On Motion Planning in Graphs

A thesis submitted
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by
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Declaration

I do hereby declare that the work contained in this thesis entitled "**On Motion Planning in Graphs**" has done by me, under the supervision of Dr. Kalpesh Kapoor, Associate Professor, Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy and this work has not been submitted elsewhere for a degree.

December 26, 2012

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Certificate

It is certified that the work contained in this thesis entitled "**On Motion Planning in Graphs**" by Mr. Biswajit Deb, a student of Department of Mathematics, Indian Institute of Technology Guwahati, for the award of the degree of Doctor of Philosophy has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

December 26, 2012

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Dedicated

To

My Family

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December 26, 2012 **Biswajit Deb**

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Abstract

Consider an undirected graph G in which a robot is placed at a vertex, say u, and obstacles are placed at all other vertices except at vertex v. The vertex without a robot or an obstacle is said to have a hole. We refer to this placement of robot and obstacles as a configuration C_v^u of G. We say that C_v^u is reachable from C_u^v by an mRJ move of the robot provided there is a u-v path $[u = u_0, u_1, u_2, \cdots u_m, u_{m+1} = v]$ of length $m + 1$ in G. For $m = 0$, an mRJ move is also known as a simple move of the robot. In this thesis, we obtain necessary conditions for trees in which any two configurations are reachable from each other by using a sequence of mRJ moves of the robot for some fixed positive integer m and simple moves of the robot as well as obstacles. We call such a tree as complete mRJ-reachable. We characterize complete mRJ-reachable trees for $m = 0, 1, 2, 3$. We introduce the concept of complete S-reachable graphs, where S is a finite set of non-negative integers. By a complete S-reachable graph we mean a graph in which any two configurations are reachable from each other by using a sequence of mRJ moves of the robot for $m \in S$ and simple moves of the obstacles. We give necessary conditions for a graph to be complete S-reachable. We also characterize the cycles that are complete $\{m\}$-reachable. In addition we identify the graphs that are complete $\{1, 2\}$-reachable. Lastly, we give expression for minimum number of simple moves to take the robot from an initial position to any other vertex in various classes of product graphs.

Contents

List of Symbols

Important symbols are listed in the following table. Within the table G and v denotes a graph and a vertex in G, respectively.

$V(G)$	:	vertex set of G
$E(G)$	:	edge set of G
C_v^u	:	a configuration with the robot at u and the hole at v.
$D(G)$	:	the diameter of G
$\delta(G)$	:	minimum degree of G
$\Delta(G)$	:	maximum degree of G
$N_G(v)$ or $N(v)$	:	neighborhood of v
$\overline{N(v)}$	:	the set $N(v) \cup \{v\}$
$G - v$	:	the induced subgraph of G with vertex set $V(G) - \{v\}$
P_n	:	the path of order n
C_n	:	the cycle of order n
K_n	:	the complete graph on n vertices
$v \overset{r}{\leftarrow} u$	:	simple robot move from u to v
$v \overset{o}{\leftarrow} u$	:	simple obstacle move from u to v
$v \underset{m}{\overset{r}{\leftarrow}} u$	:	mRJ-move of the robot from u to v
$\lvert A \rvert$	:	cardinality of the set A
$\mathbb{W}$	:	the set of whole numbers
$\mathbb{Z}$	:	the set of integers
$\mathbb{N}$	:	the set of natural numbers

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Chapter 1

Introduction

Let G be a simple connected graph. Further, let $V(G)$ and $E(G)$ be the set of vertices and edges of G, respectively. Fix two distinct vertices r and h. Now imagine placing a robot at r, a hole at h, and obstacles at the remaining vertices. We call the resulting structure a *configuration* C_h^r of G.

Consider a configuration C_v^u of G. Let $m \geq 0$ be an integer. By an mRJ *move* on C_v^u we understand the process of exchanging the places of the robot and the hole (by jumping over m obstacles), if there is a u-v path of length $m+1$ in G. In addition, in a $0RJ$ move, we allow the exchange of the places between the hole and any adjacent obstacle. We refer to a $0RJ$ move as a simple move.

