface.

Now, the finite element Galerkin method with quadrature may be stated as follows: Find $u_h^* \in V_h$ such that

$$\overline{A}_h(u_h^*,v_h)=(f,v_h)+\langle g_h,v_h\rangle_{\Gamma_h}\quad \forall v_h\in V_h. \tag{2.4.1}$$

The following lemmas will be useful for our future analysis. For a proof, we refer to Chen *et al.* [14] and Ciarlet [16].

Lemma 2.4.1 *On V_h the norms $\|.\|_{H^1(\Omega)}$ and $\|.\|_{1,h}$ are equivalent. Further, for $w, v \in V_h$*

$$|\overline{A}_h(w,v) - A(w,v)| \leq Ch\|w\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

Lemma 2.4.2 *Let $\Omega_\Gamma^* \subset \Omega$ be the union of all interface curved triangles. Then, for $w_h \in V_h$, we have*

$$|\langle g, w_h\rangle - \langle g_h, w_h\rangle_{\Gamma_h}| \leq Ch^{\frac{3}{2}}\|g\|_{H^2(\Gamma)}\|w_h\|_{H^1(\Omega_\Gamma^*)}.$$

In view of Lemma 2.4.1, we obtain

$$\begin{aligned}
\|u_h - u_h^*\|^2_{H^1(\Omega)} &\leq C\overline{A}_h(u_h - u_h^*, u_h - u_h^*) \\
&= C\left\{\overline{A}_h(u_h, u_h - u_h^*) - \overline{A}_h(u_h^*, u_h - u_h^*)\right\} \\
&= C\left\{\overline{A}_h(u_h, u_h - u_h^*) - A(u_h, u_h - u_h^*)\right\} \\
&\quad +C\left\{A(u_h, u_h - u_h^*) - \overline{A}_h(u_h^*, u_h - u_h^*)\right\} \\
&= C\left\{\overline{A}_h(u_h, u_h - u_h^*) - A(u_h, u_h - u_h^*)\right\} \\
&\quad +C\left\{A(u, u_h - u_h^*) - \overline{A}_h(u_h^*, u_h - u_h^*)\right\},
\end{aligned}$$

in the last equality, we used the Galerkin orthogonality (2.2.3).

Further, using (2.1.4) and (2.4.1), we obtain

$$\begin{aligned}
\|u_h - u_h^*\|^2_{H^1(\Omega)} &\leq C\left\{\overline{A}_h(u_h, u_h - u_h^*) - A(u_h, u_h - u_h^*)\right\} \\
&\quad +C\left\{(f, u_h - u_h^*) + \langle g, u_h - u_h^*\rangle_\Gamma\right\} \\
&\quad -C\left\{(f, u_h - u_h^*) + \langle g_h, u_h - u_h^*\rangle_{\Gamma_h}\right\} \\
&= C\left\{\overline{A}_h(u_h, u_h - u_h^*) - A(u_h, u_h - u_h^*)\right\} \\
&\quad +C\left\{\langle g, u_h - u_h^*\rangle_\Gamma - \langle g_h, u_h - u_h^*\rangle_{\Gamma_h}\right\} \\
&=: (I)_1 + (I)_2. \qquad (2.4.2)
\end{aligned}$$

To estimate $(I)_1$, apply Lemma 2.4.1, Theorem 2.3.2 and Theorem 1.2.1 to have

$$\begin{aligned}
|(I)_1| &\leq C|\overline{A}_h(u_h, u_h - u_h^*) - A(u_h, u_h - u_h^*)| \\
&\leq Ch\|u_h\|_{H^1(\Omega)}\|u_h - u_h^*\|_{H^1(\Omega)} \\
&\leq Ch\{\|u_h - u\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)}\}\|u_h - u_h^*\|_{H^1(\Omega)} \\
&\leq Ch\left\{\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)}\right\}\|u_h - u_h^*\|_{H^1(\Omega)}. \qquad (2.4.3)
\end{aligned}$$

For $(I)_2$, use Lemma 2.4.2 to obtain

$$\begin{aligned} |(I)_2| &\leq Ch^{\frac{3}{2}} \|g\|_{H^2(\Gamma)} \|u_h - u_h^*\|_{H^1(\Omega_\Gamma^*)} \\ &\leq Ch^{\frac{3}{2}} \|g\|_{H^2(\Gamma)} \|u_h - u_h^*\|_{H^1(\Omega)} \end{aligned} \tag{2.4.4}$$

which, together with (2.4.2)-(2.4.3), yields

$$\|u_h - u_h^*\|_{H^1(\Omega)} \leq Ch \left\{ \|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)} \right\}. \tag{2.4.5}$$

Now, (2.4.5) combine with Theorem 2.3.2 leads to the following theorem.

Theorem 2.4.1 *Let u and u_h^* be the solutions of (2.1.1)-(2.1.3) and (2.4.1), respectively. Then, for $f \in L^2(\Omega)$ and $g \in H^2(\Omega)$, there exists a positive constant C independent of h such that*

$$\|u - u_h^*\|_{H^1(\Omega)} \leq Ch \left(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)} \right).$$

Next, for the L^2 norm error estimate, we proceed by the duality argument. To do so, we consider the following problem. Let $\phi \in X \cap H_0^1(\Omega)$ be the solution of the following auxiliary problem

$$\mathcal{L}\phi = u_h - u_h^* \quad \text{in } \Omega \tag{2.4.6}$$

$$\phi = 0 \quad \text{on } \partial\Omega \tag{2.4.7}$$

$$[\phi] = 0, \quad \left[\mathcal{A} \frac{\partial \phi}{\partial \mathbf{n}} \right] = 0 \quad \text{along} \quad \Gamma. \tag{2.4.8}$$

Let ϕ_h be its Galerkin approximation as defined through (2.4.1) with $f = u_h - u_h^*$ and $g_h = 0$. We then have

$$\begin{aligned} \|u_h - u_h^*\|_{L^2(\Omega)}^2 &= \overline{A}_h(\phi_h, u_h - u_h^*) \\ &= \overline{A}_h(\phi_h, u_h) - \overline{A}_h(\phi_h, u_h^*) \\ &= \overline{A}_h(\phi_h, u_h) - (f, \phi_h) - \langle g_h, \phi_h \rangle_{\Gamma_h} \\ &= \overline{A}_h(\phi_h, u_h) - A(\phi_h, u_h) + A(\phi_h, u_h) \\ &\quad -(f, \phi_h) - \langle g_h, \phi_h \rangle_{\Gamma_h} \\ &= \overline{A}_h(\phi_h, u_h) - A(\phi_h, u_h) + A(\phi_h, u) \\ &\quad -(f, \phi_h) - \langle g_h, \phi_h \rangle_{\Gamma_h} \\ &= \overline{A}_h(\phi_h, u_h) - A(\phi_h, u_h) + (f, \phi_h) + \langle g, \phi_h \rangle_\Gamma \\ &\quad -(f, \phi_h) - \langle g_h, \phi_h \rangle_{\Gamma_h} \\ &= \{\overline{A}_h(\phi_h, u_h) - A(\phi_h, u_h)\} + \{\langle g, \phi_h \rangle_\Gamma - \langle g_h, \phi_h \rangle_{\Gamma_h}\} \\ &=: (II)_1 + (II)_2. \end{aligned} \tag{2.4.9}$$

If $\Omega_\Gamma^* \subset \Omega$ is the union of all curved triangles in Ω, it follows from Ciarlet [16] that

$$\begin{aligned} |(II)_1| &\leq Ch^2\|u_h\|_{H^1(\Omega)}\|\phi_h\|_{H^1(\Omega)} + C\sum_{K\in\mathcal{T}_C^*} h\|u_h\|_{H^1(K)}\|\phi_h\|_{H^1(K)} \\ &\leq Ch^2\{\|u_h - u\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)}\}\{\|\phi_h - \phi\|_{H^1(\Omega)} + \|\phi\|_{H^1(\Omega)}\} \\ &\quad + Ch\{\|u_h - u\|_{H^1(\Omega_\Gamma^*)} + \|u\|_{H^1(\Omega_\Gamma^*)}\} \\ &\quad \times\{\|\phi_h - \phi\|_{H^1(\Omega_\Gamma^*)} + \|\phi\|_{H^1(\Omega_\Gamma^*)}\}. \end{aligned} \tag{2.4.10}$$

Recall $\mathcal{T}_C^*$ is the set of all curved triangles in Ω. For the union of all curved triangles Ω_Γ^*, we have (cf. Huang and Zou [27], p.570)

$$\|u\|_{H^1(\Omega_\Gamma^*)} \leq Ch^{\frac{1}{2}}\|u\|_X, \quad \|\phi\|_{H^1(\Omega_\Gamma^*)} \leq Ch^{\frac{1}{2}}\|\phi\|_X.$$

The above estimates, together with (2.4.10) and Theorems 1.2.1, 2.3.2, yield

$$\begin{aligned} |(II)_1| &\leq Ch^2\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)}\right)\{\|\phi_h - \phi\|_{H^1(\Omega)} + \|\phi\|_{H^1(\Omega)}\} \\ &\quad + Ch^{\frac{3}{2}}\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)}\right)\{\|\phi_h - \phi\|_{H^1(\Omega_\Gamma^*)} + h^{\frac{1}{2}}\|\phi\|_X\}. \end{aligned}$$

Finally, applying Theorem 1.2.1 and Theorem 2.4.1 for the problem (2.4.6)-(2.4.8), we obtain

$$\begin{aligned} |(II)_1| &\leq Ch^2\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)}\right)\{\|\phi\|_X + \|u_h - u_h^*\|_{L^2(\Omega)}\} \\ &\leq Ch^2\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)}\right)\|u_h - u_h^*\|_{L^2(\Omega)}. \end{aligned} \tag{2.4.11}$$

For the term $(II)_2$, use Lemma 2.4.2 and Theorem 2.4.1 for the problem (2.4.6)-(2.4.8) to obtain

$$\begin{aligned} |(II)_2| &\leq Ch^{\frac{3}{2}}\|g\|_{H^2(\Gamma)}\|\phi_h\|_{H^1(\Omega_\Gamma^*)} \\ &\leq Ch^{\frac{3}{2}}\{\|\phi_h - \phi\|_{H^1(\Omega_\Gamma^*)} + \|\phi\|_{H^1(\Omega_\Gamma^*)}\}\|g\|_{H^2(\Gamma)} \\ &\leq Ch^{\frac{3}{2}}\{h\|u_h - u_h^*\|_{L^2(\Omega)} + h^{\frac{1}{2}}\|\phi\|_X\}\|g\|_{H^2(\Gamma)} \\ &\leq Ch^2\|u_h - u_h^*\|_{L^2(\Omega)}\|g\|_{H^2(\Gamma)}. \end{aligned} \tag{2.4.12}$$

In the last inequality, we have used Theorem 1.2.1 for the problem (2.4.6)-(2.4.8). Combining (2.4.9), (2.4.11) and (2.4.12), we obtain

$$\|u_h - u_h^*\|_{L^2(\Omega)} \leq Ch^2\left\{\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)}\right\},$$

and this, together with Theorem 2.3.3, leads to the following theorem.

Theorem 2.4.2 *Let u and u_h^* be the solutions of (2.1.1)-(2.1.3) and (2.4.1), respectively. Then, for $f \in L^2(\Omega)$ and $g \in H^2(\Omega)$, there exists a positive constant C independent of h such that*

$$\|u - u_h^*\|_{L^2(\Omega)} \leq Ch^2\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Gamma)}\right).$$

Remark 2.4.1 *Here, for simplicity, we do not consider the numerical integration of the term involving f in (2.4.1). This can be treated in a similar manner as in the case of bilinear form $A(u_h, v_h)$ and the interface integral $\langle g, v_h \rangle_\Gamma$. Note that optimal error estimates in Theorems 2.4.1 and 2.4.2 require $g \in H^2(\Gamma)$ whereas in the case of exact integration (Theorems 2.3.2 and 2.3.3) $g \in H^{\frac{1}{2}}(\Gamma)$.*

Chapter 3

Finite Element Method for Elliptic Interface Problems: Part-II

In this chapter, we have derived some new *a priori* error estimates in fitted finite element method for elliptic interface problems in a two dimensional convex polygonal domain. In this method, the grid lines need not fit to the interface exactly. More precisely, when the discretization is based on straight triangulation, optimal order convergence in H^1 norm and sub-optimal in L^2 norm are derived for the elliptic interface problems.

3.1 Introduction

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ and $\Omega_1 \subset \Omega$ is an open domain with C^2 boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \backslash \Omega_1$. We shall again recall the following linear elliptic interface problems of the form

$$\mathcal{L}u = f(x) \quad \text{in } \Omega \tag{3.1.1}$$

subject to the boundary condition

$$u(x) = 0 \quad \text{on } \partial\Omega \tag{3.1.2}$$

and interface conditions

$$[u] = 0, \qquad \left[\mathcal{A}\frac{\partial u}{\partial \mathbf{n}}\right] = g(x) \quad \text{along } \Gamma, \tag{3.1.3}$$

where the symbols $[v]$ and $\mathbf{n}$ are defined as in Chapter 1. Further, $f = f(x)$ and $g = g(x)$ are real valued functions in Ω and Γ, respectively. For the simplicity of exposition, we assume $\mathcal{L}u = -\nabla \cdot (\beta \nabla u)$, where the function β is positive and piecewise constant, i.e.,

$$\beta(x) = \beta_i \quad \text{for} \quad x \in \Omega_i, \; i = 1, 2.$$

Let $H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$. To present the finite element methods for the interface problems (3.1.1)-(3.1.3), we define a bilinear form $A(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$A(w,v) = \int_\Omega \beta(x)\nabla w.\nabla v \quad \forall w, v \in H^1(\Omega).$$

The weak formulation is then defined as a function $u \in H_0^1(\Omega)$ such that

$$A(u,v) = (f,v) + \langle g, v\rangle_\Gamma \quad \forall v \in H_0^1(\Omega). \tag{3.1.4}$$

We shall recall the solution space

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$$

equipped with the norm

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}.$$

In order to derive optimal error estimate, we recall the following statement from Chen and Zou [14].

Remark 3.1.1 *It is proved in Ladyzhenskaya et al. [30] that $u \in X \cap W^{1,\infty}(\Omega_0 \cap \Omega_1) \cap W^{1,\infty}(\Omega_0 \cap \Omega_2) \cap H_0^1(\Omega)$, where Ω_0 is some neighborhood of the interface Γ. Thus, the solution for the interface problem (3.1.1)-(3.1.3) belongs to $X \cap W^{1,\infty}(\Omega_0 \cap \Omega_1) \cap W^{1,\infty}(\Omega_0 \cap \Omega_2) \cap H_0^1(\Omega)$. This assumption is reasonable if the interface is sufficiently smooth.*

The purpose of the present chapter is to study the convergence of fitted finite element solution to the exact solution for elliptic interface problem. In the previous chapter, we have assumed that the grid lines coincide with the actual interface by allowing interface triangles to be curved triangles. But, it is very costly to generate such a mesh whose grid lines follow the actual interface of general shape. Therefore, a modification of the mesh is proposed and analyzed in this chapter. In the proposed method, we approximate the interface curved triangles by straight triangles. The error estimates are shown to be of optimal in H^1 norm and sub-optimal in L^2 norm. The main crucial technical tools used in our analysis are some Sobolev embedding inequality, duality arguments, some newly established interpolation and interface approximation results, and some known results on elliptic problems on non-convex polygonal domain.

The plan of this chapter is as follows. In section 3.2, some new optimal interpolation approximation properties for linear interpolant are obtained. Subsequently, some approximation properties related to the auxiliary projection is also obtained. Section 3.3 is devoted to error estimates for the elliptic interface problem by fitted finite element method with straight triangles.

3.2 Finite Element Discretization and Some Auxiliary Estimates

For the purpose of finite element approximation of the problems (3.1.1)-(3.1.3), we now describe the triangulation $\mathcal{T}_h$ of Ω as follows. We first approximate the domain Ω_1 by a domain Ω_1^h with the polygonal boundary Γ_h whose vertices all lie on the interface Γ. Let Ω_2^h be the approximation for the domain Ω_2 with polygonal exterior and interior boundaries as $\partial\Omega$ and Γ_h, respectively.

Triangulation $\mathcal{T}_h$ of the domain Ω satisfy the following conditions:

($\mathcal{A}1$) $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$.

($\mathcal{A}2$) If K_1, $K_2 \in \mathcal{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex or edge of both triangles.

($\mathcal{A}3$) Each triangle $K \in \mathcal{T}_h$ is either in Ω_1^h or Ω_2^h and intersects Γ (interface) in at most two points.

($\mathcal{A}4$) For each triangle $K \in \mathcal{T}_h$, let r_K, $\overline{r}_K$ be the radii of its inscribed and circumscribed circles, respectively. Let $h = \max\{\overline{r}_K : K \in \mathcal{T}_h\}$. We assume that, for some fixed $h_0 > 0$, there exists two positive constants C_0 and C_1 independent of h such that

$$C_0 r_K \leq h \leq C_1 \overline{r}_K \quad \forall K \in \mathcal{T}_h, \quad \forall h \in (0, h_0).$$

The triangles with one or two vertices on Γ are called the interface triangles, the set of all interface triangles is denoted by $\mathcal{T}_\Gamma^*$ and we write $\Omega_\Gamma^* = \cup_{K\in\mathcal{T}_\Gamma^*} K$.

Let V_h be a family of finite dimensional subspaces of $H_0^1(\Omega)$ defined on $\mathcal{T}_h$ consisting of piecewise linear functions vanishing on the boundary $\partial\Omega$. Examples of such finite element spaces can be found in [12] and [16].

For the coefficients $\beta(x)$, we define its approximation $\beta_h(x)$ as follows: For each triangle $K \in \mathcal{T}_h$, let $\beta_K(x) = \beta_i$ if $K \subset \Omega_i^h$, i=1 or 2. Then β_h is defined as

$$\beta_h(x) = \beta_K(x) \quad \forall K \in \mathcal{T}_h.$$

For $g \in H^2(\Gamma)$ and $f \in L^2(\Omega)$, we define the finite element approximation as follows: Find $u_h \in V_h$ such that

$$A_h(u_h, v_h) = (f, v_h) + \langle g_h, v_h\rangle_{\Gamma_h} \quad \forall v_h \in V_h, \tag{3.2.1}$$

where $A_h(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is given by

$$A_h(w, v) = \sum_{K\in\mathcal{T}_h} \int_K \beta_K(x)\nabla w\nabla v dx \quad \forall w, v \in H^1(\Omega),$$

and $g_h \in V_h$ is the linear interpolant of g given by

$$g_h = \sum_{j=1}^{m_h} g(P_j)\Phi_j^h,$$

where $\{\Phi_j^h\}_{j=1}^{m_h}$ is the set of standard nodal basis functions corresponding to the nodes $\{P_j\}_{j=1}^{m_h}$ on the interface Γ.

By Sobolev embedding theorem, for $v \in X$, we have $v \in W^{1,p}(\Omega)\ \forall p > 2$. Therefore, the linear interpolant operator $\Pi_h : X \to V_h$ is well defined (cf. [19, 51]). As the solutions concerned are only on $H^1(\Omega)$ globally, one can not apply the standard interpolation theory directly. However, following the arguments of [14] it is possible to obtain optimal error bounds for the interpolant Π_h (see, Remark 2.4 of [14]).

Lemma 3.2.1 *Let $\Pi_h : X \to V_h$ be the linear interpolation operator and u be the solution for the interface problem (3.1.1)-(3.1.3), then the following approximation properties*

$$\|u - \Pi_h u\|_{H^m(\Omega)} \le Ch^{2-m}\{\|u\|_X + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_1)} + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_2)}\},\ m = 0, 1,$$

hold true.

Proof. For any $v \in X$, let v_i be the restriction of v on Ω_i for $i = 1, 2$. As the interface is of class C^2, we can extend the function $v_i \in H^2(\Omega_i)$ on to the whole Ω and obtain the function $\tilde{v}_i \in H^2(\Omega)$ such that $\tilde{v}_i = v_i$ on Ω_i and

$$\|\tilde{v}_i\|_{H^2(\Omega)} \le C\|v_i\|_{H^2(\Omega_i)}, i = 1, 2. \tag{3.2.2}$$

For the existence of such extensions, we refer to Stein [56].

Now, for any triangle $K \in \mathcal{T}_h \backslash \mathcal{T}_\Gamma^*$, the standard finite element interpolation theory (cf. [12, 16]) implies that

$$\|u - \Pi_h u\|_{H^m(K)} \le Ch^{2-m}\|u\|_{H^2(K)},\ m = 0, 1. \tag{3.2.3}$$

For any element $K \in \mathcal{T}_\Gamma^*$, we write $K_i = K \cap \Omega_i, i = 1, 2$, for our convenient. Again it follows from the analysis of [59] that $\mathrm{dist}(\Gamma, \Gamma_h) \le \mathrm{O}(h^2)$. Thus, without loss of generality, we can assume that $\mathrm{meas}(K_2) \le Ch^3$. Further, using the Hölder's inequality and the fact $\mathrm{meas}(K_2) \le Ch^3$ we derive that for any $p > 2$, and $m = 0, 1$,

$$\begin{aligned}
\|u - \Pi_h u\|_{H^m(K_2)} &\le Ch^{\frac{3(p-2)}{2p}}\|u - \Pi_h u\|_{W^{m,p}(K_2)} \\
&\le Ch^{\frac{3(p-2)}{2p}}\|u - \Pi_h u\|_{W^{m,p}(K)} \\
&\le Ch^{\frac{3(p-2)}{2p}+1-m}\|u\|_{W^{1,p}(K)},
\end{aligned} \tag{3.2.4}$$

in the last inequality, we used the standard interpolation theory (cf. [16]). On the other hand

$$\begin{aligned} \|u-\Pi_h u\|_{H^m(K_1)} &= \|\tilde{u}_1-\Pi_h\tilde{u}_1\|_{H^m(K_1)} \\ &\leq C\|\tilde{u}_1-\Pi_h\tilde{u}_1\|_{H^m(K)} \\ &\leq Ch^{2-m}\|\tilde{u}_1\|_{H^2(K)} \\ &\leq Ch^{2-m}\|u\|_X, \end{aligned} \tag{3.2.5}$$

in the last inequality, we used (3.2.2).

In view of (3.2.4)-(3.2.5), it now follows that

$$\begin{aligned} &\|u-\Pi_h u\|^2_{H^m(\Omega^*_\Gamma)} \\ &\leq Ch^{4-2m}\|u\|^2_X + C\sum_{K\in\mathcal{T}^*_\Gamma} h^{\frac{3(p-2)}{p}+2-2m}\|u\|^2_{W^{1,p}(K)} \\ &\leq Ch^{4-2m}\|u\|^2_X + C\sum_{K\in\mathcal{T}^*_\Gamma} h^{5-2m-\frac{6}{p}}\|u\|^2_{W^{1,p}(K)} \\ &\leq Ch^{4-2m}\|u\|^2_X + C\sum_{K\in\mathcal{T}^*_\Gamma} h^{5-2m-\frac{6}{p}}\{\|u\|^2_{W^{1,p}(K_1)}+\|u\|^2_{W^{1,p}(K_2)}\} \\ &\leq Ch^{4-2m}\|u\|^2_X + C\sum_{K\in\mathcal{T}^*_\Gamma} h^{5-2m-\frac{6}{p}}\{\|u\|^2_{W^{1,p}(K\cap\Omega_1)}+\|u\|^2_{W^{1,p}(K\cap\Omega_2)}\} \\ &\leq Ch^{4-2m}\|u\|^2_X + Ch^{5-2m-\frac{6}{p}}\{\|u\|^2_{W^{1,p}(\Omega^*_\Gamma\cap\Omega_1)}+\|u\|^2_{W^{1,p}(\Omega^*_\Gamma\cap\Omega_2)}\} \\ &\leq Ch^{4-2m}\|u\|^2_X + Ch^{5-2m-\frac{6}{p}}\{\|u\|^2_{W^{1,p}(\Omega_0\cap\Omega_1)}+\|u\|^2_{W^{1,p}(\Omega_0\cap\Omega_2)}\}, \end{aligned}$$

for sufficiently small $h>0$ such that $\Omega^*_\Gamma\subset\Omega_0$, Ω_0 is some neighborhood of Γ.