In this thesis, we address the following issues.

1. Given a graph G with an initial configuration C_v^u. We investigate the problem of moving a robot from its initial position u to all the other vertices using mRJ moves (for some fixed m) in addition to simple moves of the robot as well as obstacles.

2. Given a graph G with an initial configuration C_v^u and a set of positive integers S. We investigate the problem of moving a robot from its initial position to all the other vertices using simple moves of the obstacles and mRJ moves of the robot for $m \in S$.

3. Given an initial configuration in a product of two graphs. We compute the minimum number of moves required to take the robot from its ini-

tial position to any other vertex using simple moves of the robot as well as obstacles. We give the minimum number of moves required for the motion planning problem in Cartesian Product and Lexicographic Product of graphs. Also, we give bounds for the minimum number of moves required for the motion planning problem in Strong Product.

1.1 Scope and Basic Terminology

Throughout all graphs are finite simple connected unless otherwise stated. We use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges in a graph G. We refer [15] for the standard terms used in this thesis. We call $|V(G)|$ and $|E(G)|$ respectively the *order* and the *degree* of the graph G. The *diameter, minimum degree* and *maximum degree* of a graph G are denoted by $D(G), \delta(G)$, and $\Delta(G)$, respectively.

We write $\{u, v\} \in E(G)$ to mean the existence of an edge between the vertices u and v in the graph G. By a *u-v path* in a graph G we mean a finite sequence of distinct vertices $[u = u_1, u_2, \ldots u_k = v]$ starting and ending with u and v respectively, such that for each $i \in \{1, 2, \ldots k-1\}$ the vertices u_i and u_{i+1} are adjacent in G. A closed path is known as a cycle. We denote the *path, cycle* and *complete* graph of order n by P_n, C_n and K_n respectively. A path P in a graph G is said to be *critical* if each intermediate vertex in P is a cut vertex of degree two in G. By a *tail* we mean critical path with one of its end points is of degree one. The *girth* of a graph G, denoted by $g(G)$, is the length of a shortest cycle contained in the graph G. By the *neighborhood* of a vertex $u \in V(G)$, we mean the collection of all vertices those are adjacent to u in G and it is denoted by $N_G(u)$ or simply by $N(u)$. Also we use $\overline{N(u)}$ to denote the set $N(u) \cup \{u\}$. We sometimes write $v \in G$ in stead of $v \in V(G)$ to mean that v is a vertex in the graph G. For any vertex $v \in G$, we use $G - v$ to denote the induced subgraph of G with vertex set $V(G) - \{v\}$.

1.2 Motivation

We draw motivation for the problems stated at beginning of this chapter from the Robot Motion Planning Problem on Graphs. The problem of robot motion planning is one of the central topics of research in robotics and it has been studied by researchers using a variety of techniques [6,21,23,28,30]. Imagine a robot placed in a room that has variety of furniture. Assume that the robot can move from point to point in any direction along straight lines. The objective is to design an algorithm which can determine how the robot should move to reach a target position from its initial position without hitting any object. This is also referred to as the *Piano Movers Problem*. The feasibility of this problem is also studied in the context of directed graphs [35,36].

The work described in this thesis is concerned with the Robot Motion Planning on Graphs, which is a simple abstraction of the robot motion planning problem. The aim is to deal with this problem from two different perspective as stated below.

(I) To study the problem from combinatorial perspective by introducing more generalized robot moves, called mRJ moves.

(II) To give solution of this problem for the different classes of product graphs.

Our problem belongs to a much larger class of problems called 'Motion Planning on Graphs'. Here the term planning stands for a branch of algorithms. The problem of motion planning on graphs can be divided into the following two categories:

Robot Motion Planning on Graphs (RMPG): Given a graph G, with a robot placed at one of its vertices and movable obstacles at some of the other vertices. A vertex at which the robot or an obstacle is not placed is said to have a hole or equivalently empty. Assume that we are allowed to slide the robot and the obstacle to an adjacent empty vertex. The objective is to investigate the feasibility of taking the robot to every vertex in G. Also, if feasible, to find the minimum number of moves required to take the robot from a given (source) vertex to another (destination) vertex. Other versions of this problem like multi robot motion planning on graphs **(MRMPG)** and robot motion planning on digraphs **(RMPDG)** are also well studied.