By a simple calculation, for $i=1,2$, we obtain

$$\|u\|_{W^{1,p}(\Omega_0\cap\Omega_i)} \leq \text{meas}(\Omega_0\cap\Omega_i)^{\frac{1}{p}}\|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_i)} \leq C^{\frac{1}{p}}\|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_i)},$$

and this leads to

$$\begin{aligned} \|u-\Pi_h u\|^2_{H^m(\Omega^*_\Gamma)} \leq\ & Ch^{4-2m}\|u\|^2_X \\ & +Ch^{5-2m-\frac{6}{p}}C^{\frac{2}{p}}\{\|u\|^2_{W^{1,\infty}(\Omega_0\cap\Omega_1)}+\|u\|^2_{W^{1,\infty}(\Omega_0\cap\Omega_2)}\}. \end{aligned}$$

Which, together with (3.2.3), implies

$$\begin{aligned} \|u-\Pi_h u\|^2_{H^m(\Omega)} \leq\ & Ch^{4-2m}\|u\|^2_X \\ & +Ch^{5-2m-\frac{6}{p}}C^{\frac{2}{p}}\{\|u\|^2_{W^{1,\infty}(\Omega_0\cap\Omega_1)}+\|u\|^2_{W^{1,\infty}(\Omega_0\cap\Omega_2)}\}. \end{aligned} \tag{3.2.6}$$

Taking $p\to\infty$ both sides of (3.2.6), we deduce

$$\|u-\Pi_h u\|_{H^m(\Omega)} \leq Ch^{2-m}\{\|u\|_X+\|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_1)}+\|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_2)}\}.$$

This completes the proof of Lemma 3.2.1. □

The following lemma is proved for our future use.

Lemma 3.2.2 *Let Ω_Γ^* be the union of all interface triangles and u is a solution of (3.1.1)-(3.1.3), then we have*

$$\|u\|_{H^1(\Omega_\Gamma^*)} \le Ch^{\frac{1}{2}}\|u\|_X.$$

Proof. For any $K \in \mathcal{T}_\Gamma^*$, we have

$$\begin{aligned} \|u\|_{H^1(K)} &\le \|u\|_{H^1(K_1)} + \|u\|_{H^1(K_2)} \\ &\le \|\tilde{u}_1\|_{H^1(K_1)} + \|\tilde{u}_2\|_{H^1(K_2)} \\ &\le \|\tilde{u}_1\|_{H^1(K)} + \|\tilde{u}_2\|_{H^1(K)}, \end{aligned} \tag{3.2.7}$$

where $\tilde{u}_i \in H^2(\Omega)$ is a previously defined extension of $u|_{\Omega_i} = u_i \in H^2(\Omega_i)$, $i = 1, 2$, onto whole domain Ω such that $\tilde{u}_i = u_i$ on Ω_i.

We now recall Sobolev embedding inequality for two dimensions (cf. [50])

$$\|u\|_{L^p(\Omega)} \le Cp^{\frac{1}{2}}\|u\|_{H^1(\Omega)} \;\; \forall v \in H^1(\Omega), \;\; p > 2. \tag{3.2.8}$$

Consider an interface triangle K. Then

$$\|\tilde{u}_i\|^2_{H^1(K)} = \|\tilde{u}_i\|^2_{L^2(K)} + \|\nabla\tilde{u}_i\|^2_{L^2(K)}. \tag{3.2.9}$$

An application of (3.2.8) and Hölder's inequality yields

$$\begin{aligned} \|\tilde{u}_i\|^2_{L^2(K)} &= \int_K |\tilde{u}_i|^2 dx \\ &\le \left(\int_K 1.dx\right)^{1/2} \left(\int_K |\tilde{u}_i|^4 dx\right)^{1/2} \\ &= (\text{meas}(K))^{1/2}\|\tilde{u}_i\|^2_{L^4(K)} \\ &\le (\text{meas}(K))^{1/2} 4C\|\tilde{u}_i\|^2_{H^1(K)}. \end{aligned}$$

Since $\text{meas}(K) \le Ch^2$, we have

$$\|\tilde{u}_i\|^2_{L^2(K)} \le Ch\|\tilde{u}_i\|^2_{H^1(K)}.$$

Similarly,

$$\|\nabla\tilde{u}_i\|^2_{L^2(K)} \le Ch\|\tilde{u}_i\|^2_{H^2(K)},$$

for an interface triangle K. Thus, we now obtain from (3.2.9) that

$$\|\tilde{u}_i\|_{H^1(K)} \le Ch^{\frac{1}{2}}\|\tilde{u}_i\|_{H^2(K)}. \tag{3.2.10}$$

In view of (3.2.10), (3.2.7) and (3.2.2), it now follows that

$$\begin{aligned}\|u\|_{H^1(\Omega_\Gamma^*)} &\leq \sum_{K\in\mathcal{T}_\Gamma^*}\|u\|_{H^1(K)}\\ &\leq Ch^{\frac{1}{2}}\{\|\tilde{u}_1\|_{H^2(\Omega)}+\|\tilde{u}_2\|_{H^2(\Omega)}\}\\ &\leq Ch^{\frac{1}{2}}\|u\|_X.\end{aligned}$$

This completes the rest of the proof. □

The following lemma is on the approximation property of g_h to the interface function g. For a proof, see [14].

Lemma 3.2.3 *Assume that $g \in H^2(\Gamma)$. Then we have*

$$\left|\int_\Gamma g v_h ds - \int_{\Gamma_h} g_h v_h ds\right| \leq Ch^{\frac{3}{2}}\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega_\Gamma^*)} \quad \forall v_h \in V_h.$$

For any $v \in X$, we define

$$f^* = \begin{cases} -\beta_1\Delta v_1 & \text{in } \Omega_1,\\ -\beta_2\Delta v_2 & \text{in } \Omega_2.\end{cases}$$

With this f^*, define an operator $Q_h : X \cap H_0^1(\Omega) \to V_h$ by

$$A_h(Q_h v, \phi) = (f^*, \phi) = A(v, \phi) \quad \forall\phi \in V_h,\ v \in X \cap H_0^1(\Omega). \tag{3.2.11}$$

The error estimates obtained for Q_h in [14] are not optimal. Below, we present a proof which shows that the loss in accuracy for H^1 norm can be recovered via interpolation postprocessing technique. This lemma is very crucial for our later analysis.

Lemma 3.2.4 *Let Q_h be defined by (3.2.11). Then there exists a positive constant C independent of h such that*

$$\|u - Q_h u\|_{H^1(\Omega)} \leq C(u)h; \quad \|u - Q_h u\|_{L^2(\Omega)} \leq C(u)h^{1+\min\{\frac{1}{2},s\}}, \ \ 0 < s < 1,$$

where $C(u) = C(\|u\|_X + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_1)} + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_2)})$.

Proof. We first split $u - Q_h u$ as

$$u - Q_h u = (u - \Pi_h u) + (\Pi_h u - Q_h u).$$

From Lemma 3.2.1 and (3.2.11), we note that

$$\begin{aligned}&C\|\Pi_h u - Q_h u\|^2_{H^1(\Omega)}\\ &\leq A_h(\Pi_h u - u, \Pi_h u - Q_h u) + A_h(u - Q_h u, \Pi_h u - Q_h u)\\ &\leq Ch(\|u\|_X + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_1)} + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_2)})\|\Pi_h u - Q_h u\|_{H^1(\Omega)}\\ &\quad +\{A_h(u, \Pi_h u - Q_h u) - A(u, \Pi_h u - Q_h u)\}\\ &=: C(u)h\|\Pi_h u - Q_h u\|_{H^1(\Omega)} + (I).\end{aligned} \tag{3.2.12}$$

To estimate (I), we need the following information

$$\text{supp}\,(\beta - \beta_h) \cap K = \tilde{K},\ \tilde{K} = K_1 \text{ or } K_2.$$

Recall that $K_i = K \cap \Omega_i$ for $i = 1, 2$, and $K \in \mathcal{T}_\Gamma^*$. Then we have

$$\begin{aligned}
|(I)| &\leq \sum_{K \in \mathcal{T}_\Gamma^*} \int_{\tilde{K}} |(\beta_h - \beta)\nabla u \nabla(\Pi_h u - Q_h u)| \\
&\leq C \sum_{K \in \mathcal{T}_\Gamma^*} \left[\|\nabla u\|_{L^2(\tilde{K})} \|\nabla(\Pi_h u - Q_h u)\|_{L^2(\tilde{K})} \right] \qquad (3.2.13) \\
&\leq C \sum_{K \in \mathcal{T}_\Gamma^*} \|\nabla(u - \Pi_h u)\|_{L^2(\tilde{K})} \|\nabla(\Pi_h u - Q_h u)\|_{L^2(\tilde{K})} \\
&\quad + C \sum_{K \in \mathcal{T}_\Gamma^*} \|\nabla \Pi_h u\|_{L^2(\tilde{K})} \|\nabla(\Pi_h u - Q_h u)\|_{L^2(\tilde{K})} \\
&=: (I)_1 + (I)_2. \qquad (3.2.14)
\end{aligned}$$

In view of Lemma 3.2.1, for $(I)_1$, we have

$$|(I)_1| \leq C(u)h \|\Pi_h u - Q_h u\|_{H^1(\Omega)}. \qquad (3.2.15)$$

Since meas$(\tilde{K}) \leq Ch^3$, $|\nabla \Pi_h u|$ and $|\nabla(\Pi_h u - P_h u)|$ are constant in $K \in \mathcal{T}_h$, we have

$$\begin{aligned}
|(I)_2| &\leq Ch \sum_{K \in \mathcal{T}_\Gamma^*} \|\nabla \Pi_h u\|_{L^2(K)} \|\nabla(\Pi_h u - Q_h u)\|_{L^2(K)} \\
&\leq C(u)h \|\Pi_h u - Q_h u\|_{H^1(\Omega)}, \qquad (3.2.16)
\end{aligned}$$

in the last inequality, we have used Lemma 3.2.1. Combining (3.2.15)-(3.2.16) together with (3.2.14), we obtain

$$|(I)| \leq C(u)h \|\Pi_h u - Q_h u\|_{H^1(\Omega)}.$$

This, in combination with (3.2.12), leads to

$$\|\Pi_h u - Q_h u\|_{H^1(\Omega)} \leq C(u)h,$$

and hence, by Lemma 3.2.1 and triangle inequality, we obtain

$$\|u - Q_h u\|_{H^1(\Omega)} \leq C(u)h. \qquad (3.2.17)$$

Next, we shall use duality trick to obtain $O(h^{1+s}), 0 < s < 1$, accuracy in L^2 norm for the projection Q_h. For this purpose, we shall consider the following interface problem in $\Omega_1^h \cup \Omega_2^h \cup \Gamma_h$: Find $w \in H_0^1(\Omega)$ such that

$$A_h(w, v) = (u - Q_h u, v) \ \ \forall v \in H_0^1(\Omega) \qquad (3.2.18)$$

and its finite element approximation $w_h \in V_h$ be such that

$$A_h(w_h, v_h) = (u - Q_h u, v_h) \;\; \forall v_h \in V_h. \tag{3.2.19}$$

Note that $w \in H_0^1(\Omega)$ is the solution for the elliptic interface problem (3.2.18) with the jump condition

$$[w] = 0, \quad \left[\beta_h(x)\frac{\partial w}{\partial \tilde{\mathbf{n}}}\right] = 0 \quad \text{along } \Gamma_h,$$

where $\tilde{\mathbf{n}}$ is the outward pointing unit normal to $\partial\Omega_1^h$. Since the interface Γ is of arbitrary shape, Ω_1^h becomes non-convex polygonal domain with boundary Γ_h and hence, $w \in H^{1+s}(\Omega_1^h) \cup H^{1+s}(\Omega_2^h)$ $(0 < s < 1)$ (cf. [40]). Further, the solution w satisfies the following *a priori* estimate (cf. [22])

$$\|w\|_{H^1(\Omega)} + \|w\|_{H^{1+s}(\Omega_1^h)} + \|w\|_{H^{1+s}(\Omega_2^h)} \leq C\|u - Q_h u\|_{L^2(\Omega)}. \tag{3.2.20}$$

From (3.2.18) and (3.2.19), we have

$$A_h(w - w_h, v_h) = 0 \quad \forall v_h \in V_h. \tag{3.2.21}$$

For the linear interpolant operator Π_h, we have (cf. [19, 40])

$$\begin{aligned}
\|w - \Pi_h w\|_{H^1(\Omega)} &\leq C\{\|w - \Pi_h w\|_{H^1(\Omega_1^h)} + \|w - \Pi_h w\|_{H^1(\Omega_2^h)}\} \\
&\leq Ch^s\{\|w\|_{H^{1+s}(\Omega_1^h)} + \|w\|_{H^{1+s}(\Omega_2^h)}\}.
\end{aligned}$$

This, together with the Galerkin orthogonality (3.2.21) and (3.2.20), we have

$$\begin{aligned}
\|w - w_h\|_{H^1(\Omega)} &\leq C\|w - \Pi_h w\|_{H^1(\Omega)} \\
&\leq Ch^s\|u - Q_h u\|_{L^2(\Omega)}.
\end{aligned} \tag{3.2.22}$$

Now, setting $v = u - Q_h u$ in (3.2.18) and using (3.2.17) and (3.2.22), we have

$$\begin{aligned}
\|u - Q_h u\|_{L^2(\Omega)}^2 &= A_h(w - w_h, u - Q_h u) + A_h(w_h, u) - A(w_h, u) \\
&\leq C\|w - w_h\|_{H^1(\Omega)}\|u - Q_h u\|_{H^1(\Omega)} + (II) \\
&\leq C(u)h^{1+s}\|u - Q_h u\|_{L^2(\Omega)} + (II).
\end{aligned} \tag{3.2.23}$$

Arguing as in deriving (3.2.13), we can deduce

$$\begin{aligned}
|(II)| &\leq C\sum_{K\in\mathcal{T}_\Gamma^*} \|\nabla(u - \Pi_h u)\|_{L^2(\tilde{K})}\|\nabla w_h\|_{L^2(\tilde{K})} \\
&\quad +C\sum_{K\in\mathcal{T}_\Gamma^*} \|\nabla\Pi_h u\|_{L^2(\tilde{K})}\|\nabla w_h\|_{L^2(\tilde{K})} \\
&=: (II)_1 + (II)_2.
\end{aligned} \tag{3.2.24}$$

We shall estimate each term $(II)_i, i = 1,2$, separately. Using the fact that ∇w_h is constant in $K \in \mathcal{T}_h$ and $\text{meas}(\tilde{K}) \leq Ch^3$, we have

$$\|\nabla w_h\|^2_{L^2(\tilde{K})} \leq Ch\|\nabla w_h\|^2_{L^2(K)}.$$

This, together with Lemma 3.2.1, we have

$$\begin{aligned}
|(II)_1| &\leq C\Big\{\sum_{K\in\mathcal{T}_\Gamma^*}\|\nabla(u-\Pi_h u)\|^2_{L^2(\tilde{K})}\Big\}^{\frac{1}{2}}\Big\{\sum_{K\in\mathcal{T}_\Gamma^*}\|\nabla w_h\|^2_{L^2(\tilde{K})}\Big\}^{\frac{1}{2}} \\
&\leq Ch^{\frac{1}{2}}\|u-\Pi_h u\|_{H^1(\Omega)}\sum_{K\in\mathcal{T}_\Gamma^*}\|\nabla w_h\|_{L^2(K)} \\
&\leq C(u)h^{\frac{3}{2}}\sum_{K\in\mathcal{T}_\Gamma^*}\|\nabla(w-w_h)\|_{L^2(K)} + C(u)h^{\frac{3}{2}}\|w\|_{H^1(\Omega_\Gamma^*)} \\
&\leq C(u)h^{\frac{3}{2}}\|u-Q_h u\|_{L^2(\Omega)}, \qquad (3.2.25)
\end{aligned}$$

in the last inequality, we have used (3.2.20) and

$$\|w_h\|_{H^1(\Omega)} \leq C\|u-Q_h u\|_{L^2(\Omega)}. \qquad (3.2.26)$$

Similarly, we have

$$\begin{aligned}
&|(II)_2| \\
&\leq Ch\sum_{K\in\mathcal{T}_\Gamma^*}\|\nabla\Pi_h u\|_{L^2(K)}\|\nabla w_h\|_{L^2(K)} \\
&\leq C(u)h^2\sum_{K\in\mathcal{T}_\Gamma^*}\|\nabla w_h\|_{L^2(K)} + Ch\|u\|_{H^1(\Omega_\Gamma^*)}\|w-w_h\|_{H^1(\Omega_\Gamma^*)} \\
&\quad +Ch\|u\|_{H^1(\Omega_\Gamma^*)}\|w\|_{H^1(\Omega_\Gamma^*)} \\
&\leq C(u)h^2\|u-Q_h u\|_{L^2(\Omega)} + Ch^{\frac{3}{2}}\|u\|_X\|u-Q_h u\|_{L^2(\Omega)} \\
&\quad +Ch^{\frac{3}{2}}\|u\|_X\|u-Q_h u\|_{L^2(\Omega)} \\
&\leq C(u)h^{\frac{3}{2}}\|u-Q_h u\|_{L^2(\Omega)}, \qquad (3.2.27)
\end{aligned}$$

where we have used (3.2.20), Lemmas 3.2.1-3.2.2 and (3.2.26). Thus, it follows from (3.2.24), (3.2.25) and (3.2.27) that

$$|(II)| \leq C(u)h^{\frac{3}{2}}\|u-Q_h u\|_{L^2(\Omega)},$$

and which, combine with (3.2.23), yields

$$\|u-Q_h u\|_{L^2(\Omega)} \leq C(u)h^{1+\min\{\frac{1}{2},s\}}. \qquad (3.2.28)$$

This completes the proof. □

3.3 Convergence Analysis

In this section, we shall prove optimal order error estimate in H^1 norm and sub-optimal in L^2 norm for the elliptic interface problems (3.1.1)-(3.1.3). We state our main results of this section in the following theorem.

Theorem 3.3.1 *Let u and u_h be the solutions of (3.1.1)-(3.1.3) and (3.2.1), respectively. Then, for $f \in L^2(\Omega)$ and $g \in H^2(\Gamma)$, there exists a positive constant C independent of h such that*

$$\|u - u_h\|_{H^1(\Omega)} \leq C(u, g)h, \qquad \|u - u_h\|_{L^2(\Omega)} \leq C(u, g)h^{1+\min\{\frac{1}{2}, s\}}, \; 0 < s < 1,$$

where $C(u, g) = C(\|g\|_{H^2(\Gamma)} + \|u\|_X + \|u\|_{W^{1,\infty}(\Omega_0 \cap \Omega_1)} + \|u\|_{W^{1,\infty}(\Omega_0 \cap \Omega_2)})$.

Proof. As usual we split the error $u - u_h$ as $(u - Q_h u) + (Q_h u - u_h)$. In view of Lemma 3.2.4 it is enough to have bounds for $u_h - Q_h u$. We first proceed to estimate $\|u_h - Q_h u\|_{H^1(\Omega)}$. From (3.1.4) and (3.2.1), we note that

$$A_h(u_h, v_h) - A(u, v_h) = \langle g_h, v_h \rangle_{\Gamma_h} - \langle g, v_h \rangle_{\Gamma} \quad \forall v_h \in V_h. \tag{3.3.1}$$

By the coercivity of the bilinear form A and (3.3.1), we have

$$\begin{aligned} C\|u_h - Q_h u\|^2_{H^1(\Omega)} &\leq A(u - Q_h u, u_h - Q_u) + A(u_h - u, u_h - Q_h u) \\ &= A(u - Q_h u, u_h - Q_h u) + \{A(u_h, u_h - Q_h u) \\ &\quad - A_h(u_h, u_h - Q_h u)\} + \{A_h(u_h, u_h - Q_h u) - A(u, u_h - Q_h u)\} \\ &\leq A(u - Q_h u, u_h - Q_h u) + \{A(u_h, u_h - Q_h u) - A_h(u_h, u_h - Q_h u)\} \\ &\quad + \{\langle g_h, u_h - Q_h u \rangle_{\Gamma_h} - \langle g, u_h - Q_h u \rangle_{\Gamma}\} \\ &=: (III)_1 + (III)_2 + (III)_3. \end{aligned} \tag{3.3.2}$$

It follows from (3.2.17) that

$$|(III)_1| \leq C(u)h\|u_h - Q_h u\|_{H^1(\Omega)}. \tag{3.3.3}$$

Arguing as in deriving (3.2.16), we have

$$|(III)_2| \leq Ch\|u_h\|_{H^1(\Omega)}\|u_h - Q_h u\|_{H^1(\Omega)}. \tag{3.3.4}$$

It is easy to see from (3.3.1) that

$$\|u_h\|_{H^1(\Omega)} \leq C(\|g\|_{H^2(\Gamma)} + \|u\|_{H^1(\Omega)}).$$

This, together with (3.3.4) yields

$$|(III)_2| \leq Ch(\|g\|_{H^2(\Gamma)} + \|u\|_{H^1(\Omega)})\|u_h - Q_h u\|_{H^1(\Omega)}. \tag{3.3.5}$$

Then, Lemma 3.2.3 immediately implies

$$|(III)_3| \le Ch^{\frac{3}{2}} \|g\|_{H^2(\Gamma)} \|u_h - Q_h u\|_{H^1(\Omega)}. \tag{3.3.6}$$

Thus, it follows from (3.3.2)-(3.3.3) and (3.3.5)-(3.3.6) that

$$\|u_h - Q_h u\|_{H^1(\Omega)} \le C(u, g)h. \tag{3.3.7}$$

Next, we shall estimate the term $\|u_h - Q_h u\|_{L^2(\Omega)}$. For this purpose, we use duality trick. Consider the following interface problem : Find $\tilde{w} \in H_0^1(\Omega)$ such that

$$A_h(\tilde{w}, v) = (u_h - Q_h u, v) \;\; \forall v \in H_0^1(\Omega), \tag{3.3.8}$$

with the jump conditions

$$[\tilde{w}] = 0, \quad \left[\beta_h(x) \frac{\partial \tilde{w}}{\partial \mathbf{\tilde{n}}}\right] = 0 \quad \text{along } \Gamma_h.$$

The solution $\tilde{w}$ satisfies the following regularity estimate (cf. [22, 40])

$$\|\tilde{w}\|_{H^1(\Omega)} + \|\tilde{w}\|_{H^{1+s}(\Omega_1^h)} + \|\tilde{w}\|_{H^{1+s}(\Omega_2^h)} \le C\|u_h - Q_h u\|_{L^2(\Omega)}. \tag{3.3.9}$$

Before proceeding further, we define the projection $R_h : H_0^1(\Omega) \to V_h$ by

$$A_h(R_h w, v_h) = A_h(w, v_h) \;\; \forall v_h \in V_h, \; w \in H_0^1(\Omega). \tag{3.3.10}$$

Now, setting $v = u_h - Q_h u$ in (3.3.8) and using (3.3.10), we have

$$\begin{aligned} \|u_h - Q_h u\|^2_{L^2(\Omega)} &= A_h(\tilde{w}, u_h - Q_h u) \\ &= A_h(R_h \tilde{w}, u_h - Q_h u) \\ &= A_h(R_h \tilde{w}, u_h) - A(R_h \tilde{w}, u), \end{aligned} \tag{3.3.11}$$

in the last equality, we used (3.2.11). Further, using (3.3.1), we have

$$A_h(u_h, R_h \tilde{w}) - A(u, R_h \tilde{w}) = \langle g_h, R_h \tilde{w} \rangle_{\Gamma_h} - \langle g, R_h \tilde{w} \rangle_{\Gamma},$$

which, together with (3.3.11), we have

$$\|u_h - Q_h u\|^2_{L^2(\Omega)} = \langle g_h, R_h \tilde{w} \rangle_{\Gamma_h} - \langle g, R_h \tilde{w} \rangle_{\Gamma} =: (IV). \tag{3.3.12}$$

Using the fact $\|R_h \tilde{w}\|_{H^1(\Omega)} \le \|\tilde{w}\|_{H^1(\Omega)}$ and Lemma 3.2.3, we obtain

$$\begin{aligned} |(IV)| &\le Ch^{\frac{3}{2}} \|g\|_{H^2(\Gamma)} \|R_h \tilde{w}\|_{H^1(\Omega)} \\ &\le Ch^{\frac{3}{2}} \|g\|_{H^2(\Gamma)} \|\tilde{w}\|_{H^1(\Omega)}. \end{aligned}$$

Further, applying the regularity estimate (3.3.9), we have

$$|(IV)| \leq Ch^{\frac{3}{2}} \|g\|_{H^2(\Gamma)} \|u_h - Q_h u\|_{L^2(\Omega)}. \tag{3.3.13}$$

Thus, it follows from (3.3.12)-(3.3.13) that

$$\|u_h - Q_h u\|_{L^2(\Omega)} \leq Ch^{\frac{3}{2}} \|g\|_{H^2(\Gamma)} \|u_h - Q_h u\|_{L^2(\Omega)}, \tag{3.3.14}$$

which, together with (3.3.7) and Lemma 3.2.4, yields the desired result. This completes the proof. □

Remark 3.3.1 *In this chapter, we have proved a new interface approximation result (Lemma 3.2.2) which plays a crucial role in studying the error analysis. For the L^2-norm error estimate, we have considered the dual problem over $\Omega_1^h \cup \Omega_2^h \cup \Gamma_h$. Since the interface Γ is of arbitrary shape, Ω_i^h $(i = 1, 2)$ is no more convex, and hence one can not have full regularity in each individual subdomains Ω_i^h (see, [22]). The existing regularity result for non-convex domain is then used to derive sub-optimal L^2-norm error estimate. Therefore, the proposed technique will be useful to deal with the interface problems on nonsmooth domain. Further, the proposed technique can easily be extended to treat more general interface problem of non-selfadjoint type (cf. [55]).*

Chapter 4

Error Estimates for Spatially Discrete Schemes for Parabolic Interface Problems

In this chapter, we extend the finite element analysis of elliptic interface problems discussed in Chapters 2 and 3 to parabolic interface problems. For the spatially discrete scheme, we first establish optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms when the grid lines follow the actual interface.[2] Secondly, when the grid lines form an approximation to the interface, the semidiscrte solution is shown to converge to the exact solution at an optimal rate in $L^2(H^1)$ norm.