Reconfiguration on Graphs(CONFIG): Given a connected graph G. Let V_1 and V_2 be two subsets of $V(G)$ of equal size and not necessarily disjoint. Suppose that distinct robots are placed at each vertex of V_1 and a vertex where no robot is placed is said to be empty. The objective is to move each of the robots into a specified vertex of V_2 as moving a robot from x to y ($x, y \in V(G)$) on a path formed by edges of G so that all intermediate vertices are empty.

The well studied $(n^2 - 1)$-puzzle falls into the category of CONFIG problem. Wilson [34] introduced this notion with $n - 1$ robots on a graph of order n. Miller et al., [20] studied a more general version of CONFIG problem mentioned above.

The study of RMPG was proposed by Papadimitriou et al. [24] to strip away the geometrical considerations involved in the actual robot motion problem. It has numerous practical applications beyond robotics, such as in animation, games, manufacturing [22] as well as in computational biology [30, 32, 33]. Frederickson and Guan [11] also studied a problem of motion planning on trees which is quite different from the problem introduced by Papadimitriou. Reachability problem has many practical applications. We motivate the subject by presenting the following applications in diverse domains. For a general overview of the subject and its application, we refer the reader to [21, 22].

Track Transportation Systems: This problem is related to movement of vehicles on a system of tracks. Each track connects two distinct stations. Now there is a particular vehicle s, that has to go to the station t. Among all the stations in path some are occupied by other vehicles (obstacles). Vehicles can stop at the stations only, and not in the middle of the track. Some of the tracks will allow one way traffic to go whereas the rest are going to allow both way traffic. The objective is to prepare a plan of coordinated movement of the vehicles, that allows the special vehicle move from the source station to the destination station. Also, there do arise the question that if for a particular configuration, the problem is infeasible, what minimum change should be done to make it feasible. For more details about this we can refer [27].

Communication Network: Reachability problem also find it's use in packet transfer in communication network. We regard the graph G, as communication network and objects as indivisible packet of data. A distinguished packet has to reach the destination, as there are already some packets stored in communication buffer of nodes, in the network. The objective is to move the distinguished packet from source node s to the destination node t in the network. We refer [26, 31] for details.

Game on Graphs: Suppose there are k robots (labeled or unlabeled) placed on k different vertices on a graph G of order n with $k \leq n-1$. Suppose that we are allowed to move a robot either to an adjacent empty vertex or over m robots along a path to an empty vertex. Find a minimum sequence of moves that takes the robot from the initial configuration to a final configuration. Similar games on graphs are also discussed in [7].

1.3 Literature Survey

Researchers have studied different kinds of movement problems on graphs. One of them is the graph pebbling. In graph pebbling, given m pebbles are distributed onto the vertices of a graph G, a pebbling step consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. We say a pebble can be moved to a vertex v, if we can repeatedly apply pebbling steps so that in the resulting distribution v has one pebble. The problem is to find the minimum value of m such that however the m pebbles are distributed onto the vertices of G, one pebble can be moved to any specified vertex. The smallest number of pebbles satisfying this property is known as the *pebbling number* of G. It was first suggested by Lagarias and Saks as a tool for solving a number-theoretical conjecture of Erdös. Chung successfully used this tool to prove the conjecture and established other results concerning pebbling numbers [5]. Graph pebbling has many application in variety of subjects, namely number theory, graph theory, probability, and extremal set theory. We refer an initial survey [18] and a recent update [17] for an overview of the subject.

The goal of the Robot Motion Planning on Graphs (RMPG) is to concentrate on the combinatorial aspects and ignoring the geometric constraints of the general robot motion planning problem. More specifically, a motion plan involves determining what moves are appropriate for the robot so that it reaches the target without colliding into obstacles. It was proposed by Papadimitriou et al. [24] and they have shown that the decision version of this problem is NP-complete. Also the problem remains in the same complexity class even when restricted to planar graphs. Also, a polynomial time algorithm has been given in [24] for the case in which the graph is a tree. Later, the results of [24] were improved by Auletta and Persiano in [3].

A straight forward generalization of the problem of RMPG is the problem where we have k different robots with respective destinations, and it is also known as multi robot motion planning problem (MRMPG). Goldreich [12] introduced a restricted version of the MRMPG problem, the SMS problem, in which $k = n - 1$ and gave an elegant polynomial time reduction to show that the SMS problem is NP-hard. Ellips and Azadeh [23] studied the multi robot motion planning problem on trees and introduced the concept of minimal solvable trees. Auletta et. al., [2] also studied the feasibility of the MRMPG problem on trees and gave an algorithm that, on input of two arrangements of k robots on a tree of order n, decides in time $O(n)$ whether the two arrangements are reachable from one another.

We also investigate the feasibility of the RMPG problem for trees with respect to the newly introduced mRJ moves of the robot and call them complete mRJ-reachable trees. In this regard we characterize the class of trees that are complete $2RJ$ and $3RJ$-reachable (see Chapter 2).

If we restrict ourself to the grid graphs of order n^2 with $n^2 - 1$ robots then the $MRMPG$ problem reduces to the $(n^2 - 1)$-puzzle. The $(n^2 - 1)$-puzzle have been studied extensively in the last three decades [20, 25, 28, 34]. The objective of the $(n^2 - 1)$-puzzle is to verify whether two given configurations of the grid graph of order n^2 are reachable from each other and if they are reachable then to provide a sequence of minimum number of moves that takes one configuration to the other. A move consists of sliding a robot into a adjacent empty vertex. Ratner and Warmuth [28] have proved a decision version of the (n^2-1)-puzzle to be NP-complete, and they gave an approximate algorithm that makes no more than a (fairly large) constant factor number of moves than necessary. Miller et al. [20] presented an algorithm for the (n^2-1)-puzzle and its generalizations (multiple holes) that always runs in time $O(n^3)$. In [25] Parberry gave a real-time algorithm for this problem using greedy and divide-and-conquer techniques that uses at most $5n^3$ moves. Here we would like to mention that the grid graph of order n^2 can also viewed as the Cartesian product graph $P_n \Box P_n$. We will discuss about the product graphs in Chapter 4 in detail. The techniques used by Parberry [25] for the graph $P_n \Box P_n$ have been generalized and applied them to calculate the minimum number of moves for the motion planning problem for the Cartesian product of two given graphs.

The (n^2-1)-puzzle is also a natural extension of the 8-puzzle and the 15-puzzle [1, 19]. The existence of the solution to the 15-puzzle depends on the parity of the permutation that map the intial configuration to the final configuration and it was observed as early as 1879 [19]. Horden [16] gives an interesting history of the 15-square puzzle and its variants.

The (n^2-1)-puzzle is a special case the of the CONFIG problem as well. Călinescu et al., discussed different versions of the CONFIG in [6] for finite as well as infinite grid graphs. This includes (i) labeled version of the CONFIG problem in which all robots are distinct and (ii) unlabeled version of the CONFIG problem in which all robots are alike. The area of application of the CONFIG problem includes the modular robotic system [4, 8, 9]. A modular robotic system is a collection of robotic modules, each of which can connect to, disconnect from, and relocate relative to adjacent modules. Thus, such a robotic system has the ability to dynamically self-reconfigure. The motion-planning problem for a modular robotic system is that of computing a sequence of module motions that brings the system in a given initial configuration into a desired goal configuration.

1.4 Organization of the Thesis

There are five chapters in the thesis. **Chapter 1** contains a brief introduction of the thesis, literature survey, and motivation.

Chapter 2 We characterize the trees that are complete mRJ-reachable for $m = 0, 1, 2, 3$. Also give bound for the diameter of mRJ-reachable trees.