4.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ is an open domain with C^2 boundary Γ. Let $\Omega_2 = \Omega \backslash \Omega_1$. We shall recall the following parabolic interface problems of the form

$$u_t + \mathcal{L}u = f(x,t) \quad \text{in } \Omega \times (0,T] \tag{4.1.1}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \;\; \text{in } \Omega; \;\; u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T] \tag{4.1.2}$$

and interface conditions

$$[u] = 0, \qquad \left[\mathcal{A}\frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \quad \text{along } \Gamma \tag{4.1.3}$$

[2] *SIAM J. Numer. Anal., 43 (2005), pp. 733-749*

where $f = f(x,t)$ and $g = g(x,t)$ are real valued functions in $\Omega \times (0,T]$, and $u_t = \frac{\partial u}{\partial t}$. Further, $u_0 = u_0(x)$ is real valued function in Ω. For the ease of exposition, we assume $\mathcal{L}v = -\nabla \cdot (\mathcal{A}\nabla v)$. The symbols $[v]$ and $\mathbf{n}$ are defined as in Chapter 1, and $T < \infty$.

In the theorem below, we prove the *a priori* estimate for the solution u of the interface problem (4.1.1)-(4.1.3) under appropriate regularity conditions on f and g.

Theorem 4.1.1 *Let $f \in H^1(0,T;L^2(\Omega))$, $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$ and $u_0 \in H^1_0(\Omega)$. Then the problem (4.1.1)-(4.1.3) has a unique solution $u \in L^2(0,T;X) \cap H^1(0,T;Y) \cap H^1_0(\Omega)$. Further, u satisfies the following a priori estimate*

$$\begin{aligned} \|u\|_{L^2(0,T;X)} \leq \; & C\Big\{\|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|g(0)\|_{H^{\frac{1}{2}}(\Gamma)} \\ & +\|g(T)\|_{H^{\frac{1}{2}}(\Gamma)} + \|g\|_{H^1(0,T;H^{\frac{1}{2}}(\Gamma))}\Big\}. \end{aligned} \tag{4.1.4}$$

Proof. The proof of the existence of a unique solution is in [30]. Next, to obtain the *a priori* estimate (4.1.4), we first transform the problem (4.1.1)-(4.1.3) into the following equivalent problem.

For a.e. $t \in (0,T]$, find $u = u(x,t) \in H^1_0(\Omega) \cap X$ satisfying

$$\begin{aligned} \mathcal{L}u &= f(x,t) - u_t \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ [u] = 0, \quad \left[\mathcal{A}\frac{\partial u}{\partial \mathbf{n}}\right] &= g(x,t) \quad \text{along } \Gamma. \end{aligned} \tag{4.1.5}$$

From the elliptic regularity estimate for the elliptic interface problem (cf. Theorem 1.2.1), it follows that

$$\|u\|_X \leq C\left(\|f - u_t\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\Gamma)}\right). \tag{4.1.6}$$

Multiply both sides of (4.1.5) by u_t and then integrate over Ω to obtain

$$\|u_t\|^2_{L^2(\Omega)} + (\mathcal{L}u, u_t) = (f, u_t). \tag{4.1.7}$$

Note that $u \in H^1(0,T;X)$ and $[u] = 0$ on Γ imply $[u_t] = 0$ on Γ. Hence, an integration by parts leads to

$$\begin{aligned} (\mathcal{L}u, u_t) &= \int_{\Omega_1} \mathcal{A}_1 \nabla u \cdot \nabla u_t dx + \int_{\Omega_2} \mathcal{A}_2 \nabla u \cdot \nabla u_t dx + \int_\Gamma \left[\mathcal{A}\frac{\partial u}{\partial \mathbf{n}}\right] u_t ds \\ &= A^1(u,u_t) + A^2(u,u_t) + \langle g, u_t\rangle_\Gamma, \end{aligned} \tag{4.1.8}$$

where $A^l(.,.) : H^1(\Omega_l) \times H^1(\Omega_l) \to \mathbb{R}$ is given by

$$A^l(w,v) = \int_{\Omega_l} \mathcal{A}_l \nabla w \cdot \nabla v dx, \quad l = 1,2.$$

Equation (4.1.7), together with (4.1.8), yields

$$\|u_t\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{2} A^i(u,u)\right) \;=\; (f,u_t) - \frac{d}{dt}\langle g,u\rangle_\Gamma + \langle g_t,u\rangle_\Gamma.$$

Integrate the above equation from 0 to T. Then apply the Cauchy-Schwarz inequality and the trace theorem (cf. [1]) to obtain

$$\begin{aligned}
&\int_0^T \|u_t\|^2_{L^2(\Omega)}ds + \|u(T)\|^2_{H^1(\Omega_1)} + \|u(T)\|^2_{H^1(\Omega_2)} \\
&\le C\Bigg(\int_0^T \|f\|_{L^2(\Omega)}\|u_t\|_{L^2(\Omega)}ds \\
&\quad + \|g(T)\|_{L^2(\Gamma)}\|u(T)\|_{L^2(\Gamma)} + \|g(x,0)\|_{L^2(\Gamma)}\|u_0\|_{L^2(\Gamma)} \\
&\quad + \int_0^T \|g_t\|_{L^2(\Gamma)}\|u\|_{L^2(\Gamma)}ds + \|u_0\|^2_{H^1(\Omega_1)} + \|u_0\|^2_{H^1(\Omega_2)}\Bigg) \\
&\le C\Bigg(\int_0^T \|f\|_{L^2(\Omega)}\|u_t\|_{L^2(\Omega)}ds + \|g(T)\|_{H^{\frac{1}{2}}(\Gamma)}\|u(T)\|_{H^1(\Omega)} \\
&\qquad + \|g(x,0)\|_{H^{\frac{1}{2}}(\Gamma)}\|u_0\|_{H^1(\Omega)} + \int_0^T \|g_t\|_{H^{\frac{1}{2}}(\Gamma)}\|u\|_{H^1(\Omega)}ds + \|u_0\|^2_{H^1(\Omega)}\Bigg).
\end{aligned}$$

Use a standard kickback argument to obtain

$$\begin{aligned}
&\|u_t\|^2_{L^2(0,T;L^2(\Omega))} + \|u(T)\|^2_{H^1(\Omega)} \\
&\le C\Bigg(\int_0^T \|f\|^2_{L^2(\Omega)}ds + \|g(T)\|^2_{H^{\frac{1}{2}}(\Gamma)} \\
&\qquad + \|g(0)\|^2_{H^{\frac{1}{2}}(\Gamma)} + \int_0^T \|g_t\|^2_{H^{\frac{1}{2}}(\Gamma)}ds + \|u_0\|^2_{H^1(\Omega)}\Bigg) + C\int_0^T \|u(s)\|^2_{H^1(\Omega)}ds.
\end{aligned}$$

Finally, an application of Gronwall's lemma completes the proof. □

The purpose of the present chapter is to extend the convergence analysis of fitted finite element method for elliptic interface problems to parabolic interface problems. Due to low global regularity of the true solution, pointwise-in-time error estimates in L^2 and H^1 norms are difficult and hence, we study the convergence analysis in terms of $L^2(H^1)$ and $L^2(L^2)$ norms. The previous work on finite element analysis for parabolic problems without interface can be found in [29], [38], [58], and references therein. The key to the present analysis is the introduction of some auxiliary projections and duality arguments.

The outline of this chapter is as follows: Section 4.2 is devoted to the error estimates for the spatially discrete scheme for a finite element discretization where interface triangles follow exactly the actual interface. Optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are shown to hold even if the global regularity of the solution is low. In section 4.3, a finite element discretization based on straight triangles is discussed for the problem (4.1.1)-(4.1.3) and optimal order of convergence for the semidiscrete solution in $L^2(H^1)$ norm is obtained.

4.2 The Continuous time Galerkin Approximation with Curved Triangles

This section deals with the error analysis for the spatially discrete scheme for parabolic interface problems (4.1.1)-(4.1.3). We have seen that optimal order of convergence in L^2 and H^1 norms are possible for elliptic interface problems if we allow interface triangles to be curved triangles (cf. Chapter 2). In this section, we extend the analysis of Chapter 2 to obtain optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms for parabolic problems.

For the purpose of spatially discrete Galerkin procedure, we define $H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$. Further, let $A(.,.)$ be the symmetric bilinear form on $H^1(\Omega) \times H^1(\Omega)$ corresponding to the operator $\mathcal{L}$. Then the weak formulation of the interface problem (4.1.1)-(4.1.3) is stated as follows: Find $u(t) \in H_0^1(\Omega)$ such that

$$(u_t, v) + A(u, v) = (f, v) + \langle g, v\rangle_\Gamma \quad \forall v \in H_0^1(\Omega), \;\; t \in (0, T], \tag{4.2.1}$$

with $u(0) = u_0$. Here, $(\cdot, \cdot)$ and $\langle \cdot, \cdot\rangle$ are used to denote the inner products of the $L^2(\Omega)$ and $L^2(\Gamma)$ spaces, respectively.

Now, we shall recall the finite element space $V_h \subset H_0^1(\Omega)$ consisting of piecewise linear polynomials vanishing on the boundary $\partial\Omega$ for the triangulation which allows interface triangles to be curved triangles instead of straight triangles (cf. Chapter 2).

Further, we assume that V_h satisfy the inverse estimate

$$\|\phi\|_{H^1(\Omega)} \le Ch^{-1}\|\phi\|_{L^2(\Omega)} \;\; \forall \, \phi \in V_h, \tag{4.2.2}$$

and this follows immediately from the estimate (2.2.1).

For $v \in X$, let

$$f^* = -\nabla \cdot (\mathcal{A}_l \nabla v) \quad \text{in } \Omega_l, \; l = 1, 2.$$

Clearly, $f^* \in L^2(\Omega)$. With this f^*, define an projection $P_h : X \cap H_0^1(\Omega) \to V_h$ by

$$A(P_h v, \phi) = (f^*, \phi) \quad \forall \phi \in V_h, \; v \in X \cap H_0^1(\Omega). \tag{4.2.3}$$

It follows from the definition of f^* that

$$(f^*, \phi) = (-\nabla.(\mathcal{A}\nabla v), \phi) = A(v, \phi) \quad \forall \phi \in V_h, \; v \in X \cap H_0^1(\Omega),$$

which, combine with (4.2.3), leads to

$$A(P_h v, \phi) = (f^*, \phi) = A(v, \phi) \quad \forall \phi \in V_h, \; v \in X \cap H_o^1(\Omega). \tag{4.2.4}$$

Below, we present a proof that shows optimal error bounds for the projection operator P_h. This lemma is very crucial for our later analysis.

Lemma 4.2.1 *Let P_h be defined by (4.2.4). Then there exists a positive constant C independent of h such that*

$$\|v - P_h v\| + h\|v - P_h v\|_1 \leq Ch^2 \|v\|_X \quad \forall v \in X \cap H_0^1(\Omega).$$

Proof. By the definition of f^*, we obtain

$$A(v, \phi) = (f^*, \phi) \quad \forall \phi \in H_0^1(\Omega). \tag{4.2.5}$$

For $k = 1, 2$, define $f_k : \Omega \to \mathbb{R}$ by

$$f_k = \begin{cases} f^*|_{\Omega_k} & \text{in } \Omega_k, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $f_k \in L^2(\Omega)$ and $f^* = f_1 + f_2$ a.e. in Ω. We now consider the following interface problems: Let $w_k \in H_0^1(\Omega)$ be the solution of the interface problem

$$A(w_k, \phi) = (f_k, \phi) \quad \forall \phi \in H_0^1(\Omega). \tag{4.2.6}$$

Then by the coercivity of the operator A it follows immediately that $v = w_1 + w_2$.

Let $w_h^k \in V_h$ be the finite element approximation to w_k defined by

$$A(w_h^k, \phi) = (f_k, \phi) \quad \forall \phi \in V_h. \tag{4.2.7}$$

Again by the coercivity of the operator A and definition (4.2.4) of the P_h operator, it follows that $P_h v = w_h^1 + w_h^2$.

Since $f_k|_{\Omega_s} = 0$, $s = 1(2)$ if $k = 2(1)$, we have for the elliptic interface problem (4.2.6)-(4.2.7) (cf. Theorem 2.3.1)

$$\begin{aligned} \|w_k - w_h^k\|_{H^1(\Omega)} &\leq Ch(\|w_k\|_{H^2(\Omega_1)} + \|w_k\|_{H^2(\Omega_2)}) \\ &\leq Ch\|w_k\|_X. \end{aligned} \tag{4.2.8}$$

Then by the elliptic regularity (cf. Theorem 1.2.1) we have

$$\begin{aligned} \|w_k\|_X &\leq C\|f_k\|_{L^2(\Omega)} \\ &\leq C\|f^*\|_{L^2(\Omega_k)} \leq C\|v\|_{H^2(\Omega_k)}. \end{aligned}$$

This, together with (4.2.8), yields

$$\|w_k - w_h^k\|_{H^1(\Omega)} \leq Ch\|v\|_{H^2(\Omega_k)}. \tag{4.2.9}$$

Then

$$\begin{aligned}
\|v - P_h v\|_{H^1(\Omega)} &\leq \|w_1 - w_h^1\|_{H^1(\Omega)} + \|w_2 - w_h^2\|_{H^1(\Omega)} \\
&\leq C\left(h\|w_1\|_X + h\|w_2\|_X\right) \\
&\leq Ch\left(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\right) \\
&\leq Ch\|v\|_X \quad \forall v \in X \cap H_0^1(\Omega).
\end{aligned} \tag{4.2.10}$$

For the L^2 norm estimate, let us consider the following problem: Find $w \in H_0^1(\Omega)$ such that

$$A(w, \phi) = (v - P_h v, \phi) \quad \forall \phi \in H_0^1(\Omega). \tag{4.2.11}$$

By setting $\phi = v - P_h v \in H_0^1(\Omega)$ in (4.2.11) and using definition (4.2.4) of the P_h operator, we have

$$\begin{aligned}
\|v - P_h v\|_{L^2(\Omega)}^2 &= A(w, v - P_h v) \\
&= A(w - P_h w, v - P_h v) + A(P_h w, v - P_h v) \\
&= A(w - P_h w, v - P_h v) \\
&\leq C\|w - P_h w\|_{H^1(\Omega)}\|v - P_h v\|_{H^1(\Omega)} \\
&\leq Ch^2\|v\|_X\|w\|_X,
\end{aligned}$$

in the last inequality, we used (4.2.10). Then applying the elliptic regularity estimate (cf. Theorem 1.2.1) for the interface problem (4.2.11), we get

$$\begin{aligned}
\|v - P_h v\|_{L^2(\Omega)}^2 &\leq Ch^2\|v\|_X\|w\|_X \\
&\leq Ch^2\|v - P_h v\|_{L^2(\Omega)}\|v\|_X.
\end{aligned}$$

This completes the proof of Lemma 4.2.1. □

Let $L_h : L^2(\Omega) \to V_h$ be the standard L^2 projection defined by

$$(L_h v, \phi) = (v, \phi) \quad \forall v \in L^2(\Omega), \quad \phi \in V_h. \tag{4.2.12}$$

It is well known that $L_h v \in V_h$ is the best approximation in the L^2 norm to $v \in L^2(\Omega)$. The following lemma shows that $L_h v$ is a quasi-best approximation to $v \in X \cap H_0^1(\Omega)$ in the H^1 norm.

Lemma 4.2.2 *Let P_h and L_h be defined by (4.2.4) and (4.2.12), respectively. Then we have*

$$\|L_h v - v\|_{H^1(\Omega)} \leq C\|P_h v - v\|_{H^1(\Omega)} \quad \forall v \in X \cap H_0^1(\Omega).$$

Proof. For any $v \in X \cap H_0^1(\Omega)$, we know that there exists a unique solution $w \in H_0^1(\Omega)$ for the elliptic interface problem

$$A(w, \phi) = (P_h v - v, \phi) \quad \forall \phi \in H_0^1(\Omega). \tag{4.2.13}$$

Equation (4.2.13), together with (4.2.4) and Lemma 4.2.1, leads to

$$\begin{aligned} \|P_h v - v\|^2_{L^2(\Omega)} &= A(w - P_h w, P_h v - v) + A(P_h w, P_h v - v) \\ &\leq C\|w - P_h w\|_{H^1(\Omega)} \|v - P_h v\|_{H^1(\Omega)} \\ &\leq Ch\|w\|_X \|v - P_h v\|_{H^1(\Omega)} \\ &\leq Ch\|v - P_h v\|_{L^2(\Omega)} \|v - P_h v\|_{H^1(\Omega)}. \end{aligned} \tag{4.2.14}$$

Here, we used the fact that $\|w\|_X \leq C\|v - P_h v\|_{L^2(\Omega)}$. Use of the triangle inequality and (4.2.2) yields

$$\begin{aligned} \|L_h v - v\|_{H^1(\Omega)} &\leq \|P_h v - v\|_{H^1(\Omega)} + \|L_h v - P_h v\|_{H^1(\Omega)} \\ &\leq \|P_h v - v\|_{H^1(\Omega)} + Ch^{-1}\|L_h v - P_h v\|_{L^2(\Omega)} \\ &\leq \|P_h v - v\|_{H^1(\Omega)} + Ch^{-1}\{\|v - P_h v\|_{L^2(\Omega)} \\ &\quad + \|L_h v - v\|_{L^2(\Omega)}\}. \end{aligned} \tag{4.2.15}$$

We know $L_h v$ is the best approximation of $v \in L^2(\Omega)$ with respect to the L^2 norm. Since $v \in X \cap H_0^1(\Omega)$, $P_h v \in V_h$. Thus, $\|v - L_h v\|_{L^2(\Omega)} \leq C\|v - P_h v\|_{L^2(\Omega)}$. Therefore, (4.2.15) implies

$$\|L_h v - v\|_{H^1(\Omega)} \leq \|P_h v - v\|_{H^1(\Omega)} + Ch^{-1}\|v - P_h v\|_{L^2(\Omega)}. \tag{4.2.16}$$

The desired estimate now follows from (4.2.14) and (4.2.16). □

The following Lemma is crucial for our subsequent error analysis.

Lemma 4.2.3 *Let L_h be defined by (4.2.12). Then we have*

$$\|L_h v\|_{H^1(\Omega)} \leq C\|v\|_{H^1(\Omega)} \quad \forall v \in H_0^1(\Omega).$$

Proof. We define a projection $\overline{R}_h : H_0^1(\Omega) \to V_h$ by

$$A(\overline{R}_h v, v_h) = A(v, v_h) \quad \forall v_h \in V_h, \ v \in H_0^1(\Omega). \tag{4.2.17}$$

For any $v \in H_0^1(\Omega)$, we know that there exists a unique solution $w \in X \cap H_0^1(\Omega)$ for the elliptic interface problem

$$A(w, \phi) = (\overline{R}_h v - v, \phi) \quad \forall \phi \in H_0^1(\Omega). \tag{4.2.18}$$

Setting $\phi = \overline{R}_h v - v$ in (4.2.18) and using Lemma 4.2.1, we obtain

$$\begin{aligned} \|\overline{R}_h v - v\|^2_{L^2(\Omega)} &= A(w - P_h w, \overline{R}_h v - v) + A(P_h w, \overline{R}_h v - v) \\ &\le C\|w - P_h w\|_{H^1(\Omega)} \|v - \overline{R}_h v\|_{H^1(\Omega)} \\ &\le Ch\|w\|_X \|v - \overline{R}_h v\|_{H^1(\Omega)} \\ &\le Ch\|v - \overline{R}_h v\|_{L^2(\Omega)} \|v - \overline{R}_h v\|_{H^1(\Omega)}. \end{aligned} \tag{4.2.19}$$

Here, we used the fact that $\|w\|_X \le C\|v - \overline{R}_h v\|_{L^2(\Omega)}$. This, together with (4.2.12), we have

$$\begin{aligned} \|\overline{R}_h v - L_h v\|^2_{L^2(\Omega)} &= (\overline{R}_h v - L_h v, \overline{R}_h v - L_h v) \\ &= (\overline{R}_h v - v, \overline{R}_h v - L_h v) + (v - L_h v, \overline{R}_h v - L_h v) \\ &\le C\|v - \overline{R}_h v\|_{L^2(\Omega)} \|\overline{R}_h v - L_h v\|_{L^2(\Omega)} \\ &\le Ch\|v - \overline{R}_h v\|_{H^1(\Omega)} \|L_h v - \overline{R}_h v\|_{L^2(\Omega)}. \end{aligned} \tag{4.2.20}$$

Then, inverse inequality and (4.2.20), leads to

$$\begin{aligned} \|L_h v\|_{H^1(\Omega)} &\le \|L_h v - \overline{R}_h v\|_{H^1(\Omega)} + \|\overline{R}_h v\|_{H^1(\Omega)} \\ &\le C\{h^{-1}\|L_h v - \overline{R}_h v\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega)}\} \\ &\le C\|v - \overline{R}_h v\|_{H^1(\Omega)} + C\|v\|_{H^1(\Omega)} \le C\|v\|_{H^1(\Omega)} \quad \forall v \in H^1_0(\Omega) \end{aligned}$$

where, we used the fact that $\|\overline{R}_h v\|_{H^1(\Omega)} \le C\|v\|_{H^1(\Omega)}$ $\forall v \in H^1_0(\Omega)$, and this completes the rest of the proof. □

Now, we are in a position to discuss the continuous time Galerkin finite element approximation to (4.2.1) which may be stated as follows: Find $u_h(t) \in V_h$ such that

$$(u_{ht}, v_h) + A(u_h, v_h) = (f, v_h) + \langle g, v_h\rangle_\Gamma \quad \forall v_h \in V_h, \;\; t \in (0, T], \tag{4.2.21}$$

with $u_h(0) = L_h u_0$.

Subtracting (4.2.21) from (4.2.1) we have

$$(u_t - u_{ht}, v_h) + A(u - u_h, v_h) = 0 \quad \forall v_h \in V_h. \tag{4.2.22}$$

Define the error $e(t)$ as $e(t) = u(t) - u_h(t)$. Setting $v_h = L_h u$ in (4.2.22) and using (4.2.12), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|e\|^2_{L^2(\Omega)} &+ A(e, e) = A(u - u_h, u - L_h u) + (u_t - u_{ht}, u - L_h u) \\ &= A(u - u_h, u - L_h u) + (u_t - L_h u_t, u - L_h u) \\ &\quad + (L_h u_t - u_{ht}, u - L_h u) \\ &= A(u - u_h, u - L_h u) + (u_t - (L_h u)_t, u - L_h u) \\ &\quad + (L_h u_t - u_{ht}, u - L_h u) \\ &= A(u - u_h, u - L_h u) + \frac{1}{2}\frac{d}{dt}(u - L_h u, u - L_h u). \end{aligned} \tag{4.2.23}$$

In the last equality we used the fact that $L_h u_t - u_{ht} \in V_h$ and the definition (4.2.12) of the L_h operator. Integrate the above equation from 0 to t. Then apply the Cauchy-Schwarz inequality and Young's inequality to obtain

$$\begin{aligned}\|e\|^2_{L^2(\Omega)} &+ \int_0^t \|e\|^2_{H^1(\Omega)} ds \\ &\leq C\Big(\int_0^t \|u - L_h u\|^2_{H^1(\Omega)} ds + \|u - L_h u\|^2_{L^2(\Omega)} + \|u_0 - L_h u_0\|^2_{L^2(\Omega)} \Big).\end{aligned}$$

An application of Lemma 4.2.2 leads to

$$\begin{aligned}\|e\|^2_{L^2(\Omega)} &+ \int_0^t \|e\|^2_{H^1(\Omega)} ds \\ &\leq C\Big(\int_0^t \|u - P_h u\|^2_{H^1(\Omega)} ds + \|u - L_h u\|^2_{L^2(\Omega)} + \|u_0 - L_h u_0\|^2_{L^2(\Omega)} \Big),\end{aligned}$$

which, together with Lemma 4.2.1 and the fact $\|L_h v - v\|_{L^2(\Omega)} \leq Ch\|v\|_{H^1(\Omega)} \quad \forall v \in H^1_0(\Omega)$, yields

$$\int_0^t \|e\|^2_{H^1(\Omega)} ds \leq Ch^2 \left\{ \|u_0\|^2_{H^1(\Omega)} + \|u\|^2_X + \|u\|^2_{L^2(0,T;X)} \right\}.$$

Thus, we have proved the following optimal H^1 norm estimate.

Theorem 4.2.1 *Let u and u_h be the solutions of (4.1.1)-(4.1.3) and (4.2.21), respectively. Then, for $u_0 \in H^1_0(\Omega)$, $f \in H^1(0,T;L^2(\Omega))$, and $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$, we have*

$$\|u - u_h\|_{L^2(0,T;H^1(\Omega))} \leq Ch \left\{ \|u_0\|_{H^1(\Omega)} + \|u\|_X + \|u\|_{L^2(0,T;X)} \right\}.$$

Next, for the L^2 norm error estimate, we shall use the duality argument. For this purpose, we now consider the following auxiliary problem: Find $z \in H^1_0(\Omega)$ such that

$$A(z, v) = (u - u_h, v) \quad \forall v \in H^1_0(\Omega), \quad t \in (0,T], \tag{4.2.24}$$

with $\left[\mathcal{A}\frac{\partial z}{\partial \mathbf{n}}\right] = 0$ across the interface Γ. Then its finite element approximation is defined to be a function $z_h \in V_h$ satisfying

$$A(z_h, v_h) = (u - u_h, v_h) \quad \forall v_h \in V_h, \quad t \in (0,T]. \tag{4.2.25}$$

Setting $v = u - u_h$ in (4.2.24) and using (4.2.22), we obtain

$$\begin{aligned}\|u - u_h\|^2_{L^2(\Omega)} &= A(z, u - u_h) \\ &= A(z - z_h, u - u_h) + A(z_h, u - u_h) \\ &= A(z - z_h, u - u_h) - (u_t - u_{ht}, z_h). \qquad (4.2.26)\end{aligned}$$

Differentiating (4.2.25) with respect to t, we obtain

$$A(z_{ht}, v_h) = (u_t - u_{ht}, v_h).$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}A(z_h, z_h) = A(z_{ht}, z_h) = (u_t - u_{ht}, z_h),$$

and hence, integrating (4.2.26) from 0 to T we obtain

$$\begin{aligned}
&\|u - u_h\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}A(z_h, z_h) \\
&\leq C\int_0^T \|z - z_h\|_{H^1(\Omega)}\|u - u_h\|_{H^1(\Omega)}ds + \frac{1}{2}A(z_h(0), z_h(0)).
\end{aligned}$$

Further, using Theorem 1.2.1 for the elliptic interface problem (4.2.24) and Lemma 4.2.1, we obtain

$$\begin{aligned}
\|z - z_h\|_{H^1(\Omega)} &\leq C\|z - P_h z\|_{H^1(\Omega)} \\
&\leq Ch\|z\|_X \\
&\leq Ch\|u - u_h\|_{L^2(\Omega)},
\end{aligned}$$

and hence

$$\begin{aligned}
&\|u - u_h\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leq C\int_0^T h\|u - u_h\|_{L^2(\Omega)}\|u - u_h\|_{H^1(\Omega)}ds + \frac{1}{2}A(z_h(0), z_h(0)). \qquad (4.2.27)
\end{aligned}$$

Taking $t \to 0$, it now follows from (4.2.25) that

$$A(z_h(0), z_h(0)) = (u_0 - L_h u_0, z_h(0)) = 0.$$

This, together with (4.2.27) and Theorem 4.2.1, leads to

$$\begin{aligned}
&\|u - u_h\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leq Ch\left(\int_0^T \|u - u_h\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\left(\int_0^T \|u - u_h\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}} \\
&\leq Ch^2\|u - u_h\|_{L^2(0,T;L^2(\Omega))}\left\{\|u_0\|_{H^1(\Omega)} + \|u\|_X + \|u\|_{L^2(0,T;X)}\right\}.
\end{aligned}$$

Thus, we have proved the following optimal L^2 norm estimate.

Theorem 4.2.2 *Let u and u_h be the solutions of (4.1.1)-(4.1.3) and (4.2.21), respectively. Then, for $u_0 \in H_0^1(\Omega)$, $f \in H^1(0,T;L^2(\Omega))$, and $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$, we have*

$$\|u - u_h\|_{L^2(0,T;L^2(\Omega))} \leq Ch^2\left\{\|u_0\|_{H^1(\Omega)} + \|u\|_X + \|u\|_{L^2(0,T;X)}\right\}.$$

4.3 The Continuous time Galerkin Approximation with Straight Triangles

This section is concerned with *a priori* error estimate for the semidiscrete solution of the parabolic interface problems (4.1.1)-(4.1.3) in a convex polygonal domain Ω. We have shown error estimate of optimal order in $L^2(H^1)$ norm for fitted finite element method. In this method, grid lines need not fit to the interface exactly.

Let $A(\cdot,\cdot)$ be the symmetric bilinear form on $H^1(\Omega)\times H^1(\Omega)$ as defined in Chapter 3. Then the weak formulation of the interface problem (4.1.1)-(4.1.3) is stated as follows: Find $u(t) \in H_0^1(\Omega)$ such that

$$(u_t, v) + A(u, v) = (f, v) + \langle g, v\rangle_\Gamma \quad \forall v \in H_0^1(\Omega), \;\; t \in (0, T], \tag{4.3.1}$$

with $u(0) = u_0$.

Let V_h be a family of finite element subspaces of $H_0^1(\Omega)$ defined on straight triangulation $\mathcal{T}_h$ consisting of piecewise linear polynomials vanishing on the boundary $\partial\Omega$. We recall that the triangulation $\mathcal{T}_h$ consists of straight triangles (cf. Chapter 3).

We now turn to the semidiscrete finite element approximation to the problem (4.3.1). Before, we need the standard L^2 projection $L_h : L^2(\Omega) \to V_h$ defined by

$$(L_h v, \phi) = (v, \phi) \quad \forall v \in L^2(\Omega), \;\; \phi \in V_h, \tag{4.3.2}$$

satisfying the stability estimate (cf. [14])

$$\|L_h v\|_{H^1(\Omega)} \le C\|v\|_{H^1(\Omega)} \quad \forall v \in H_0^1(\Omega). \tag{4.3.3}$$

It is well known that $L_h v \in V_h$ is the best approximation of $v \in L^2(\Omega)$ with respect to the L^2 norm. Thus, Lemma 3.2.1 immediately implies

$$\|L_h u - u\|_{L^2(\Omega)} \le C(u)h^2, \quad C(u) = C(\|u\|_X + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_1)} + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_2)}),$$

where Ω_0 is some neighborhood of the interface Γ. Further, using the inverse inequality

$$\|\phi\|_{H^1(\Omega)} \le Ch^{-1}\|\phi\|_{L^2(\Omega)} \;\; \forall \phi \in V_h,$$

we have

$$\begin{aligned}
\|L_h u - u\|_{H^1(\Omega)} &\le \|\Pi_h u - u\|_{H^1(\Omega)} + \|L_h u - \Pi_h u\|_{H^1(\Omega)} \\
&\le \|\Pi_h u - u\|_{H^1(\Omega)} + Ch^{-1}\|L_h u - \Pi_h u\|_{L^2(\Omega)} \\
&\le C(u)h + Ch^{-1}\|u - \Pi_h u\|_{L^2(\Omega)} + Ch^{-1}\|L_h u - u\|_{L^2(\Omega)} \\
&\le C(u)h.
\end{aligned}$$

Here, we used the fact $\|u-\Pi_h u\|_{H^1(\Omega)} \leq C(u)h$ for the linear interpolant Π_h (cf. Lemma 3.2.1). Thus, we have

$$\|L_h u - u\|_{H^m(\Omega)} \leq C(u)h^{2-m},\ m = 0, 1. \tag{4.3.4}$$

Now, the standard continuous in time Galerkin finite element approximation to (4.3.1) is stated as follows: Find $u_h : [0, T] \to V_h$ such that $u_h(0) = L_h u_0$ and

$$(u_{ht}, v_h) + A_h(u_h, v_h) = (f, v_h) + \langle g_h, v_h\rangle_{\Gamma_h} \quad \forall v_h \in V_h,\ \ t \in (0, T], \tag{4.3.5}$$

where $A_h(\cdot,\cdot)$ is the suitable approximation of A as defined in Chapter 3.

We shall need the following stability result for the semidiscrete solution u_h satisfying (4.3.5) for our future use.

Lemma 4.3.1 *Let $f \in L^2(\Omega)$, $u_0 \in H_0^1(\Omega)$ and $g \in H^2(\Gamma)$. Then we have*

$$\int_0^t \|u_h\|^2_{H^1(\Omega)} ds \leq C\bigg(\int_0^t \{\|f\|^2_{L^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\} ds + \|u_0\|^2_{H^1(\Omega)}\bigg).$$

Proof. The lemma can be proved easily by setting $v_h = u_h$ in (4.3.5). We omit the details. □

Subtracting (4.3.5) from (4.3.1) we have

$$\begin{aligned}(u_t - u_{ht}, v_h) + A(u-u_h, v_h) &= \langle g, v_h\rangle_\Gamma - \langle g_h, v_h\rangle_{\Gamma_h} \\ &\quad + A_h(u_h, v_h) - A(u_h, v_h) \quad \forall v_h \in V_h. \end{aligned} \tag{4.3.6}$$

Define the error $e(t)$ as $e(t) = u(t) - u_h(t)$. Setting $v_h = L_h u$ in (4.3.6) and using (4.3.2), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e(t)\|^2_{L^2(\Omega)} &+ A(e, e) \\ &= (I)_1 + (I)_2 + (I)_3 + \frac{1}{2}\frac{d}{dt}\|u - L_h u\|^2_{L^2(\Omega)},\end{aligned} \tag{4.3.7}$$

where the terms $(I)_1$-$(I)_3$ are given by

$$\begin{aligned}(I)_1 &= \langle g, L_h u - u_h\rangle_\Gamma - \langle g_h, L_h u - u_h\rangle_{\Gamma_h},\\ (I)_2 &= A_h(u_h, L_h u - u_h) - A(u_h, L_h u - u_h),\\ (I)_3 &= A(u_h - u, L_h u - u).\end{aligned}$$

Now, we estimate the terms $(I)_1$, $(I)_2$ and $(I)_3$ one by one. By Lemma 3.2.3, we have

$$\begin{aligned}|(I)_1| &\leq Ch^{\frac{3}{2}}\|g\|_{H^2(\Gamma)}\|L_h u - u_h\|_{H^1(\Omega)} \\ &\leq Ch^{\frac{5}{2}}\|g\|_{H^2(\Gamma)}\{\|u\|_X + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_1)} + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_2)}\} \\ &\quad + Ch^{\frac{3}{2}}\|g\|_{H^2(\Gamma)}\|e(t)\|_{H^1(\Omega)},\end{aligned}$$

in the second inequality we used (4.3.4). Now, applying the Cauchy-Schwarz inequality, we have

$$|(I)_1| \leq C(u)h^{\frac{5}{2}}\|g\|_{H^2(\Gamma)} + Ch^3\|g\|^2_{H^2(\Gamma)} + \frac{1}{4}\|e(t)\|^2_{H^1(\Omega)}. \qquad (4.3.8)$$

Arguing as in deriving (3.2.16) and using (4.3.4), we can deduce that

$$\begin{aligned} |(I)_2| &\leq Ch\|u_h\|_{H^1(\Omega)}\|L_h u - u_h\|_{H^1(\Omega)} \\ &\leq C(u)h^2\|u_h\|_{H^1(\Omega)} + Ch\|u_h\|_{H^1(\Omega)}\|e(t)\|_{H^1(\Omega)} \\ &\leq C(u)h^2\|u_h\|_{H^1(\Omega)} + Ch^2\|u_h\|^2_{H^1(\Omega)} + \frac{1}{4}\|e(t)\|^2_{H^1(\Omega)} \\ &\leq \{C(u)\}^2h^2 + Ch^2\|u_h\|^2_{H^1(\Omega)} + \frac{1}{4}\|e(t)\|^2_{H^1(\Omega)}. \qquad (4.3.9) \end{aligned}$$

Then, the last term $(I)_3$ can be bounded by using (4.3.4) as

$$\begin{aligned} |(I)_3| &\leq C(u)h\|e(t)\|_{H^1(\Omega)} \\ &\leq \{C(u)\}^2h^2 + \frac{1}{4}\|e(t)\|^2_{H^1(\Omega)}. \qquad (4.3.10) \end{aligned}$$

Integrating the identity (4.3.7) from 0 to t and using the estimates (4.3.8)-(4.3.10), we obtain

$$\begin{aligned} \int_0^t \|e\|^2_{H^1(\Omega)}ds &\leq C(u,g)h^2 + Ch^2\int_0^t \|u_h\|^2_{H^1(\Omega)}ds + \frac{3}{4}\int_0^t \|e\|^2_{H^1(\Omega)} \\ &\quad + \|u - L_h u\|^2_{L^2(\Omega)}. \qquad (4.3.11) \end{aligned}$$

It is known that

$$\|v - L_h v\|_{L^2(\Omega)} \leq Ch\|v\|_{H^1(\Omega)} \ \ \forall v \in H^1_0(\Omega).$$

This, together with Lemma 4.3.1, leads to the following optimal H^1 norm error estimate.

Theorem 4.3.1 *Let u and u_h be the solutions of (4.1.1)-(4.1.3) and (4.3.5), respectively. Then, for $u_0 \in H^1_0(\Omega), f \in H^1(0,T;L^2(\Omega))$, and $g \in H^1(0,T;H^2(\Gamma))$, there exists a positive constant C independent of h such that*

$$\|u - u_h\|_{L^2(0,T;H^1(\Omega))} \leq C(u_0, u, f, g)h.$$

Chapter 5

Fully Discrete Schemes for Parabolic Interface Problems

In this chapter, a discrete time discontinuous Galerkin (DG) method is used to analyze the fully discrete scheme for parabolic interface problems. The error estimates are shown to be of optimal order in $L^2(L^2)$ and $L^2(H^1)$ norms when the grid lines coincide with the interface. Further, the fully discrete solution converges to the exact solution at an optimal rate in $L^2(H^1)$ norm if we use straight triangles instead of curved interface triangles.

5.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ is an open domain with C^2 boundary Γ. Let $\Omega_2 = \Omega \backslash \Omega_1$. We shall again recall the following linear parabolic interface problems of the form

$$u_t + \mathcal{L}u = f(x,t) \quad \text{in } \Omega \times (0,T] \tag{5.1.1}$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \;\text{ in }\; \Omega, \qquad u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T] \tag{5.1.2}$$

and interface conditions

$$[u] = 0, \qquad \left[\mathcal{A}\frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \quad \text{along } \Gamma, \tag{5.1.3}$$

where $f = f(x,t)$ and $g = g(x,t)$ are real valued functions in $\Omega \times (0,T]$, and $u_t = \frac{\partial u}{\partial t}$. Further, $u_0 = u_0(x)$ is real valued function in Ω. Here, the operator $\mathcal{L}(v) = -\nabla \cdot (\mathcal{A}\nabla v)$. The symbols $[v]$ and $\mathbf{n}$ are defined as in Chapter 1, and $T < \infty$.

In the present chapter, we shall discuss the time discretization of the semidiscrete equations (4.2.21) and (4.3.5) for fitted finite element method. The discontinuous Galerkin method is used in the direction of the time variable. In this method an approximation to the solution is sought as a piecewise constant polynomial function in t, which is not necessarily continuous at the nodes of the defining partition. Optimal order error estimates is obtained in both $L^2(L^2)$ and $L^2(H^1)$ norms if the grid lines coincide the actual interface. Further, we proved optimal order of convergence for the fully discrete solution in $L^2(H^1)$ norm when the grid lines approximate the interface. The key to the present analysis is the introduction of parabolic dual problems and newly established convergence results for elliptic projection. For the literature on the discontinuous Galerkin methods, we refer to [4], [31], [43], [44], and [58].

In order to discretize (4.2.21) and (4.3.5) in time, we first divide the interval $[0, T]$ into M equally spaced subintervals by the points

$$0 = t^0 < t^1 < \cdots < t^M = T$$

with $t^n = nk$, $k = T/M$ be the time step. Let $I_n = (t^{n-1}, t^n]$ be the nth subinterval. To present the DG method for the equations (4.2.21) and (4.3.5), we shall use the finite dimensional space

$$V_{hk} = \{\phi : [0, T] \rightarrow V_h \ : \phi|_{I_n} \in V_h \text{ is constant in time}\}.$$

For $\phi \in V_{hk}$, we denote ϕ^n and ϕ^n_+ to be the value of ϕ and its limit from the above at t^n, respectively. Further, we write V^n_{hk} for the restriction to I_n of the functions in V_{hk}.

Remark 5.1.1 *The functions belonging to V_{hk} need not be continuous at the nodes, but are taken to be continuous to the left there.*

Now we introduce the backward difference quotient

$$\Delta_k \phi^n = \frac{\phi^n - \phi^{n-1}}{k}$$

for a given sequence $\{\phi^n\}_{n=0}^M \subset L^2(\Omega)$. For a given Banach space $\mathcal{B}$ and some function $\xi \in L^2(0, T; \mathcal{B})$, we write

$$\overline{\xi}^{\,n} = k^{-1} \int_{I_n} \xi(x, t) dt. \tag{5.1.4}$$

A brief outline of this chapter is as follows. In section 5.2, some new optimal *a priori* error estimates are derived for the fully discrete solution when the interface triangles are assumed to be curved triangles. Section 5.3 is devoted to the convergence of fully discrete solution to the exact solution when the finite element discretization consist of straight triangles.

5.2 The Discrete time DG Method with Curved Triangles

This section is concerned with the error estimates for the backward Euler scheme for a mesh where the grid lines coincide with the interface. We have derived error estimates of optimal orders in $L^2(L^2)$ and $L^2(H^1)$ norms.

Let $H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$. Further, let $A(\cdot,\cdot)$ be the symmetric bilinear form on $H^1(\Omega)\times H^1(\Omega)$ corresponding to operator $\mathcal{L}$. Then the weak formulation of the interface problem (5.1.1)-(5.1.3) is stated as follows: Find $u(t) \in H_0^1(\Omega)$ such that

$$(u_t, v) + A(u, v) = (f, v) + \langle g, v\rangle_\Gamma \quad \forall v \in H_0^1(\Omega), \ \ t \in (0, T], \tag{5.2.1}$$

with $u(0) = u_0$. Here, $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are used to denote the inner products of the $L^2(\Omega)$ and $L^2(\Gamma)$ spaces, respectively.

Now, we shall again recall the finite element space $V_h \subset H_0^1(\Omega)$ defined on $\mathcal{T}_h^*$ consisting of piecewise linear polynomials vanishing on the boundary $\partial\Omega$. The triangulation $\mathcal{T}_h^*$ is such that interface triangles follow exactly the actual interface (cf. Chapter 2).

We now turn to the fully discrete finite element approximation to the problem (4.2.21). In this case, the fully discrete finite element approximation is defined as follows: Find $U^n \in V_{hk}$, for $n = 1, 2, ..., M$, such that

$$(\Delta_k U^n, v_h) + A(U^n, v_h) = (\overline{f}^{\,n}, v_h) + \langle \overline{g}^{\,n}, v_h\rangle_\Gamma \quad \forall v_h \in V_{hk}^n \tag{5.2.2}$$

with $U^0 = P_h u_0$, P_h is given by (4.2.4). The symbols $\overline{f}^{\,n}$ and $\overline{g}^{\,n}$ are defined as in (5.1.4).

Below we prove the stability result for the solution U^n satisfying (5.2.2) for later analysis.

Lemma 5.2.1 *Let U^n be satisfy* (5.2.2). *Then we have*

$$\|U^M\|^2_{L^2(\Omega)} + \sum_{n=1}^{M} k\|U^n\|^2_{H^1(\Omega)} \leq C\left(\|f\|^2_{L^2(0,T;L^2(\Omega))} + \|u_0\|^2_{H^1(\Omega)} + \|g\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}\right).$$

Proof. Setting $v_h = U^n$ in (5.2.2) and then using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\|U^n\|^2_{L^2(\Omega)} &+ k\|U^n\|^2_{H^1(\Omega)} \\ &\leq k(\overline{f}^{\,n}, U^n) + k\langle \overline{g}^{\,n}, U^n\rangle_\Gamma + (U^{n-1}, U^n) \\ &\leq k\|\overline{f}^{\,n}\|_{L^2(\Omega)}\|U^n\|_{L^2(\Omega)} + k\|\overline{g}^{\,n}\|_{L^2(\Gamma)}\|U^n\|_{L^2(\Gamma)} \\ &\quad +\|U^{n-1}\|_{L^2(\Omega)}\|U^n\|_{L^2(\Omega)}.\end{aligned}$$

Applying Young's inequality, summing over n from $n = 1$ to $n = M$, and noting that $\|P_h u_0\|_{H^1(\Omega)} \le C\|u_0\|_{H^1(\Omega)}$, we obtain

$$\begin{aligned}\|U^M\|^2_{L^2(\Omega)} &+ \sum_{n=1}^{M} k\|U^n\|^2_{H^1(\Omega)} \\ &\le C\left(\|u_0\|^2_{H^1(\Omega)} + \sum_{n=1}^{M} k\|\overline{f}^{\,n}\|^2_{L^2(\Omega)} + \sum_{n=1}^{M} k\|\overline{g}^{\,n}\|^2_{L^2(\Gamma)}\right).\end{aligned}$$

It follows from simple calculation that

$$\sum_{n=1}^{M} k\|\overline{f}^{\,n}\|^2_{L^2(\Omega)} \le C\|f\|^2_{L^2(0,T;L^2(\Omega))} \quad \text{and} \quad \sum_{n=1}^{M} k\|\overline{g}^{\,n}\|^2_{L^2(\Gamma)} \le C\|g\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}.$$

Altogether these estimates lead to the desired result and complete the proof. □

Now we introduce the interpolant $P_k \in V_{hk}$ of u defined by

$$\begin{aligned}\int_{I_n} A(P_k - u, \phi)\, ds &= 0 \quad \forall \phi \in V_{hk}, \\ \text{i.e. } P_k|_{I_n} &= k^{-1}\int_{I_n} P_h u ds \\ &= \overline{P}_k^{\,n}. \end{aligned} \tag{5.2.3}$$

It is easy to notice from Lemma 4.2.1 that

$$\left(\sum_{n=1}^{M} k\|\overline{u}^{\,n} - \overline{P}_k^{\,n}\|^2_{H^m(\Omega)}\right)^{\frac{1}{2}} \le Ch^{2-m}\|u\|_{L^2(0,T;X)}, \quad m = 0, 1. \tag{5.2.4}$$

Now we state the main results of this section in the following theorems.

Theorem 5.2.1 *Let u and U be the solutions of (5.1.1)-(5.1.3) and (5.2.2), respectively. Then, for $u_0 \in X \cap H^1_0(\Omega)$, $f \in H^1(0,T;L^2(\Omega))$, and $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$, there exists a constant C independent of h and k such that*

$$\|u - U\|_{L^2(0,T;L^2(\Omega))} \le C(k + h^2)\left\{\|u_0\|_X + \|u\|_{L^2(0,T;X)} + \|u_t\|_{L^2(0,T;L^2(\Omega))}\right\}.$$

Theorem 5.2.2 *Let u and U be the solutions of (5.1.1)-(5.1.3) and (5.2.2), respectively. Then, for $u_0 \in X \cap H^1_0(\Omega)$, $f \in H^1(0,T;L^2(\Omega))$, and $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$, there exists a positive constant C independent of h and k such that*

$$\|u - U\|_{L^2(0,T;H^1(\Omega))} \le C(k + h)\left\{\|u_0\|_X + \|u\|_{L^2(0,T;X)} + \|u_t\|_{L^2(0,T;Y)}\right\}.$$

The proofs of the above theorems require some preparation. We now appeal to the parabolic duality arguments. Consider the following auxiliary problem: Find $z^n \in V_{hk}$ such that

$$(-\nabla_k z^n, v_h) + A(z^{n-1}_+, v_h) = (\overline{u}^{\,n} - U^n, v_h) \quad \forall v_h \in V^n_{hk},\ 1 \le n \le M, \tag{5.2.5}$$

with $z_+^M = 0,\ \nabla_k z^n = \frac{z_+^n - z_+^{n-1}}{k}$.

The following stability result of the solution z^n of (5.2.5) is very crucial for the convergence analysis.

Lemma 5.2.2 *Let z^n be the solution of* (5.2.5). *Then we have*

$$\sum_{n=1}^{M} k\|\nabla_k z^n\|^2_{L^2(\Omega)} + \|z_+^0\|^2_{H^1(\Omega)} \le C \sum_{n=1}^{M} k\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)}.$$

Proof. Taking $v_h = -k\nabla_k z^n$ in (5.2.5) and applying the Cauchy-Schwarz inequality, we obtain

$$k\|\nabla_k z^n\|^2_{L^2(\Omega)} + A(z_+^{n-1}, z_+^{n-1} - z_+^n) \le Ck\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)}. \tag{5.2.6}$$

It is easy to notice that

$$\begin{aligned} A(z_+^{n-1}, z_+^{n-1} - z_+^n) &= \frac{k^2}{2}A(\nabla_k z^n, \nabla_k z^n) \\ &\quad -\frac{1}{2}A(z_+^n, z_+^n) + \frac{1}{2}A(z_+^{n-1}, z_+^{n-1}). \end{aligned}$$

This, combined with (5.2.6), yields

$$\begin{aligned} k\|\nabla_k z^n\|^2_{L^2(\Omega)} &+ \frac{k^2}{2}A(\nabla_k z^n, \nabla_k z^n) - \frac{1}{2}A(z_+^n, z_+^n) + \frac{1}{2}A(z_+^{n-1}, z_+^{n-1}) \\ &\le Ck\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)}. \end{aligned}$$

Summing over n form $n = 1$ to $n = M$, we obtain

$$\sum_{n=1}^{M} k\|\nabla_k z^n\|^2_{L^2(\Omega)} + A(z_+^0, z_+^0) \le \sum_{n=1}^{M} k\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)}.$$

This completes the proof. □

Proof of Theorem 5.2.1. Choose $v_h = k(\overline{P}_k^{\,n} - U^n) \in V_{hk}^n$ in (5.2.5). Observing that $A(z_+^{n-1}, \overline{P}_k^{\,n}) = A(z_+^{n-1}, \overline{u}^{\,n})$, we have

$$\begin{aligned} &k\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)} \\ &= k(\overline{u}^{\,n} - U^n, \overline{u}^{\,n} - \overline{P}_k^{\,n}) + k(-\nabla_k z^n, \overline{P}_k^{\,n} - U^n) + kA(z_+^{n-1}, \overline{P}_k^{\,n} - U^n) \\ &= k(\overline{u}^{\,n} - U^n, \overline{u}^{\,n} - \overline{P}_k^{\,n}) + k(-\nabla_k z^n, \overline{P}_k^{\,n} - U^n) + kA(z_+^{n-1}, \overline{u}^{\,n} - U^n) \\ &= k(\overline{u}^{\,n} - \overline{P}_k^{\,n}, \overline{u}^{\,n} - U^n) + k(-\nabla_k z^n, \overline{P}_k^{\,n} - u^n) \\ &\quad + k(-\nabla_k z^n, u^n - U^n) + kA(z_+^{n-1}, \overline{u}^{\,n} - U^n). \end{aligned} \tag{5.2.7}$$

Now summing over n from $n = 1$ to $n = M$, we obtain

$$\begin{aligned} \sum_{n=1}^{M} k\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)} &= \sum_{n=1}^{M} \left\{k(\overline{u}^{\,n} - \overline{P}_k^{\,n}, \overline{u}^{\,n} - U^n)\right\} + \sum_{n=1}^{M} \left\{k(-\nabla_k z^n, \overline{P}_k^{\,n} - u^n)\right\} \\ &\quad + \sum_{n=1}^{M} k\left\{(-\nabla_k z^n, u^n - U^n) + A(z_+^{n-1}, \overline{u}^{\,n} - U^n)\right\} \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{5.2.8}$$

Before estimating the three terms in (5.2.8), we first rewrite the term I_3. Note that for all $v \in H_0^1(\Omega)$, we have

$$(\Delta_k u^n, v) + A(\overline{u}^{\,n}, v) = (\overline{f}^{\,n}, v) + \langle \overline{g}^{\,n}, v\rangle_\Gamma, \quad 1 \le n \le M. \tag{5.2.9}$$

Now, taking $v_h = v = z_+^{n-1}$ in both (5.2.2) and (5.2.9), subtracting one from the other, and summing the resulting equation over n, we obtain

$$\sum_{n=1}^{M} k\left\{(\Delta_k(u^n - U^n), z_+^{n-1}) + A(\overline{u}^{\,n} - U^n, z_+^{n-1})\right\} = 0 \tag{5.2.10}$$

which, together with (5.2.8), yields

$$\begin{aligned}
&\sum_{n=1}^{M} k\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)} \\
&= I_1 + I_2 + \sum_{n=1}^{M} k\Big((-\nabla_k z^n, u^n - U^n) + (-\Delta_k(u^n - U^n), z_+^{n-1})\Big) \\
&=: I_1 + I_2 + I_4.
\end{aligned} \tag{5.2.11}$$

Using the fact that $z_+^M = 0$, and applying the identity

$$\sum_{n=1}^{M}(a_n - a_{n-1})b_n = a_M b_M - a_0 b_0 - \sum_{n=1}^{M} a_{n-1}(b_n - b_{n-1})$$

to I_4 with $a_n = z_+^n$ and $b_n = u^n - U^n$, we obtain

$$\begin{aligned}
I_4 &= \sum_{n=1}^{M} k\left\{(-\nabla_k z^n, u^n - U^n) + (-\Delta_k(u^n - U^n), z_+^{n-1})\right\} \\
&= (z_+^0, u_0 - P_h u_0).
\end{aligned} \tag{5.2.12}$$

Using (5.2.4) and the Cauchy-Schwarz inequality, it now follows that

$$\begin{aligned}
|I_1| &\le \left(\sum_{n=1}^{M} k\|\overline{u}^{\,n} - \overline{P}_k^{\,n}\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{M} k\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \\
&\le Ch^2\|u\|_{L^2(0,T;X)} \left(\sum_{n=1}^{M} k\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.2.13}$$

Similarly, for I_2, use of (5.2.4) leads to

$$\begin{aligned}
|I_2| &\le C\left(\sum_{n=1}^{M} k\|\nabla_k z^n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \\
&\quad \times \left[\left(\sum_{n=1}^{M} k\|\overline{P}_k^{\,n} - \overline{u}^{\,n}\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{M} k\|\overline{u}^{\,n} - u^n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}\right] \\
&\le C(k + h^2)\left(\|u\|^2_{L^2(0,T;X)} + \|u_t\|^2_{L^2(0,T;L^2(\Omega))}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{M} k\|\nabla_k z^n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}
\end{aligned}$$

Further, using Lemma 5.2.2 we obtain

$$|I_2| \leq C(k+h^2)\left(\|u\|^2_{L^2(0,T;X)} + \|u_t\|^2_{L^2(0,T;L^2(\Omega))}\right)^{\frac{1}{2}} \times \left(\sum_{n=1}^{M} k\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \quad (5.2.14)$$

Finally, using Lemmas 4.2.1 and 5.2.2, the term I_4 is estimated as

$$|I_4| \leq \|z^0_+\|_{H^1(\Omega)}\|u_0 - P_h u_0\|_{L^2(\Omega)} \leq Ch^2\|u_0\|_X \left(\sum_{n=1}^{M} k\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \quad (5.2.15)$$

By simple calculation, it follows that

$$\|u - U\|_{L^2(0,T;L^2(\Omega))} \leq Ck\|u_t\|_{L^2(0,T;L^2(\Omega))} + C\left(\sum_{n=1}^{M} k\|\overline{u}^{\,n} - U^n\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \quad (5.2.16)$$

and hence, the desired result now follows from (5.2.11) and the estimates (5.2.13)-(5.2.16). This completes the proof. □

For the H^1 norm estimate, as for the L^2 norm, we analyze the following auxiliary discrete problem: For $1 \leq n \leq M$, find $w^n \in V_{hk}$ such that

$$(-\nabla_k w^n, v_h) + A(w^{n-1}_+, v_h) = (\nabla(\overline{u}^{\,n} - U^n), \nabla v_h) \quad \forall v_h \in V^n_{hk}, \quad (5.2.17)$$

with $w^M_+ = 0$. Applying the standard arguments (cf. [14]) and Lemma 4.2.3, we have the following stability result of the solution w satisfying (5.2.17).

Lemma 5.2.3 *Let w^n satisfy* (5.2.17). *Then the following stability results hold:*

$$\max_{1\leq n\leq M}\|w^{n-1}_+\|^2_{L^2(\Omega)} + \sum_{n=1}^{M} k\|w^{n-1}\|^2_{H^1(\Omega)} \leq \sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)},$$

$$\sum_{n=1}^{M} k\|\nabla_k w^n\|^2_{H^{-1}(\Omega)} \leq \sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)}.$$

Proof of Theorem 5.2.2. Now choose $v_h = k(\overline{P}_k^{\,n} - U^n)$ in (5.2.17) and repeat the same analysis as for deriving I_4 in (5.2.12), to obtain

$$\begin{aligned}
&\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)} \\
&= \sum_{n=1}^{M} k(\nabla(\overline{u}^{\,n} - \overline{P}_k^{\,n}), \nabla(\overline{u}^{\,n} - U^n)) + \sum_{n=1}^{M} k(-\nabla_k w^n, \overline{P}_k^{\,n} - u^n) \\
&\quad + \sum_{n=1}^{M}\left(k(-\nabla_k w^n, u^n - U^n) + kA(w^{n-1}_+, \overline{u}^{\,n} - U^n)\right) \\
&= \sum_{n=1}^{M} k(\nabla(\overline{u}^{\,n} - \overline{P}_k^{\,n}), \nabla(\overline{u}^{\,n} - U^n)) + \sum_{n=1}^{M} k(-\nabla_k w^n, \overline{P}_k^{\,n} - u^n) + (w^0_+, u_0 - P_h u_0) \\
&=: II_1 + II_2 + II_3. \qquad (5.2.18)
\end{aligned}$$

For the term II_1, use the Cauchy-Schwarz inequality and (5.2.4) to obtain

$$
\begin{aligned}
|II_1| &\leq \sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - \overline{P}_k^{\,n})\|_{L^2(\Omega)}\|\nabla(\overline{u}^{\,n} - U^n)\|_{L^2(\Omega)} \\
&\leq \left\{\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - \overline{P}_k^{\,n})\|^2_{L^2(\Omega)}\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)}\right\}^{\frac{1}{2}} \\
&\leq Ch\|u\|_{L^2(0,T;X)} \left\{\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)}\right\}^{\frac{1}{2}}. \qquad (5.2.19)
\end{aligned}
$$

The term II_2 is estimated in a manner similar to I_2 as in (5.2.14). Thus, using (5.2.4) and Lemma 5.2.3, we obtain

$$
\begin{aligned}
|II_2| &\leq \left(\sum_{n=1}^{M} k\|\nabla_k w^n\|^2_{H^{-1}(\Omega)}\right)^{\frac{1}{2}} \left[\left\{\sum_{n=1}^{M} k\|\overline{u}^{\,n} - \overline{P}_k^{\,n}\|^2_{H^1(\Omega)}\right\}^{\frac{1}{2}}\right. \\
&\quad + \left.\left\{\sum_{n=1}^{M} k\|\overline{u}^{\,n} - u^n\|^2_{H^1(\Omega)}\right\}^{\frac{1}{2}}\right] \\
&\leq C(k+h)\left(\|u\|^2_{L^2(0,T;X)} + \|u_t\|^2_{L^2(0,T;Y)}\right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \qquad (5.2.20)
\end{aligned}
$$

An application of Lemmas 4.2.1 and 5.2.3 yields

$$
|II_3| \leq Ch\|u_0\|_X \left(\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \qquad (5.2.21)
$$

Again, by an easy calculation (cf. [14]), it follows that

$$
\|u - U\|_{L^2(0,T;H^1(\Omega))} \leq k\|u_t\|_{L^2(0,T;Y)} + \left(\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \qquad (5.2.22)
$$

and hence, the desired estimate now follows from (5.2.18)-(5.2.22). This completes the proof. □

5.3 The Discrete time DG Method with Straight Triangles

In this section, we have derived optimal order error estimate in $L^2(H^1)$ norm for the backward Euler scheme in a convex polygonal domain Ω. The finite element discretization is based on a triangulation consisting of straight triangles.

Let $A(\cdot,\cdot)$ be the symmetric bilinear form on $H^1(\Omega)\times H^1(\Omega)$ as defined in Chapter 3. Then the weak formulation of the interface problem (5.1.1)-(5.1.3) is stated as follows: Find $u(t)\in H_0^1(\Omega)$ such that

$$(u_t, v) + A(u,v) = (f,v) + \langle g, v\rangle_\Gamma \quad \forall v\in H_0^1(\Omega),\;\; t\in(0,T], \tag{5.3.1}$$

Now, we shall recall the finite element space $V_h\subset H_0^1(\Omega)$ defined on $\mathcal{T}_h$ consisting of piecewise linear functions vanishing on the boundary $\partial\Omega$. The triangulation $\mathcal{T}_h$ is as described in Chapter 3.

Then the fully discrete finite element approximation to the problem (4.3.5) in this case may be stated as follows: Find $U^n\in V_{hk}^n$, for $n=1,2,...,M$, such that

$$(\Delta_k U^n, v_h) + A_h(U^n, v_h) = (\overline{f}^{\,n}, v_h) + \langle \overline{g}_h^n, v_h\rangle_{\Gamma_h} \quad \forall v_h\in V_{hk}^n \tag{5.3.2}$$

with $U^0 = L_h u_0$, L_h is the L^2 projection defined by (4.3.2). The symbols $\overline{f}^{\,n}$ and $\overline{g}_h^n$ are defined as in (5.1.4). The bilinear form $A_h(\cdot,\cdot)$ is as defined in Chapter 3.

Arguing as in deriving Lemma 5.2.1, we have the stability result of the solution U^n satisfying (5.3.2). We omit the details.

Lemma 5.3.1 *Let U^n be the solution for the problem* (5.3.2)*. Then we have*

$$\|U^M\|^2_{L^2(\Omega)} + \sum_{n=1}^M k\|U^n\|^2_{H^1(\Omega)} \le C\left(\|f\|^2_{L^2(0,T;L^2(\Omega))} + \|u_0\|^2_{H^1(\Omega)} + \|g\|^2_{L^2(0,T;H^2(\Gamma))}\right).$$

Now we introduce the interpolant $Q_k\in V_{hk}$ of u defined by

$$\begin{aligned}\int_{I_n} A(Q_k - u, \phi)\, ds &= 0 \quad \forall\phi\in V_{hk},\\ \text{i.e. } Q_k|_{I_n} &= k^{-1}\int_{I_n} Q_h u ds = \overline{Q}_k^{\,n}.\end{aligned} \tag{5.3.3}$$

We recall the operator Q_h given by (3.2.11). It is easy to notice from Lemma 3.2.4 that

$$\left(\sum_{n=1}^M k\|\overline{u}^{\,n} - \overline{Q}_k^{\,n}\|^2_{H^1(\Omega)}\right)^{\frac12} \le C(u)h, \tag{5.3.4}$$

where $C(u) = C(\|u\|_{L^2(X)} + \|u\|_{L^2(W^{1,\infty}(\Omega_0\cap\Omega_1))} + \|u\|_{L^2(W^{1,\infty}(\Omega_0\cap\Omega_2))})$, Ω_0 is a neighborhood of the interface Γ.

Now, we are in a position to discuss the main result of this section which is stated in the following theorem.

Theorem 5.3.1 *Let u and U be the solutions of (5.1.1)-(5.1.3) and (5.3.2), respectively. Then, for $f\in H^1(0,T;L^2(\Omega))$, $g\in H^1(0,T;H^2(\Gamma))$, and $u_0\in H_0^1(\Omega)$, there exists a positive constant C independent of h and k such that*

$$\|u-U\|_{L^2(0,T;H^1(\Omega))} \le (k+h)C(u,u_t,g).$$

The proof of the above theorem require some preparation. We shall again appeal to parabolic duality arguments. Consider the following auxiliary problem: For $1 \leq n \leq M$, find $v^n \in V_{hk}^n$ such that

$$(-\nabla_k v^n, v_h) + A_h(v_+^{n-1}, v_h) = (\nabla(\overline{u}^{\,n} - U^n), \nabla v_h) \quad \forall v_h \in V_{hk}^n, \tag{5.3.5}$$

with $v_+^M = 0, \ \nabla_k v^n = \frac{v_+^n - v_+^{n-1}}{k}$.

Applying the standard arguments (cf. [14]), we have the following stability result of the solution v satisfying (5.3.5). We omit the details.

Lemma 5.3.2 *Let v^n satisfy* (5.3.5)*. Then the following stability results*

$$\|v_+^0\|_{L^2(\Omega)}^2 + \sum_{n=1}^{M} k\|v^{n-1}\|_{H^1(\Omega)}^2 \ \leq \ \sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|_{L^2(\Omega)}^2$$

and

$$\sum_{n=1}^{M} k\|\nabla_k v^n\|_{H^{-1}(\Omega)}^2 \ \leq \ \sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|_{L^2(\Omega)}^2$$

hold true.

Proof of the Theorem 5.3.1 Now choose $v_h = k(\overline{Q}_k^{\,n} - U^n)$ in (5.3.5). Observing that $A_h(z_+^{n-1}, \overline{Q}_k^{\,n}) = A(z_+^{n-1}, \overline{u}^{\,n})$, we obtain

$$\begin{aligned}
&k\|\nabla(\overline{u}^{\,n} - U^n)\|_{L^2(\Omega)}^2 \\
&= k(\nabla(\overline{u}^{\,n} - U^n), \nabla(\overline{u}^{\,n} - \overline{Q}_k^{\,n})) + k(-\nabla_k v^n, \overline{Q}_k^{\,n} - U^n) + kA_h(v_+^{n-1}, \overline{Q}_k^{\,n} - U^n) \\
&= k(\nabla(\overline{u}^{\,n} - U^n), \nabla(\overline{u}^{\,n} - \overline{Q}_k^{\,n})) + k(-\nabla_k v^n, \overline{Q}_k^{\,n} - u^n) + k(-\nabla_k v^n, u^n - U^n) \\
&\quad + kA(v_+^{n-1}, \overline{u}^{\,n}) - kA_h(v_+^{n-1}, U^n).
\end{aligned}$$

Now summing over n from $n = 1$ to $n = M$, we obtain

$$\begin{aligned}
\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|_{L^2(\Omega)}^2 \ &= \ \sum_{n=1}^{M} \Big[\big\{k(\nabla(\overline{u}^{\,n} - U^n), \nabla(\overline{u}^{\,n} - \overline{Q}_k^{\,n}))\big\} \\
&\quad + \big\{k(-\nabla_k v^n, \overline{Q}_k^{\,n} - u^n)\big\} \\
&\quad + k\big\{(-\nabla_k v^n, u^n - U^n) + A(v_+^{n-1}, \overline{u}^{\,n}) \\
&\quad - A_h(v_+^{n-1}, U^n)\big\}\Big] \\
&=: \ (III)_1 + (III)_2 + (III)_3.
\end{aligned} \tag{5.3.6}$$

Before estimating the three terms in (5.3.6), we first rewrite term $(III)_3$ as I_3 appearing in (5.2.8). Note that for all $w \in H_0^1(\Omega)$, we have

$$(\Delta_k u^n, w) + A(\overline{u}^{\,n}, w) = (\overline{f}^{\,n}, w) + \langle \overline{g}^{\,n}, w\rangle_\Gamma, \quad 1 \leq n \leq M. \tag{5.3.7}$$

Now, taking $v_h = w = v_+^{n-1}$ in both (5.3.7) and (5.3.2), subtracting one from the other, we obtain

$$\{A(\overline{u}^{\,n}, v_+^{n-1}) - A_h(U^n, v_+^{n-1})\} = \{\langle \overline{g}^{\,n}, v_+^{n-1}\rangle_\Gamma - \langle \overline{g}_h^{\,n}, v_+^{n-1}\rangle_{\Gamma_h}\} + (-\Delta_k(u^n - U^n), v_+^{n-1}),$$

which leads to

$$\begin{aligned}(III)_3 &= \sum_{n=1}^{M} k\{\langle \overline{g}^{\,n}, v_+^{n-1}\rangle_\Gamma - \langle \overline{g}_h^{\,n}, v_+^{n-1}\rangle_{\Gamma_h}\} \\ &\quad + \sum_{n=1}^{M} k\{(-\Delta_k(u^n - U^n), v_+^{n-1}) + (-\nabla_k v^n, u^n - U^n)\} \\ &=: (IV)_1 + (IV)_2. \end{aligned} \tag{5.3.8}$$

Using the fact that $v_+^M = 0$, and applying the identity

$$\sum_{n=1}^{M}(a_n - a_{n-1})b_n = a_M b_M - a_0 b_0 - \sum_{n=1}^{M} a_{n-1}(b_n - b_{n-1})$$

to $(IV)_2$ with $a_n = v_+^n$ and $b_n = u^n - U^n$, we obtain

$$\begin{aligned}(IV)_2 &= \sum_{n=1}^{M} k\left\{\left(-\frac{v_+^n - v_+^{n-1}}{k}, u^n - U^n\right) + (-\Delta_k(u^n - U^n), v_+^{n-1})\right\} \\ &= (v_+^0, u_0 - L_h u_0) = 0. \end{aligned} \tag{5.3.9}$$

This leads to

$$\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)} = \sum_{i=1}^{2}(III)_i + (IV)_1. \tag{5.3.10}$$

For the term $(III)_1$, use the Cauchy-Schwarz inequality and (5.3.4) to obtain

$$\begin{aligned}|(III)_1| &\le \left\{\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - \overline{Q}_k^{\,n})\|^2_{L^2(\Omega)}\right\}^{\frac{1}{2}} \left\{\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)}\right\}^{\frac{1}{2}} \\ &\le C(u)h\left\{\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)}\right\}^{\frac{1}{2}}. \end{aligned} \tag{5.3.11}$$

Applying Lemma 5.3.2 and (5.3.4), it is easy to notice (cf. [54])

$$\begin{aligned}|(III)_2| &\le \left(\sum_{n=1}^{M} k\|\nabla_k w^n\|^2_{H^{-1}(\Omega)}\right)^{\frac{1}{2}} \left[\left\{\sum_{n=1}^{M} k\|\overline{u}^{\,n} - \overline{Q}_k^{\,n}\|^2_{H^1(\Omega)}\right\}^{\frac{1}{2}} \right. \\ &\quad \left. + \left\{\sum_{n=1}^{M} k\|\overline{u}^{\,n} - u^n\|^2_{H^1(\Omega)}\right\}^{\frac{1}{2}}\right] \\ &\le C(u, u_t)(k + h)\left(\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n} - U^n)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \end{aligned} \tag{5.3.12}$$

An Application of Lemmas 3.2.3 and 5.3.2 yields

$$\begin{aligned} |(IV)_1| &\leq Ch^{\frac{3}{2}}\Big(\sum_{n=1}^{M} k\|\overline{g}^{\,n}\|^2_{H^2(\Gamma)}\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{M} k\|v_+^{n-1}\|^2_{H^1(\Omega)}\Big)^{\frac{1}{2}} \\ &\leq Ch^{\frac{3}{2}}\|g\|_{L^2(0,T;H^2(\Gamma))}\left(\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n}-U^n)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}. \end{aligned} \tag{5.3.13}$$

Again by a simple calculation (cf. [54]), it follows that

$$\|u-U\|_{L^2(0,T;H^1(\Omega))} \leq Ck\|u_t\|_{L^2(0,T;Y)} + \left(\sum_{n=1}^{M} k\|\nabla(\overline{u}^{\,n}-U^n)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \tag{5.3.14}$$

and hence, the desired estimate now follows from (5.3.10)-(5.3.14). This completes the proof. □

Chapter 6

Unfitted Finite Element Method for Elliptic and Parabolic Interface Problems

In this chapter, a finite element discretization independent of the location of the interface is proposed and analyzed for linear elliptic and time dependent parabolic interface problems. For an unfitted finite element method, we establish error estimates of optimal order in H^1 norm and almost optimal order in L^2 norm for elliptic problems. Moreover, we have shown that the proposed method can be used to derive optimal rate of convergence in $L^2(H^1)$ norm and almost optimal in $L^2(L^2)$ norm in the spatially discrete scheme for parabolic problems. A fully discrete scheme based on backward Euler method is also discussed and related error estimate is obtained.[3]

6.1 Introduction

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ and $\Omega_1 \subset \Omega$ is an open domain with C^2 smooth boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \backslash \Omega_1$. To begin with, we first recall the following elliptic interface problems

$$\mathcal{L}u = f(x) \quad \text{in } \Omega \tag{6.1.1}$$

with boundary condition

$$u(x) = 0 \quad \text{on } \partial\Omega \tag{6.1.2}$$

and interface conditions

$$[u] = 0, \qquad \left[\mathcal{A}\frac{\partial u}{\partial \mathbf{n}}\right] = g(x) \quad \text{along } \Gamma, \tag{6.1.3}$$

[3] Revised version submitted to *IMA J. Numer. Anal.*.

where the symbols $[v]$ and $\mathbf{n}$ are defined as in Chapter 1. For the simplicity of the exposition, we assume $\mathcal{L}u = -\nabla \cdot (\beta \nabla u)$, where the function β is positive and piecewise constant, i.e.,

$$\beta(x) = \beta_i \quad \text{for} \quad x \in \Omega_i, \ i = 1, 2.$$

Further, an attempt has also been made to study the finite element approximation to the following parabolic interface problems

$$u_t - \mathcal{L}(t)u = f(x,t) \quad \text{in } \Omega \times (0, T] \tag{6.1.4}$$

subject to the initial and boundary conditions

$$u(x,0) = u_0(x) \ \text{ in } \Omega \quad ; \quad u = 0 \quad \text{on } \partial\Omega \times (0, T] \tag{6.1.5}$$

and interface conditions

$$[u] = 0, \qquad \left[\mathcal{A} \frac{\partial u}{\partial \mathbf{n}} \right] = g(x,t) \quad \text{along } \Gamma. \tag{6.1.6}$$

Here, $T < \infty$ and the operator $\mathcal{L}(t)$ is a second order time dependent elliptic partial differential operator of the form

$$\mathcal{L}(t)u = -\nabla.(\beta(x,t)\nabla u).$$

The coefficient function $\beta(x,t)$ is assumed to be positive and piecewise constant across Γ, i.e., $\beta(x,t) = \beta_k$ for $(x,t) \in \Omega_k \times [0,T], \ k = 1, 2$. The symbols $[v]$ and $\mathbf{n}$ are specified as in Chapter 1.

For the purpose of finite element approximation, we shall recall the space $H_0^1(\Omega) = \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}$. Then the weak formulation of the problem (6.1.1)-(6.1.3) may be stated as: Find $u \in H_0^1(\Omega)$ such that u satisfies

$$A(u,v) = (f,v) + \langle g, v \rangle_\Gamma \quad \forall \, v \in H_0^1(\Omega). \tag{6.1.7}$$

Here, $(\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle_\Gamma$ are used to denote the inner products of $L^2(\Omega)$ and $L^2(\Gamma)$ spaces, respectively.

Further, let $A(t,\cdot,\cdot)$ be the bilinear form corresponding to the operator $\mathcal{L}(t)$. Then the weak formulation of the problem (6.1.4)-(6.1.6) is defined as: Find $u(t) \in H_0^1(\Omega)$ such that $u(0) = u_0$ and satisfies

$$(u_t, v) + A(t; u, v) = (f, v) + \langle g, v \rangle_\Gamma \quad \forall v \in H_0^1(\Omega). \tag{6.1.8}$$

The purpose of the present chapter is to study the convergence of finite element solutions to the exact solutions of elliptic and parabolic interface problems by means of unfitted finite element method. The main tools used in our analysis are Sobolev embedding inequality, duality arguments, interpolation and interface approximation results. Earlier works on the unfitted finite element method for elliptic interface problems can be found in [5, 6, 7, 8, 26, 37]. Present work not only generalize the previous works on unfitted finite element method for elliptic interface problems but also analyze time dependent parabolic interface problems for unfitted finite element method.

The layout of this Chapter is as follows: Section 6.2 is devoted to the unfitted finite element discretization and state some interpolation, and interface approximation properties needed for the error analysis. In Section 6.3, optimal H^1 and almost optimal L^2 error estimates for the elliptic interface problems are derived. Finally, the time dependent parabolic interface problems are discussed in Section 6.4 and related error estimates are obtained.

6.2 Unfitted Finite Element Discretization

In this section, we introduce the notion of unfitted mesh and state some necessary interpolation approximation properties.

In order to study the unfitted finite element approximations of (6.1.7) and (6.1.8), we now describe the triangulation $\tilde{\mathcal{T}}_h$ of Ω. The triangulations are assumed to be independent of the location of the interface Γ. We shall use the following notation for the mesh related quantities. Let h_K be the diameter of an element $K \in \tilde{\mathcal{T}}_h$ and $h = \max_{K\in\tilde{\mathcal{T}}_h} h_K$. For any element $K \in \tilde{\mathcal{T}}_h$, let $K_i = K \cap \Omega_i$, for $i = 1, 2$. By $\mathcal{T}_\Gamma^*$ we denote the set of all elements that are intersected by the interface Γ, i.e., $\mathcal{T}_\Gamma^* = \{K \in \tilde{\mathcal{T}}_h : \ K \cap \Gamma \neq \emptyset\}$. Any element $K \in \mathcal{T}_\Gamma^*$ is called an interface triangle and we write $\Omega_h^* = \cup_{K\in\mathcal{T}_\Gamma^*} K$.

Triangulation $\tilde{\mathcal{T}}_h$ of the domain Ω be such that it satisfies the following conditions:

($\mathcal{A}1$) $\overline{\Omega} = \cup_{K\in\tilde{\mathcal{T}}_h} K$.

($\mathcal{A}2$) If $\overline{K}_1$, $\overline{K}_2 \in \tilde{\mathcal{T}}_h$ and $\overline{K}_1 \neq \overline{K}_2$, then either $\overline{K}_1 \cap \overline{K}_2 = \emptyset$ or $\overline{K}_1 \cap \overline{K}_2$ is a common vertex, or edge of both triangles.

($\mathcal{A}3$) For any triangle $K \in \mathcal{T}_\Gamma^*$, we assume that Γ intersects K at most twice, and each (open) edge at most once.

($\mathcal{A}4$) For each triangle $K \in \tilde{\mathcal{T}}_h$, let $r_K, \overline{r}_K$ be the radii of its inscribed and circumscribed circles, respectively. We assume that, for some fixed $h_0 > 0$, there exists two positive constants C_0 and C_1 independent of h such that

$$C_0 r_K \leq h \leq C_1 \overline{r}_K \quad \forall K \in \tilde{\mathcal{T}}_h, \quad \forall h \in (0, h_0).$$

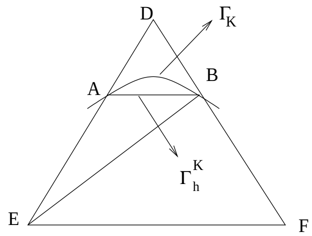

Figure 6.1: *A typical element* $K \in \mathcal{T}_\Gamma^\star$.

Here, the smooth interface Γ is approximated by the line segment $\overline{AB}$, A and B be the points of intersection of the interface Γ with two edges DE and DF in the interface triangles as shown in the Figure 6.1. The union of such line segments form an approximate interface Γ_h. Thus, Γ_h divides Ω into two polygonal subdomains Ω_1^h and Ω_2^h which are approximations of Ω_1 and Ω_2, respectively. Let V_h be a family of finite element subspaces of $H_0^1(\Omega)$ defined on $\tilde{\mathcal{T}}_h$ consisting of piecewise linear polynomials vanishing on the boundary $\partial\Omega$.

We shall again recall the following solution space

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$$

equipped with the norm

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}.$$

By Sobolev embedding theorem, for $v \in X$, we have $v \in W^{1,p}(\Omega)\ \forall p > 2$. Therefore, the linear interpolant operator $\Pi_h : X \to V_h$ is well defined (cf. [14, 19]). Following the lines of proof for Lemma 3.2.1, it is possible to obtain the following optimal error bounds for linear interpolant Π_h.

Lemma 6.2.1 *Let* $\Pi_h : X \to V_h$ *be the linear interpolation operator and* u *be the solution for the interface problem (6.1.1)-(6.1.3), then the following approximation properties*

$$\|u - \Pi_h u\|_{H^m(\Omega)} \leq Ch^{2-m}\{\|u\|_X + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_1)} + \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_2)}\},\ m = 0, 1,$$

hold true.

Proof. For any triangle $K \in \tilde{\mathcal{T}}_h \backslash \mathcal{T}_\Gamma^*$, the standard finite element interpolation theory (cf. [12] and [16]), implies that

$$\|u - \Pi_h u\|_{H^m(K)} \leq Ch^{2-m}\|u\|_{H^2(K)},\ \text{for } m = 0, 1. \tag{6.2.1}$$

Next, we consider any element $K \in \mathcal{T}_\Gamma^*$, as shown in Figure 6.1. Further, using the Holder's inequality and the fact meas$(K_i) \leq Ch^2,\ i = 1, 2$, we derive that for any $p > 2$ and $m = 0, 1$,

$$\begin{aligned} \|u - \Pi_h u\|_{H^m(K_i)} &\leq Ch^{\frac{(p-2)}{p}} \|u - \Pi_h u\|_{W^{m,p}(K_i)} \\ &\leq Ch^{\frac{(p-2)}{p}} \|u - \Pi_h u\|_{W^{m,p}(K)} \\ &\leq Ch^{\frac{(p-2)}{p}+1-m} \|u\|_{W^{1,p}(K)}, \end{aligned} \tag{6.2.2}$$

in the last inequality we have used the standard interpolation theory (cf. Ciarlet [16]).

Now, by (6.2.1)-(6.2.2) we obtain

$$\begin{aligned} \|u - \Pi_h u\|^2_{H^m(\Omega)} &\leq Ch^{4-2m}\|u\|^2_X \\ &+ C \sum_{K \in \mathcal{T}_\Gamma^*} h^{4-2m-\frac{4}{p}} \{\|u\|^2_{W^{1,p}(K_1)} + \|u\|^2_{W^{1,p}(K_2)}\} \\ &\leq Ch^{4-2m}\|u\|^2_X \\ &+ C \sum_{K \in \mathcal{T}_\Gamma^*} h^{4-2m-\frac{4}{p}} \{\|u\|^2_{W^{1,p}(K\cap\Omega_1)} + \|u\|^2_{W^{1,p}(K\cap\Omega_2)}\} \\ &\leq Ch^{4-2m}\|u\|^2_X \\ &+ Ch^{4-2m-\frac{4}{p}} \{\|u\|^2_{W^{1,p}(\Omega_h^*\cap\Omega_1)} + \|u\|^2_{W^{1,p}(\Omega_h^*\cap\Omega_2)}\} \\ &\leq Ch^{4-2m}\|u\|^2_X \\ &+ Ch^{4-2m-\frac{4}{p}} \{\|u\|^2_{W^{1,p}(\Omega_0\cap\Omega_1)} + \|u\|^2_{W^{1,p}(\Omega_0\cap\Omega_2)}\} \end{aligned} \tag{6.2.3}$$

for sufficiently small $h > 0$ such that $\Omega_h^* \subset \Omega_0$, Ω_0 is some neighborhood of Γ.

A simple calculation yields, for $i = 1, 2$,

$$\begin{aligned} \|u\|_{W^{1,p}(\Omega_0\cap\Omega_i)} &\leq \text{meas}(\Omega_0 \cap \Omega_1)^{\frac{1}{p}} \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_i)} \\ &\leq C^{\frac{1}{p}} \|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_i)}. \end{aligned}$$

Which together with (6.2.3) implies

$$\begin{aligned} &\|u - \Pi_h u\|^2_{H^m(\Omega)} \leq Ch^{4-2m}\|u\|^2_X \\ &+Ch^{4-2m-\frac{4}{p}} C^{\frac{2}{p}} \{\|u\|^2_{W^{1,\infty}(\Omega_0\cap\Omega_1)} + \|u\|^2_{W^{1,\infty}(\Omega_0\cap\Omega_2)}\}. \end{aligned} \tag{6.2.4}$$

The desired estimate follows by taking $p \to \infty$ both sides of (6.2.4). ■

Remark 6.2.1 *In Lemma 6.2.1, it is assumed that the solution* $u \in X \cap W^{1,\infty}(\Omega_0 \cap \Omega_1) \cap W^{1,\infty}(\Omega_0 \cap \Omega_2)$. *However, if* $u \in X$ *then it is easy to derive the following approximation results (cf. [14])*

$$\|u - \Pi_h u\|_{H^m(\Omega)} \leq Ch^{2-m} \log h \|u\|_X, \quad m = 0, 1. \tag{6.2.5}$$

Arguing as in deriving Lemma 3.2.2, we have the following result which will be useful for our later analysis.

Lemma 6.2.2 *Let Ω_h^* be the union of all interface triangles, then we have*

$$\|v\|_{H^1(\Omega_h^*)} \leq Ch^{\frac{1}{2}}\|v\|_X \quad \forall v \in X.$$

6.3 Convergence Analysis for Elliptic Interface Problems

This section is concerned with the convergence of unfitted finite element solution to the exact solution of elliptic interface problem. While the H^1 norm error estimate is shown to be sharp, the L^2 norm error estimate is optimal upto a factor $\log h$.

We now turn to the unfitted finite element approximation to elliptic interface problems (6.1.1)-(6.1.3). For $g \in H^{\frac{1}{2}}(\Gamma)$ and $f \in L^2(\Omega)$, we now define the finite element approximation as follows: Find $u_h \in V_h$ such that

$$\tilde{A}_h(u_h, v_h) = (f, v_h) + \langle g, v_h\rangle_\Gamma \quad \forall v_h \in V_h, \tag{6.3.1}$$

where $\tilde{A}_h(.,.) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is given by

$$\tilde{A}_h(w, v) = \sum_{K\in\tilde{\mathcal{T}}_h} \int_K \beta_h(x)\nabla w \nabla v dx \quad \forall w, v \in H^1(\Omega), \;\; \beta_h = \beta_i \text{ in } \Omega_i^h. \tag{6.3.2}$$

Note that the integrals involve in the bilinear form $\tilde{A}_h$ are evaluated by some quadrature rule over a triangle $K \in \tilde{\mathcal{T}}_h$. For the interface triangle, shown in the Figure 6.1, it may cause some technical difficulties to evaluate the integrals over the region $AEFB$ using Lagrange finite element space. The reason is that the points of intersection A and B between the element edges and the interface are not used to represent the basis functions in Lagrange finite element space. But, one can overcome this difficulty by using immersed finite element space for unfitted mesh (cf. [37]), where two more freedoms are added at A and B. More precisely, if $V_h(K)$ be a local finite element space on each element $K \in \tilde{\mathcal{T}}_h$ then the dimension of $V_h(K)$ is three for a non-interface triangle and the dimension of $V_h(K)$ is five if $K \in \mathcal{T}_\Gamma^*$. This enable us to use the well known quadrature rule over those sub-triangles.

We shall need the following Lemma which will be useful for our later analysis.

Lemma 6.3.1 *For all $v_h, w_h \in V_h$, we have*

$$|A(v_h, w_h) - \tilde{A}_h(v_h, w_h)| \leq Ch \sum_{K\in\mathcal{T}_\Gamma^*} \|\nabla v_h\|_{L^2(K)}\|\nabla w_h\|_{L^2(K)}.$$

Proof. Let Γ_K be the arc common to the interface Γ and the interface triangle K, and Γ_h^K be restriction of Γ_h in $K \in \mathcal{T}_\Gamma^*$. Let $\tilde{K} \subset K$ be the region enclosed by Γ_K and Γ_h^K for an interface triangle K. Without loss of generality, we can assume that meas$(\tilde{K}) \leq Ch^3$.

Before proceeding to prove Lemma 6.3.1, we need the following information

$$\text{supp}\,(\beta - \beta_h) \cap K = \tilde{K} \quad \forall K \in \mathcal{T}_\Gamma^*.$$

Then we have

$$\begin{aligned}
|A(v_h, w_h) - \tilde{A}_h(v_h, w_h)| &\leq \sum_{K \in \mathcal{T}_\Gamma^*} \int_{\tilde{K}} |(\beta_h - \beta)\nabla v_h \nabla w_h| \\
&\leq C \sum_{K \in \mathcal{T}_\Gamma^*} \left[\int_{\tilde{K}} |\nabla v_h \nabla w_h| \right] \\
&\leq C \sum_{K \in \mathcal{T}_\Gamma^*} \left[\|\nabla v_h\|_{L^2(\tilde{K})} \|\nabla w_h\|_{L^2(\tilde{K})} \right] \\
&\leq Ch \sum_{K \in \mathcal{T}_\Gamma^*} \|\nabla v_h\|_{L^2(K)} \|\nabla w_h\|_{L^2(K)}
\end{aligned}$$

where we used the fact that ∇v_h and ∇w_h are constant in $K \in \tilde{\mathcal{T}}_h$, and meas$(\tilde{K}) \leq Ch^3$. This completes the proof. □

Now, we are in a position to discuss the following H^1 norm convergence result.

Theorem 6.3.1 *Let u and u_h be the solutions of (6.1.1)-(6.1.3) and (6.3.1), respectively. Then, for $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma)$, there exists a positive constant C independent of h such that*

$$\|u - u_h\|_{H^1(\Omega)} \leq C(u)h,$$

where $C(u) = C(\|u\|_X + \|u\|_{W^{1,\infty}(\Omega_0 \cap \Omega_1)} + \|u\|_{W^{1,\infty}(\Omega_0 \cap \Omega_2)})$.

Proof. In view of Lemma 6.2.1, it is enough to have bounds for the term $\Pi_h u - u_h$. From (6.1.7) and (6.3.1) we note that

$$A(u, v_h) - \tilde{A}_h(u_h, v_h) = 0 \quad \forall v_h \in V_h. \tag{6.3.3}$$

Using Lemma 6.2.1 and (6.3.3), we obtain

$$\begin{aligned}
C\|\Pi_h u - u_h\|^2_{H^1(\Omega)} &\leq \tilde{A}_h(\Pi_h u - u, \Pi_h u - u_h) + \tilde{A}_h(u - u_h, \Pi_h u - u_h) \\
&\leq Ch(\|u\|_X + \|u\|_{W^{1,\infty}(\Omega_0 \cap \Omega_1)} + \|u\|_{W^{1,\infty}(\Omega_0 \cap \Omega_2)}) \\
&\quad \times \|\Pi_h u - u_h\|_{H^1(\Omega)} \\
&\quad + \{\tilde{A}_h(u - \Pi_h u, \Pi_h u - u_h) - A(u - \Pi_h u, \Pi_h u - u_h)\} \\
&\quad + \{\tilde{A}_h(\Pi_h u, \Pi_h u - u_h) - A(\Pi_h u, \Pi_h u - u_h)\} \\
&=: C(u)h\|\Pi_h u - u_h\|_{H^1(\Omega)} + (I)_1 + (I)_2.
\end{aligned} \tag{6.3.4}$$

Arguing as in deriving Lemma 6.3.1 and using Lemma 6.2.1, we have

$$
\begin{aligned}
|(I)_1| &\leq \sum_{K\in\mathcal{T}_\Gamma^*}\int_{\tilde{K}} |(\beta_h-\beta)\nabla(u-\Pi_h u)\nabla(\Pi_h u-u_h)| \\
&\leq C\sum_{K\in\mathcal{T}_\Gamma^*}\left[\int_{\tilde{K}} |\nabla(u-\Pi_h u)\nabla(\Pi_h u-u_h)|\right] \\
&\leq C\|u-\Pi_h u\|_{H^1(\Omega)}\|\Pi_h u-u_h\|_{H^1(\Omega)} \\
&\leq Ch(\|u\|_X+\|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_1)}+\|u\|_{W^{1,\infty}(\Omega_0\cap\Omega_2)})\|\Pi_h u-u_h\|_{H^1(\Omega)} \\
&=: C(u)h\|\Pi_h u-u_h\|_{H^1(\Omega)}. \qquad (6.3.5)
\end{aligned}
$$

For $(I)_2$, Lemma 6.3.1 immediately implies

$$|(I)_2| \leq Ch\|\Pi_h u\|_{H^1(\Omega)}\|\Pi_h u-u_h\|_{H^1(\Omega)} \leq C(u)h\|\Pi_h u-u_h\|_{H^1(\Omega)}, \qquad (6.3.6)$$

where, in the last inequality, we have used Lemma 6.2.1.

Combine (6.3.6) and (6.3.5) with (6.3.4), to obtain

$$\|\Pi_h u-u_h\|_{H^1(\Omega)} \leq C(u)h. \qquad (6.3.7)$$

Now, the desired result follows from (6.3.7), Lemma 6.2.1 and the triangle inequality. □

Remark 6.3.1 *Theorem 6.3.1 yields optimal order error estimate in H^1 norm. If $u \in X$ then one can derive the following almost optimal H^1 norm error estimate*

$$\|u-u_h\|_{H^1(\Omega)} \leq Ch\log h\|u\|_X \leq Ch\log h\left\{\|f\|_{L^2(\Omega)}+\|g\|_{H^{\frac{1}{2}}(\Gamma)}\right\}. \qquad (6.3.8)$$

This follows from the argument of Theorem 6.3.1 and (6.2.5). A similar result is established in [14] for the fitted finite element method.

For the L^2 norm error estimate, we shall use the duality trick. For this purpose, we consider the following interface problem: Find $w \in H_0^1(\Omega)$ such that

$$A(w,v) = (u-u_h,v) \ \ \forall v \in H_0^1(\Omega) \qquad (6.3.9)$$

and its finite element approximation is defined to be the function $w_h \in V_h$ such that

$$\tilde{A}_h(w_h,v_h) = (u-u_h,v_h) \ \ \forall v_h \in V_h. \qquad (6.3.10)$$

Note that $w \in X \cap H_0^1(\Omega)$ is the solution of the elliptic interface problem (6.3.9) with the jump conditions $[w]=0$, $\left[\beta(x)\frac{\partial w}{\partial \mathbf{n}}\right]=0$ along Γ. Further, w satisfies the *a priori* estimate

$$\|w\|_X \leq C\|u-u_h\|_{L^2(\Omega)}. \qquad (6.3.11)$$

Further, applying (6.3.8) for the interface problem (6.3.9)-(6.3.10), we obtain

$$\|w - w_h\|_{H^1(\Omega)} \le Ch \log h \|w\|_X. \tag{6.3.12}$$

Setting $v = u - u_h \in H_0^1(\Omega)$ in (6.3.9) and using (6.3.3), we obtain

$$\begin{aligned}
\|u - u_h\|^2_{L^2(\Omega)} &= A(w, u - u_h) \\
&= A(w - w_h, u - u_h) + A(w_h, u - u_h) \\
&= A(w - w_h, u - u_h) + \tilde{A}_h(w_h, u_h) - A(w_h, u_h) \\
&\le C\|w - w_h\|_{H^1(\Omega)}\|u - u_h\|_{H^1(\Omega)} \\
&\quad + Ch \sum_{K \in \mathcal{T}_\Gamma^*} \|\nabla w_h\|_{L^2(K)}\|\nabla u_h\|_{L^2(K)},
\end{aligned}$$

where, in the last inequality, we used Lemma 6.3.1. Now, apply Theorem 6.3.1, Lemma 6.2.2 and (6.3.12), to obtain

$$\begin{aligned}
\|u - u_h\|^2_{L^2(\Omega)} &\le Ch \log h \|w\|_X C(u) h + Ch\|w - w_h\|_{H^1(\Omega)}\|u - u_h\|_{H^1(\Omega)} \\
&\quad + Ch\|u\|_{H^1(\Omega)}\|w - w_h\|_{H^1(\Omega)} + Ch\|u - u_h\|_{H^1(\Omega)}\|w\|_{H^1(\Omega)} \\
&\quad + Ch\|u\|_{H^1(\Omega_h^*)}\|w\|_{H^1(\Omega_h^*)} \\
&\le C(u)h^2 \log h\|w\|_X + Ch^2 \log h\|w\|_X C(u)h \\
&\quad + Ch^2 \log h\|w\|_X\|u\|_{H^1(\Omega)} + C(u)h^2\|w\|_{H^1(\Omega)} \\
&\quad + Chh^{\frac{1}{2}}\|w\|_X h^{\frac{1}{2}}\|u\|_X \\
&\le C(u)h^2 \log h\|w\|_X.
\end{aligned}$$

Finally, using (6.3.11) we obtain

$$\|u - u_h\|^2_{L^2(\Omega)} \le C(u)h^2 \log h\|u - u_h\|_{L^2(\Omega)}.$$

Thus, we have proved the following almost optimal order error estimate in L^2 norm.

Theorem 6.3.2 *Let u and u_h be the solutions of (6.1.1)-(6.1.3) and (6.3.1), respectively. Then, for $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma)$, there exists a positive constant C independent of h such that*

$$\|u - u_h\|_{L^2(\Omega)} \le C(u)h^2 \log h.$$

6.4 Time Dependent Parabolic Interface Problems

This section is devoted to the error analysis of the time dependent parabolic interface problems. Both spatially discrete and fully discrete schemes are analyzed. In the absence of interface the analysis of both semidiscrete and fully discrete schemes are described in Douglas and Dupont [18], Luskin and Rannacher [38], Thomée [58], and references therein.

For the subsequent analysis, we need the following space

$$Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2),$$

equipped with the norm

$$\|v\|_Y = \|v\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega_1)} + \|v\|_{H^1(\Omega_2)} \ \ \forall v \in Y.$$

We recall the following regularity result for the parabolic interface problem (6.1.4)-(6.1.6) (cf. [30]).

Theorem 6.4.1 *Let $f \in H^1(0,T;L^2(\Omega))$, $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$ and $u_0 \in H_0^1(\Omega)$. Then the problem (6.1.4)-(6.1.6) has a unique solution $u \in L^2(0,T;X) \cap H^1(0,T;Y) \cap H_0^1(\Omega)$.*

Let $L_h : L^2(\Omega) \to V_h$ be the standard L^2 projection defined by

$$(L_h v, \phi) = (v, \phi) \quad \forall v \in L^2(\Omega), \ \ \phi \in V_h. \tag{6.4.1}$$

It now follows from [14] that

$$\|L_h v\|_{H^1(\Omega)} \le \|v\|_{H^1(\Omega)} \quad \forall v \ \in H_0^1(\Omega). \tag{6.4.2}$$

It is well known that $L_h v \in V_h$ is the best approximation in the L^2 norm to $v \in L^2(\Omega)$. Thus, Lemma 6.2.1 and further using the inverse inequality, we have

$$\|L_h v - v\|_{H^m(\Omega)} \le C(v)h^{2-m}, \ m = 0, 1. \tag{6.4.3}$$

For $v \in X$, it is easy to notice from (6.2.5) that

$$\|L_h v - v\|_{H^m(\Omega)} \le Ch^{2-m} \log h \|v\|_X, \ m = 0, 1. \tag{6.4.4}$$

6.4.1 The continuous time Galerkin approximation

This subsection is devoted to the continuous time Galerkin approximation to time dependent parabolic interface problems (6.1.4)-(6.1.6). We establish error estimates of optimal order in $L^2(H^1)$ norm and almost optimal order in $L^2(L^2)$ norm.

The continuous time Galerkin finite element approximation to (6.1.8) is stated as follows: Find $u_h : [0,T] \to V_h$ such that $u_h(0) = L_h u_0$ and satisfies

$$(u_{ht}, v_h) + \tilde{A}_h(t; u_h, v_h) = (f, v_h) + \langle g, v_h \rangle_\Gamma \quad \forall v_h \in V_h. \tag{6.4.5}$$

The bilinear form $\tilde{A}_h$ is defined in section 6.3.

We shall need the following stability estimate for the semidiscrete solution u_h satisfying (6.4.5).

Lemma 6.4.1 *Assume that $u_h(0) = L_h u_0$. Then, for $u_0 \in H_0^1(\Omega)$, we have*

$$\int_0^t \|u_{ht}\|^2_{L^2(\Omega)} ds + \|u_h\|^2_{H^1(\Omega)} \leq C\left\{ \|u_0\|^2_{H^1(\Omega)} + \int_0^t \left(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{H^{\frac{1}{2}}(\Gamma)} \right) ds \right\}.$$

Proof. Set $v_h = u_{ht}$ in (6.4.5). Using the standard argument of [38] and (6.4.2), the desired result is easily obtained. We omit the details. □

Subtracting (6.4.5) from (6.1.8) we obtain

$$(u_t - u_{ht}, v_h) + A(t; u - u_h, v_h) = \tilde{A}_h(t; u_h, v_h) - A(t; u_h, v_h) \quad \forall v_h \in V_h. \tag{6.4.6}$$

Define the error $e(t)$ as $e(t) = u(t) - u_h(t)$. Setting $v_h = L_h u$ in (6.4.6) and using (6.4.1) we obtain the following error equation

$$\frac{1}{2}\frac{d}{dt}\|e(t)\|^2_{L^2(\Omega)} + A(e, e) = (III)_1 + (III)_2 + \frac{1}{2}\frac{d}{dt}\|u - L_h u\|^2_{L^2(\Omega)}, \tag{6.4.7}$$

where $(III)_1 = \tilde{A}_h(t; u_h, L_h u - u_h) - A(t; u_h, L_h u - u_h)$ and $(III)_2 = A(t; u_h - u, L_h u - u)$. Now we estimate the terms $(III)_1$ and $(III)_2$ one by one. Applying Lemma 6.3.1 and further, using the estimate (6.4.3), it follows that

$$\begin{aligned} |(III)_1| &\leq Ch\|u_h\|_{H^1(\Omega)}\|L_h u - u_h\|_{H^1(\Omega)} \\ &\leq C(u)h^2\|u_h\|_{H^1(\Omega)} + Ch\|u_h\|_{H^1(\Omega)}\|e(t)\|_{H^1(\Omega)} \\ &\leq \{C(u)\}^2 h^2 + Ch^2\|u_h\|^2_{H^1(\Omega)} + \frac{1}{4}\|e(t)\|^2_{H^1(\Omega)}. \end{aligned} \tag{6.4.8}$$

For the term $(III)_2$, again use of (6.4.3) leads to

$$|(III)_2| \leq C(u)h\|e(t)\|_{H^1(\Omega)} \leq \{C(u)\}^2 h^2 + \frac{1}{4}\|e(t)\|^2_{H^1(\Omega)}. \tag{6.4.9}$$

Now integrating the identity (6.4.7) from 0 to t. Then using (6.4.8)-(6.4.9), we obtain

$$\begin{aligned} \int_0^t \|e\|^2_{H^1(\Omega)} ds \;&\leq\; C\left\{ h^2 \int_0^t \{C(u)\}^2 ds + h^2 \int_0^t \|u_h\|^2_{H^1(\Omega)} ds + \right. \\ &\qquad \left. + \|u - L_h u\|^2_{L^2(\Omega)} \right\} + \frac{1}{2}\int_0^t \|e\|^2_{H^1(\Omega)} ds. \end{aligned}$$

Then, (6.4.3) and Lemma 6.4.1 yield the following optimal H^1 norm error estimate.

Theorem 6.4.2 *Let u and u_h be the solutions of (6.1.4)-(6.1.6) and (6.4.5), respectively. Then, for $u_0 \in H_0^1(\Omega), f \in H^1(0, T; L^2(\Omega))$, and $g \in H^1(0, T; H^{\frac{1}{2}}(\Gamma))$, there exists a positive constant C independent of h such that*

$$\|u - u_h\|_{L^2(0,T;H^1(\Omega))} \leq C(u_0, u, f, g)h.$$

Next, to derive the L^2 norm error estimate we shall use the parabolic duality trick. For any time $t > 0$ and $e = u - u_h$, let $w(s) \in H_0^1(\Omega)$ and $w_h(s) \in V_h$, respectively, be the solutions of the backward problems

$$\begin{aligned}(\phi, w_s) - A(s;\phi, w) &= (\phi, e) \quad \forall \phi \in H_0^1(\Omega), \quad s < t, \qquad (6.4.10)\\ w(t) &= 0;\\ (\phi_h, w_{hs}) - \tilde{A}_h(s;\phi_h, w_h) &= (\phi_h, e) \quad \forall \phi_h \in V_h, \quad s < t, \qquad (6.4.11)\\ w_h(t) &= 0.\end{aligned}$$

We note that $[w] = 0$ and $g(x,t) = 0$ across the interface Γ. From (6.4.10) and (6.4.11), we obtain

$$(\phi_h, w_s - w_{hs}) - A(s;\phi_h, w - w_h) = A(s;\phi_h, w_h) - \tilde{A}_h(s;\phi_h, w_h) \quad \forall v_h \in V_h. \qquad (6.4.12)$$

Following the standard argument of [38], it is easy to show that

$$\int_0^t \|w_s - w_{hs}\|^2_{L^2(\Omega)} \leq C \int_0^t \|e\|^2_{L^2(\Omega)} ds. \qquad (6.4.13)$$

Further, we assume the following identity

$$\int_0^t \left(h^{-2} (\log h)^{-2} \|w - w_h\|^2_{H^1(\Omega)} \right) ds \leq C \int_0^t \|e\|^2_{L^2(\Omega)} ds \qquad (6.4.14)$$

holds true. The estimate (6.4.14) is obtained by reversing time and applying (6.4.4) instead (6.4.3) in the proof of Theorem 6.4.2.

Set $\phi = e$ in (6.4.10). Then, using the identity (6.4.6), we obtain

$$\begin{aligned}\|e\|^2_{L^2(\Omega)} &= (e, w_s) - A(s;e,w)\\ &= (e, w_{hs}) + (e, w_s - w_{hs}) - A(s;e, w - w_h) - A(s;e,w_h)\\ &= \frac{d}{ds}(e, w_h) + (e, w_s - w_{hs}) - A(s;e, w - w_h) - (e_s, w_h) - A(s;e,w_h)\\ &= \frac{d}{ds}(e, w_h) + (e, w_s - w_{hs}) - A(s;e, w - w_h) + A(s;u_h,w_h) - \tilde{A}_h(s;u_h,w_h).\end{aligned}$$

With an aid of (6.4.12), the above equation may be rewritten as

$$\begin{aligned}\|e\|^2_{L^2(\Omega)} &= \frac{d}{ds}(e, w_h) + (u - \Pi_h u, w_s - w_{hs}) - A(s;u - \Pi_h u, w - w_h)\\ &\quad + (\Pi_h u - u_h, w_s - w_{hs}) - A(s;\Pi_h u - u_h, w - w_h)\\ &\quad + \{A(s;u_h, w_h) - \tilde{A}_h(s;u_h,w_h)\}\\ &= \frac{d}{ds}(e, w_h) + (u - \Pi_h u, w_s - w_{hs}) - A(s;u - \Pi_h u, w - w_h)\\ &\quad + \{A(s;\Pi_h u - u_h, w_h) - \tilde{A}_h(s;\Pi_h u - u_h, w_h)\}\\ &\quad + \{A(s;u_h, w_h) - \tilde{A}_h(s;u_h,w_h)\}\\ &= \frac{d}{ds}(e, w_h) + (u - \Pi_h u, w_s - w_{hs})\\ &\quad - A(s;u - \Pi_h u, w - w_h) + (IV), \qquad (6.4.15)\end{aligned}$$

where $(IV) = A(s;\Pi_h u, w_h) - \tilde{A}_h(s;\Pi_h u, w_h)$.

We integrate (6.4.15) from 0 to t to have

$$\begin{aligned}
\int_0^t \|e\|^2_{L^2(\Omega)} ds &= \int_0^t \{(u-\Pi_h u, w_s - w_{hs}) - A(s; u-\Pi_h u, w-w_h)\} ds \\
&\quad + \int_0^t (IV) ds \\
&\le \epsilon \int_0^t \{\|w_s - w_{hs}\|^2_{L^2(\Omega)} + h^{-2}(\log h)^{-2}\|w-w_h\|^2_{H^1(\Omega)}\} ds \\
&\quad + \frac{C}{\epsilon}\int_0^t \{\|u-\Pi_h u\|^2_{L^2(\Omega)} + h^2(\log h)^2\|u-\Pi_h u\|^2_{H^1(\Omega)}\} ds \\
&\quad + \int_0^t |(IV)| ds. \qquad (6.4.16)
\end{aligned}$$

To estimate $\int_0^t |(IV)| ds$, use Lemmas 6.2.1- 6.2.2 and 6.3.1 to have

$$\begin{aligned}
|(IV)| &\le Ch\|\Pi_h u\|_{H^1(\Omega_h^*)}\|w_h\|_{H^1(\Omega_h^*)} \\
&\le Ch\|\Pi_h u - u\|_{H^1(\Omega_h^*)}\|w_h\|_{H^1(\Omega_h^*)} + Ch\|u\|_{H^1(\Omega_h^*)}\|w_h\|_{H^1(\Omega_h^*)} \\
&\le Ch\|\Pi_h u - u\|_{H^1(\Omega_h^*)}\|w_h\|_{H^1(\Omega_h^*)} + Ch^{\frac{3}{2}}\|u\|_X\|w-w_h\|_{H^1(\Omega_h^*)} \\
&\quad + Ch^{\frac{3}{2}}\|u\|_X\|w\|_{H^1(\Omega_h^*)} \\
&\le Ch\|u-\Pi_h u\|_{H^1(\Omega_h^*)}\|w-w_h\|_{H^1(\Omega_h^*)} \\
&\quad + Ch\|u-\Pi_h u\|_{H^1(\Omega_h^*)}\|w\|_{H^1(\Omega_h^*)} \\
&\quad + Ch^{\frac{3}{2}}\|u\|_X\|w-w_h\|_{H^1(\Omega_h^*)} + Ch^2\|u\|_X\|w\|_X.
\end{aligned}$$

Integrating from 0 to t, we obtain

$$\begin{aligned}
\int_0^t |(IV)| ds &\le Ch\int_0^t \|u-\Pi_h u\|_{H^1(\Omega_h^*)}\|w-w_h\|_{H^1(\Omega_h^*)} ds \\
&\quad + Ch^{\frac{3}{2}}\int_0^t \|u-\Pi_h u\|_{H^1(\Omega_h^*)}\|w\|_X ds \\
&\quad + Ch^{\frac{3}{2}}\int_0^t \|u\|_X\|w-w_h\|_{H^1(\Omega_h^*)} ds + Ch^2\int_0^t \|u\|_X\|w\|_X ds \\
&\le \frac{C}{\epsilon}h^4(\log h)^2\int_0^t \|u-\Pi_h u\|^2_{H^1(\Omega)} ds \\
&\quad + \frac{\epsilon}{2}h^{-2}(\log h)^{-2}\int_0^t \|w-w_h\|^2_{H^1(\Omega)} ds \\
&\quad + \frac{C}{\epsilon}h^3\int_0^t \|u-\Pi_h u\|^2_{H^1(\Omega)} ds + \frac{\epsilon}{2}\int_0^t \|w\|^2_X ds \\
&\quad + \frac{C}{\epsilon}h^5(\log h)^2\int_0^t \|u\|^2_X ds + \frac{\epsilon}{2}h^{-2}(\log h)^{-2}\int_0^t \|w-w_h\|^2_{H^1(\Omega)} ds \\
&\quad + \frac{C}{\epsilon}h^4\int_0^t \|u\|^2_X ds + \frac{\epsilon}{2}\int_0^t \|w\|^2_X ds. \qquad (6.4.17)
\end{aligned}$$

Combine (6.4.16)-(6.4.17) together with the regularity result for the parabolic problem (6.4.10) (cf. Theorem 4.1.1), estimates (6.4.13)-(6.4.14) and Lemma 6.2.1 to obtain

$$\int_0^t \|e\|^2_{L^2(\Omega)} ds \leq C\epsilon \int_0^t \|e\|^2_{L^2(\Omega)} ds + \frac{C}{\epsilon} h^4 (\log h)^2 \int_0^t \|u\|^2_X ds.$$

Chose $\epsilon > 0$ appropriately so that $(1 - C\epsilon) > 0$. This leads to the following almost optimal L^2 norm error estimate.

Theorem 6.4.3 *Let u and u_h be the solutions of (6.1.4)-(6.1.6) and (6.4.5), respectively. Then, for $u_0 \in H^1_0(\Omega), f \in H^1(0,T;L^2(\Omega))$, and $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$, there exists a positive constant C independent of h such that*

$$\|u - u_h\|_{L^2(0,T;L^2(\Omega))} \leq C(u,f,g) h^2 \log h.$$

6.4.2 Discrete time Galerkin method

A discrete-in-time scheme based on backward Euler method is considered and analyzed in this subsection. Optimal error estimate in $L^2(H^1)$ norm is established.

We first divide the interval $[0,T]$ into M equally spaced subintervals by the points

$$0 = t^0 < t^1 < \cdots < t^M = T,$$

with $t^n = nk$, $k = T/M$ be the time step. Let $I_n = (t^{n-1}, t^n]$ be the nth subinterval and ϕ^n denote the value of a function $\phi = \phi(x,t)$ at t^n. For a given sequence $\{\phi^n\}_{n=1}^M \subset L^2(\Omega)$, we introduce the backward difference quotient

$$\Delta_k \phi^n = \frac{\phi^n - \phi^{n-1}}{k}.$$

The fully discrete finite element approximation to the problem (6.4.5) is defined as follows: Find $U^n \in V_h$, $1 \leq n \leq M$, such that

$$\begin{aligned} (\Delta_k U^n, v_h) + \tilde{A}_h(t^n; U^n, v_h) &= (f^n, v_h) + \langle g^n, v_h \rangle_\Gamma \quad \forall v_h \in V_h, \qquad (6.4.18) \\ U^0 &= L_h u_0 \text{ in } \Omega. \end{aligned}$$

For convenience, let us define the piecewise constant function U_{hk} in time by $U_{hk}(x,t) = U^n(x)\ \forall t \in I_n$, $n = 1,2,3,\ldots,M$.

The main result on the convergence of fully discrete solution to the exact solution in $L^2(H^1)$ norm is stated below.

Theorem 6.4.4 *Let u and U_{hk} be the solutions of (6.1.4)-(6.1.6) and (6.4.18), respectively. Assume that $u_0 \in H^1_0(\Omega)$, $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$ and $f \in H^1(0,T;L^2(\Omega))$. Then there exists a positive constant C independent of h and k such that*

$$\|u - U_{hk}\|_{L^2(0,T;H^1(\Omega))} \leq C(u_0, u, u_t, u_{tt})\{k + h\}.$$

The proof of the above theorem is required some preparation. Now, we shall introduce an auxiliary projection which is crucial to our convergence analysis. For any $v \in X$, we define

$$f^* = \begin{cases} -\beta_1 \Delta v_1 & \text{in } \Omega_1, \\ -\beta_2 \Delta v_2 & \text{in } \Omega_2, \end{cases}$$

where $v_i = v|_{\Omega_i}$. With this f^*, define an operator $\tilde{P}_h : X \cap H_0^1(\Omega) \to V_h$ by

$$\tilde{A}_h(t; \tilde{P}_h v, \phi) = (f^*, \phi) \quad \forall \phi \in V_h,\ v \in X \cap H_0^1(\Omega). \tag{6.4.19}$$

It follows from the definition of f^* that

$$(f^*, \phi) = (-\nabla.(\beta \nabla v), \phi) = A(t; v, \phi) \quad \forall \phi \in V_h,\ v \in X \cap H_0^1(\Omega),$$

which combine with (6.4.19) leads to

$$\tilde{A}_h(t; \tilde{P}_h v, \phi) = (f^*, \phi) = A(t; v, \phi) \quad \forall \phi \in V_h,\ v \in X \cap H_0^1(\Omega). \tag{6.4.20}$$

From the error analysis of elliptic interface problems for unfitted mesh, we obtain

$$\|u - \tilde{P}_h u\|_{H^1(\Omega)} \leq C(u)h. \tag{6.4.21}$$

Proof of the Theorem 6.4.4 At $t = t^n$, (6.1.8) is of the form

$$(u_t^n, v) + A(t^n; u^n, v) = (f^n, v) + \langle g^n, v \rangle_\Gamma \ \ \forall v \in H_0^1(\Omega). \tag{6.4.22}$$

For simplicity, we write $w^n = u^n - \tilde{P}_h u^n$ and $e^n = u^n - U^n$. Using (6.4.18) and (6.4.22), and the fact $\tilde{A}_h(t^n; \tilde{P}_h u^n, \tilde{P}_h u^n - U^n) = A(t^n; u^n, \tilde{P}_h u^n - U^n)$ it follows that

$$\begin{aligned} (\Delta_k e^n, e^n) + \tilde{A}_h(t^n; e^n, e^n) \;\; &= \;\; (\Delta_k e^n, w^n) + \tilde{A}_h(t^n; e^n, w^n) + (\Delta_k u^n - u_t^n, \tilde{P}_h u^n - U^n) \\ &\quad + \tilde{A}_h(t^n; w^n, \tilde{P}_h u^n - U^n). \end{aligned}$$

Summing this identity over n from $n = 1$ to $n = M$, we obtain

$$\begin{aligned} \sum_{n=1}^{M} \|e^n\|_{L^2(\Omega)}^2 + \|e^M\|_{L^2(\Omega)}^2 \;\; &+ \;\; \sum_{n=1}^{M} \|e^n - e^{n-1}\|_{L^2(\Omega)}^2 + \sum_{n=1}^{M} k \tilde{A}_h(t^n; e^n, e^n) \\ &\leq \;\; C \left(\|e^0\|_{L^2(\Omega)}^2 + \sum_{j=1}^{4} (V)_j \right), \end{aligned} \tag{6.4.23}$$

where

$$\begin{aligned} (V)_1 \;\; &= \;\; k \sum_{n=1}^{M} (\Delta_k e^n, w^n),\ (V)_2 = k \sum_{n=1}^{M} \tilde{A}_h(t^n; e^n, w^n), \\ (V)_3 \;\; &= \;\; k \sum_{n=1}^{M} (\Delta_k u^n - u_t^n, \tilde{P}_h u^n - U^n),\ (V)_4 = k \sum_{n=1}^{M} \tilde{A}_h(t^n; w^n, \tilde{P}_h u^n - U^n). \end{aligned}$$

We now proceed to estimate each term separately. For $(V)_1$, in view of (6.4.21), we have

$$\begin{aligned} |(V)_1| &\leq \sum_{n=1}^{M} \|e^n - e^{n-1}\|_{L^2(\Omega)} \|w^n\|_{L^2(\Omega)} \\ &\leq \frac{1}{2}\sum_{n=1}^{M} \|e^n - e^{n-1}\|^2_{L^2(\Omega)} + Ch^2 \sum_{n=1}^{M} \{\|u^n\|^2_X + \sum_{i=1}^{2} \|u^n\|^2_{W^{1,\infty}(\Omega_0 \cap \Omega_i)}\} \\ &\leq \frac{1}{2}\sum_{n=1}^{M} \|e^n - e^{n-1}\|^2_{L^2(\Omega)} + \tilde{C}(u)h^2, \end{aligned} \tag{6.4.24}$$

where $\tilde{C}(u) = C\sum_{n=1}^{M}\{\|u^n\|^2_X + \sum_{i=1}^{2}\|u^n\|^2_{W^{1,\infty}(\Omega_0\cap\Omega_i)}\}$.

Similarly, for the term $(V)_2$, we obtain

$$\begin{aligned} |(V)_2| &\leq C\sum_{n=1}^{M} k\|e^n\|_{H^1(\Omega)}\|w^n\|_{H^1(\Omega)} \\ &\leq \frac{1}{4}\sum_{n=1}^{M} k\|e^n\|^2_{H^1(\Omega)} + C\sum_{n=1}^{M} k\|w^n\|^2_{H^1(\Omega)} \\ &\leq \frac{1}{4}\sum_{n=1}^{M} k\|e^n\|^2_{H^1(\Omega)} + \tilde{C}(u)h^2. \end{aligned} \tag{6.4.25}$$

For $(V)_3$, use Theorem 6.4.1 and (6.4.21) to obtain

$$\begin{aligned} |(V)_3| &\leq \sum_{n=1}^{M} k\|\Delta_k u^n - u_t^n\|_{L^2(\Omega)}\|w^n\|_{L^2(\Omega)} + \sum_{n=1}^{M} k\|\Delta_k u^n - u_t^n\|_{L^2(\Omega)}\|e^n\|_{L^2(\Omega)} \\ &\leq C\left(k^2\sum_{n=1}^{M}\|\Delta_k u^n - u_t^n\|^2_{L^2(\Omega)} + \sum_{n=1}^{M}\|w^n\|^2_{H^1(\Omega)}\right) + \frac{1}{4}\sum_{n=1}^{M}\|e^n\|^2_{L^2(\Omega)} \\ &\leq \tilde{C}(u_{tt})k^2 + \tilde{C}(u)h^2 + \frac{1}{4}\sum_{n=1}^{M}\|e^n\|^2_{L^2(\Omega)}, \end{aligned} \tag{6.4.26}$$

where $\tilde{C}(u_{tt}) = C\sum_{n=1}^{M}\|\Delta_k u^n - u_t^n\|^2_{L^2(\Omega)} \leq C\|u_{tt}\|^2_{L^2(0,T;L^2(\Omega))}$.

Finally, for the term $(V)_4$, apply (6.4.21) to obtain

$$\begin{aligned} |(V)_4| &\leq C\sum_{n=1}^{M} k\|w^n\|^2_{H^1(\Omega)} + C\sum_{n=1}^{M} k\|w^n\|_{H^1(\Omega)}\|e^n\|_{H^1(\Omega)} \\ &\leq C\sum_{n=1}^{M} k\|w^n\|^2_{H^1(\Omega)} + \frac{1}{4}\sum_{n=1}^{M} k\|e^n\|^2_{H^1(\Omega)} \\ &\leq \tilde{C}(u)h^2 + \frac{1}{4}\sum_{n=1}^{M} k\|e^n\|^2_{H^1(\Omega)}. \end{aligned} \tag{6.4.27}$$

Altogether these estimates (6.4.23)-(6.4.27) now yield

$$\sum_{n=1}^{M} k\|e^n\|^2_{H^1(\Omega)} \leq \{\tilde{C}(u,u_t) + \tilde{C}(u)\}\{k^2 + h^2\} + \|u_0 - L_h u_0\|^2_{L^2(\Omega)}. \tag{6.4.28}$$

By a simple calculation (cf. [14]) it follows that

$$\|u - U_{hk}\|_{L^2(0,T;H^1(\Omega))} \leq Ck\|u_t\|_{L^2(0,T;Y)} + C\left(\sum_{n=1}^{M} k\|e^n\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}, \tag{6.4.29}$$

and the desired estimate now follows from (6.4.28)-(6.4.29) and using the fact $\|u_0 - L_h u_0\|_{L^2(\Omega)} \leq Ch\|u_0\|_{H^1(\Omega)}$. □

Chapter 7

Numerical Results

In this chapter, we shall present some numerical experiment of one dimensional test problems to illustrate our theoretical findings. All computations have been carried out using the software MATLAB-6.

For each example, we compute the error between the exact solution and the finite element solution in L^2 and H^1 norms. Numerical results for both fitted and unfitted finite element methods are presented in this chapter.

7.1 Example 1

We consider the following two-point boundary value problem.

$$-\frac{d}{dx}\Big(\beta\frac{du}{dx}\Big) = 1 \ \text{ in } (0,1), \tag{7.1.1}$$

$$u(0) = u(1) = 0, \tag{7.1.2}$$

$$[u] = 0, \quad \Big[\beta\frac{du}{dx}\Big] = 0 \quad \text{at} \quad x = \frac{1}{2}. \tag{7.1.3}$$

Here, the domain is the interval $(0,1)$ with the interface at $x = \frac{1}{2}$. The problem (7.1.1)-(7.1.3) has a closed form solution (cf. [26]) given by

$$\begin{aligned} u_1(x) &= \frac{(3\beta_1+\beta_2)x}{4\beta_1^2+4\beta_1\beta_2} - \frac{x^2}{2\beta_1} \\ u_2(x) &= \frac{(\beta_2-\beta_1)+(3\beta_1+\beta_2)x}{4\beta_2^2+4\beta_1\beta_2} - \frac{x^2}{2\beta_2}. \end{aligned}$$

Clearly, (7.1.1)-(7.1.3) is a one dimensional version of the problem (1.1.1)-(1.1.3) with $g(x) = 0$. With $\beta_1 = \frac{1}{2}$ and $\beta_2 = 3$, we perform a numerical convergence test for the proposed finite element methods. Tables 7.1 and 7.2 present the convergence of the fitted and unfitted finite element solutions to the exact solutions, respectively.

h	$\|u-u_h\|_{H^1(\Omega)}$	$\|u-u_h\|_{L^2(\Omega)}$
$\frac{1}{2}$	0.206939	0.032730
$\frac{1}{4}$	0.103469	0.008191
$\frac{1}{6}$	0.068980	0.003624
$\frac{1}{8}$	0.051735	0.002062
$\frac{1}{10}$	0.041389	0.001303

Table 7.1: *Numerical results for the problems (7.1.1)-(7.1.3) using fitted finite element method.*

h	$\|u-u_h\|_{H^1(\Omega)}$	$\|u-u_h\|_{L^2(\Omega)}$
$\frac{1}{3}$	0.134142	0.018618
$\frac{1}{5}$	0.101111	0.010899
$\frac{1}{7}$	0.084059	0.007702
$\frac{1}{9}$	0.073406	0.005967
$\frac{1}{11}$	0.065801	0.004855

Table 7.2: *Numerical results for the problems (7.1.1)-(7.1.3) using unfitted finite element method.*

7.2 Example 2

We consider the following two-point initial boundary value problem

$$u_t - \frac{\partial}{\partial x}\Big(\beta\frac{\partial u}{\partial x}\Big) = f \quad \text{in } (0,1)\times(0,1], \tag{7.2.4}$$

$$u(x,0) = u_0(x), \qquad u(0) = u(1) = 0, \tag{7.2.5}$$

$$[u] = 0, \quad \Big[\beta\frac{\partial u}{\partial x}\Big] = 0 \quad \text{at} \quad x = \frac{1}{2}. \tag{7.2.6}$$

Clearly, (7.2.4)-(7.2.6) is a one dimensional version of a problem of the form (1.1.4)-(1.1.6) with $g(x,t)=0$. Let $v(x)$ be the following function

$$v(x) = \begin{cases} \frac{(3\beta_1+\beta_2)x}{4\beta_1^2+4\beta_1\beta_2} - \frac{x^2}{2\beta_1} =: v_1(x) & \text{for} \quad x \in \Big[0, \frac{1}{2}\Big), \\ \frac{(\beta_2-\beta_1)+(3\beta_1+\beta_2)x}{4\beta_2^2+4\beta_1\beta_2} - \frac{x^2}{2\beta_2} =: v_2(x) & \text{for} \quad x \in \Big(\frac{1}{2}, 1\Big]. \end{cases}$$

Then $u(x,t) = e^{\sin t}v(x)$ is the solution for the interface problem (7.2.4)-(7.2.6) with

$$f(x,t) = \begin{cases} \cos t e^{\sin t} v_1(x) + e^{\sin t} & \text{for } (x,t) \in \Big(0, \frac{1}{2}\Big) \times (0,1], \\ \cos t e^{\sin t} v_2(x) + e^{\sin t} & \text{for } (x,t) \in \Big(\frac{1}{2}, 1\Big) \times (0,1] \end{cases}$$

and $u_0(x) = v(x)$.

We choose $\beta_1 = \frac{1}{2}$, $\beta_2 = 3$ and perform a numerical convergence test for the proposed fitted and unfitted finite element methods. The convergence of the fully discrete solution to the exact solution for five different mesh parameters at time $t = \frac{1}{130}$ with $k = h^2$ are presented. Tables 7.3 and 7.4 show the convergence of fitted and unfitted finite element solutions to the exact solutions, respectively.

h	$\|u - U\|_{H^1(\Omega)}$	$\|u - U\|_{L^2(\Omega)}$
$\frac{1}{2}$	0.215678	0.025799
$\frac{1}{4}$	0.108580	0.004936
$\frac{1}{6}$	0.071227	0.002110
$\frac{1}{8}$	0.052883	0.001125
$\frac{1}{10}$	0.042102	0.000673

Table 7.3: *Numerical results for the fully discrete solution at $t = \frac{1}{130}$ for fitted finite element method.*

h	$\|u - U\|_{H^1(\Omega)}$	$\|u - U\|_{L^2(\Omega)}$
$\frac{1}{3}$	0.129511	0.014135
$\frac{1}{5}$	0.100278	0.006339
$\frac{1}{7}$	0.083586	0.003941
$\frac{1}{9}$	0.072915	0.002804
$\frac{1}{11}$	0.063640	0.002102

Table 7.4: *Numerical results for the fully discrete solution at $t = \frac{1}{130}$ for unfitted finite element method.*

Chapter 8

Conclusion and Extensions

This chapter is devoted to a critical assessment of the results highlighting the contributions made by this thesis and technique used in deriving these. It also provides information for the scope of possible extensions and future investigations.

8.1 Critical Review of the Results

In this thesis, we have studied linear elliptic and parabolic interface problems by means of classical finite element methods. Both fitted and unfitted finite element methods are considered and analyzed.

In Chapter 2, the finite element solution is shown to converge at optimal rate in L^2 and H^1 norms when the global regularity of the solution is low (cf. Theorems 2.3.2-2.3.3). The finite element discretization in this case is such that the grid lines coincide with the actual interface. Further, for the purpose of numerical computations we have also discussed the effect of numerical quadrature on the finite element solution and related optimal order estimates are established (cf. Theorems 2.4.1-2.4.2). Compared to [8], the present method not only assumes low global regularity on the solution but also improves upon the earlier results of Chen and Zou [14]. The main idea of our error analysis is to construct a suitable subspace $S^{\perp}_{h,\Omega_k}$ such that the estimate (2.3.11) holds true. Then, we use Lemma 2.3.1 to prove that the error $\|u - u_h\|_{H^1(\Omega)}$ is bounded by the best approximation error $\|u - v_h^{(s)}\|_{H^1(\Omega_k)}$ in Ω_k. Since the grid line coincide with the actual interface, it turns out that the error $\|u - v_h^{(s)}\|_{H^1(\Omega_k)}$ can be used to obtain the optimal H^1 norm estimate. The main crucial technical tools used in our analysis are some Sobolev embedding inequality, extension theorem and Nitsche's trick.

Since it is costly to generate the mesh whose grid lines coincide with the actual interface of general shape, a modification of the mesh is introduced in Chapter 3.

We consider a finite element method in which the domains Ω_1 and Ω_2 are replaced by polygonal domains Ω_1^h and Ω_2^h, respectively. Then, the interface function g is transferred to the polygonal boundary Γ_h of the domain Ω_1^h via its interpolant g_h. The proposed method yields optimal order convergence in H^1 norm and sub-optimal in L^2 norm for the elliptic interface problems (cf. Theorem 3.3.1). First, we have established some new optimal interpolation approximation property for the linear interpolant under minimal regularity assumption of the true solution (cf. Lemma 3.2.1). An interpolation post-processing technique in conjunction with the duality argument is introduced to obtain sub-optimal error estimates in L^2 norm. The main crucial technical tools used in our analysis are some Sobolev embedding inequality, duality arguments and some previously proved results for interface problems in non-convex domain (cf. [22, 40]).

Chapter 4 is devoted to the continuous time Galerkin approximation for the spatially discrete scheme for parabolic interface problems. We have used fitted finite element discretization to study the convergence of semidiscrete solution to the exact solution. Optimal error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are established if the grid lines coincide with the interface (cf. Theorems 4.2.1-4.2.2). Further, the error in $L^2(H^1)$ norm is shown to be optimal when the finite element discretization is based on straight triangulation (cf. Theorem 4.3.1). The standard L^2 projection played an important role in the treatment of the term $u_t - u_{ht}$ appearing in the error equations (cf. (4.2.23) and (4.3.7)) for the continuous-in-time scheme. Then, we used the convergence results of elliptic interface problem and some newly established optimal approximation results for the elliptic projection (cf. Lemma 4.2.1) to study the convergence in $L^2(H^1)$ norm for the spatially discrete scheme. Finally, the duality trick is used to obtain optimal error estimate in $L^2(L^2)$ norm.

In Chapter 5, we have discussed time discretization scheme based on backward Euler type discontinuous Galerkin method of semidiscrete equations (4.2.21) and (4.3.5). In this chapter, we recalled the optimal approximation properties for the elliptic projection derived in Chapter 4. These estimates and parabolic duality technique are used to prove optimal error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms when the curved triangles follow exactly the actual interface (cf. Theorems 5.2.1-5.2.2). Further, we have shown that the fully discrete solution converges to the true solution at an optimal rate in $L^2(H^1)$ norm if we used straight triangles instead curved interface triangles (cf. Theorem 5.3.1). The results presented in this chapter improves upon the earlier results of Chen and Zou [14].

Chapter 6 is concerned with the *a priori* error analysis of an unfitted finite element method for both linear elliptic and time dependent parabolic interface problems. Here, the finite element discretization is independent of the location of the interface. The proposed method yields optimal order error estimate in H^1 norm and almost optimal in L^2 norm for elliptic interface problems (cf. Theorems 6.3.1-6.3.2). For the time dependent parabolic interface problems, we derive optimal order error estimate in $L^2(H^1)$ norm and almost optimal order error estimate in $L^2(L^2)$ norm for the spatially discrete scheme (cf. Theorems 6.4.2-6.4.3). While the standard L^2 projection and some newly established approximation properties for the linear interpolant plays a crucial role to derive error estimate in $L^2(H^1)$ norm, the parabolic duality technique is used to obtain the almost optimal error estimate in $L^2(L^2)$ norm. Finally, the discrete in time scheme based on backward Euler method is analyzed. The standard energy technique is used to derive optimal error estimate in $L^2(H^1)$ norm for the fully discrete case (cf. Theorem 6.4.4).

In Chapter 7, we have carried out numerical experiment for one dimensional test problems. We have applied both fitted and unfitted finite element methods to check the performance of our algorithms. As our main objective is to study the theoretical aspects of finite element Galerkin methods for elliptic and parabolic problems with discontinuous coefficients, we have performed numerical experiments only for one dimensional test problems for the completeness of this work.

8.2 Extensions and Remarks

In this section, we make some informal observations pertaining to the possible extensions of our results to different problems. We shall briefly outline some interesting problems to be taken up in future.

Numerical Quadrature for Linear Parabolic Equations with Discontinuous Coefficients: The algorithm discussed in Chapter 4 and Chapter 5 require exact calculation of integrals involved in the finite element methods. This may causes some technical difficulties in practice for the evaluation of the integrals over those curved elements near the interface. It would make the numerical implementation much easier if we can replace these integrals over the straight elements by some quadrature schemes. Numerical quadrature for parabolic problems without interface have been studied by several authors, see [13, 48] and references therein. It will be useful to explore the effect of quadrature for both spatially discrete and fully discrete schemes. We wish to take up this issue in future.

Crank-Nicolson Scheme. A fully discrete scheme based on backward Euler method with uniform step sizes for the parabolic interface problems have been studied in this work.

We have seen that optimal order error estimates for the fully discrete solution of parabolic interface problem can be generated by the backward Euler time stepping. More precisely, error estimate of order $O(k + h^{2-m}),\ m = 0, 1$ are derived in $L^2(H^m)$ norms for the fully discrete scheme based on backward Euler time discretization. Note that because of the non-symmetric choice of the discretization in time, the backward Euler method is only first order in k. However, second order accurate approximation in k can be derived if the semidiscrete equation is discretized in a symmetric fashion around the point $t_{n-\frac{1}{2}} = (n - \frac{1}{2})k$ in the absence of interface condition (cf. [29, 58]). In literature, such scheme is known as Crank-Nicolson scheme.

For the parabolic interface problem Crank-Nicolson scheme may be read as: Find $U^n \in V_h$, for $n = 1, 2, ..., M$, such that

$$(\Delta_k U^n, v_h) + A\left(\frac{U^n + U^{n-1}}{2}, v_h\right) = (f(t_{n-\frac{1}{2}}), v_h) + \langle g(t_{n-\frac{1}{2}}), v_h\rangle_\Gamma \quad \forall v_h \in V_h, \quad (8.2.1)$$

with $U^0 = u_{0,h}$, where $u_{0,h}$ is an suitable approximation of u_0 and $\Delta_k U^n = \frac{U^n - U^{n-1}}{k}$. In future, we shall make an effort to see whether the standard analysis of Thomée [58] with some modification can be applied to study the convergence of fully discrete solution given by (8.2.1) to the exact solution of parabolic interface problem. The convergence analysis for Crank-Nicolson scheme under minimal regularity assumption of the true solution would be a very interesting extension of this work.

Nonlinear Elliptic Interface Problems: Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We consider the following elliptic interface problems of the form

$$\mathcal{L}u = f(x) \quad \text{in } \Omega \quad (8.2.2)$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega, \quad (8.2.3)$$

where the operator $\mathcal{L}$ is a second order elliptic partial differential operator of the form

$$\mathcal{L}u = \nabla(a(x, u(x), \nabla u(x))) + a_0(x, u(x), \nabla u(x)).$$

The functions $a, a_0 : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are such that the operator $\mathcal{L}$ is strongly monotone and Lipschitz continuous. We assume that a is of the form

$$a(x, \xi) = a^k(x, \xi) \quad x \in \Omega_k,\ \xi \in \mathbb{R}^2,\ k = 1, 2,$$

where $a^k : \Omega_k \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, Ω_1 is a subset of Ω with sufficiently smooth boundary and $\Omega_2 = \Omega \backslash (\Omega_1 \cup \Gamma)$.

In this case, equation (8.2.2) is satisfied only for $x \in \Omega_k,\ k = 1, 2$, and on the interface Γ the following transition conditions are prescribed

$$u_1 = u_2, \tag{8.2.4}$$

$$a^1(x, u_1(x), \nabla u_1(x))\mathbf{n}_1(x) - a^2(x, u_2(x), \nabla u_2(x))\mathbf{n}_2(x) = g(x), \tag{8.2.5}$$

where $u_k(x) = u(x)|_{\Omega_k},\ k = 1, 2$ and $(\mathbf{n}_1, \mathbf{n}_2)$ is the outer normal to interface Γ.

Although a good number of article is devoted to the finite element approximation of linear elliptic interface problems, the literature seems to lack on the convergence for nonlinear elliptic interface problems. It will be interesting and challenging to extend the results for linear elliptic interface problems to the nonlinear problems.

Parabolic-Hyperbolic Interface Problems: Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Further, let $\Omega_1 \subset \Omega$ be an open domain with C^2 smooth boundary Γ and $\Omega_2 = \Omega \backslash \Omega_1$. We now consider the following problem

$$\sigma u_t = \Delta u \quad \text{in } \Omega_1; \quad \beta u_{tt} = \Delta u \quad \text{in } \Omega_2,$$

with initial conditions

$$u(x, 0) = u_0(x) \ \text{ in } \Omega; \ \ u_t(x, 0) = v_0(x) \quad \text{in } \Omega_2$$

and interface conditions

$$[u] = 0, \quad \left[\frac{\partial u}{\partial \mathbf{n}}\right] = g(x, t) \quad \text{along } \Gamma.$$

The symbol $[v]$ is a jump of a quantity v across the interface Γ, i.e., $[v](x) = v_1(x) - v_2(x),\ x \in \Gamma$, where $v_i(x) = v(x)|_{\Omega_i},\ i = 1, 2$ and $\mathbf{n}$ denotes the unit outward normal to the boundary $\partial\Omega_1$. The quantities σ and β are positive constants.

Such problem arises in electromagnetism when we study the production of eddy currents in a metallic cylinder due to a particular type of external electromagnetic field. This problem leads to a parabolic equation inside the body and a hyperbolic equation outside. The two equations interact through jump relations across the interface Γ. The physical problem and the derivation of the equations are given in [41]. The existence, uniqueness and regularity results for the above problem is contained in [2]. In future, we would like to study the convergence analysis of this problem by means of finite element methods.

Moving Interface Problems: In this work, we have assumed that the interface does not change its topology during the computation. Currently, the most difficult problems in computational science involve moving interfaces between flowing or deforming media.

Biological fluid dynamics is a rich source of problems with complex geometry and frequently the interaction of fluids with moving elastic structures, e.g. the study of blood flow in flexible tubes, cell dynamics, phase change problems etc. Different approaches are being studied in the context of many different applications areas. Most methods involve some transformations either for the differential equations or the coordinate system, which complicates the problem in some way. Various approaches have been used to solve moving interface problems numerically using the level set approach. For level set methods, one represent an interface as a level set of some function for which an evolution equation must then be derived and solved. Complex geometries and topological changes can often be easily handled with this approach. Interface fitting is an another approach in which the computational grid deforms in order to follow the motion of an interface. No significant development has been made in the direction of finite element methods. We believe that unfitted mesh would be a natural candidate to tackle this type of problems. At present we do not know how to generalize this work for the moving interface problem. But, we would like to take up this issue in future.

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