Chapter 3 We introduce the notion of complete S-reachability. We discuss necessary conditions for a graph to be complete $\{m\}$-reachable. In addition we characterize the cycles that are complete $\{m\}$-reachable. Also, we characterize the complete $\{1, 2\}$-reachable graphs.

Chapter 4 We study product graphs and give minimum number of moves for the problem of RMPG with a single hole in Cartesian product and Lexicographic Product of graphs. Also, we give bounds for the same in Strong Product.

Chapter 5 We give concluding remark about the contents of the thesis. We also state some problem that are on the card for future.

Chapter 2

mRJ-Reachability in Trees

In this chapter our focus will be only on trees. In the section 2.1 we introduce the mRJ-reachability problem on graphs and discuss some basic results to make the base for further discussion. In section 2.2 we introduce the notion of mRJ-reachable trees and complete mRJ-reachable trees. This is a generalization of the reachability problem introduced in [24]. We then provide bounds on the diameter of mRJ-reachable trees and complete mRJ-reachable trees. Finally, we provide a characterization of the complete $2RJ$-reachable trees and the complete $3RJ$-reachable trees in the sections 2.3 and 2.4, respectively.

2.1 Background

Let G be a graph with n vertices. A k-*configuration* of G means that a robot is placed at one of the vertices and there are k empty vertices, also referred to as *holes*. The remaining $n-(k+1)$ vertices have obstacles.

A simple move is defined as moving an obstacle or the robot to an adjacent empty vertex. A graph G is said to be k-*reachable* if there exists a k-configuration such that the robot can reach any vertex of the graph in a finite number of simple moves.

In addition to a simple move, we define another kind of move in which a robot is allowed to jump over obstacles. Let u be v be two vertices having a robot and a hole, respectively. Further, let $[u, u_1, u_2, \ldots, u_m, v]$ be a path having obstacles at the vertices $u_1 \ldots u_m$. An mRJ **move** from the vertex u to the empty vertex v is defined as movement of the robot to empty vertex v

by jumping over m obstacles $u_1, u_2, \ldots, u_m$. After a simple or an mRJ move the source vertex becomes empty.

Our study is restricted to one empty vertex and having obstacles in the remaining vertices unless otherwise stated. We refer to 1-configuration and 1-reachable as simply configuration and reachable, respectively. A graph is said to be *mRJ-reachable* if there exists a configuration from which the robot can be moved to any vertex using simple or mRJ moves. A graph, G, is said to be *complete mRJ-reachable* if it is mRJ-reachable from every configuration of G. Also, two configurations in G are said to be mRJ-reachable from each other if one can be reached from the other by using finite number of mRJ or simple moves. Note that the graph K_2 is complete mRJ-reachable for any $m \geq 0$. So all the trees that we consider now onwards are different from K_2 unless otherwise stated.

The notation C_v^u is used to denote the configuration of a graph with the robot and the hole at u and v, respectively. The other remaining vertices will have obstacles. Moving an obstacle from a vertex x to an empty vertex y is denoted by $x \xrightarrow{o} y$ or $y \xleftarrow{o} x$. Similarly, moving the robot from a vertex x' to an adjacent empty vertex y' is denoted by $x' \xrightarrow{r} y'$ or $y' \xleftarrow{r} x'$. Let $[u, u_1, u_2, \ldots, u_m, v]$ be a path in a graph such that u and v have a robot and a hole, respectively, and $u_1, \ldots, u_m$ have obstacles. An mRJ move from vertex u to v is denoted by $u \xrightarrow[m]{r} v$ or $v \xleftarrow[m]{r} u$.

Example 2.1. Consider the tree, T^*, shown in Figure 2.1. The tree T^* is $2RJ$-reachable from the configuration C_2^1. For instance, we can take the robot to the vertex 4 using the following sequence of moves $2 \xleftarrow{o} 3 \xleftarrow{o} 4 \xleftarrow[2]{r} 1$. But the tree T^* is not $2RJ$-reachable from the configuration C_3^2. Hence T^* is $2RJ$-reachable but not complete $2RJ$-reachable. ♯

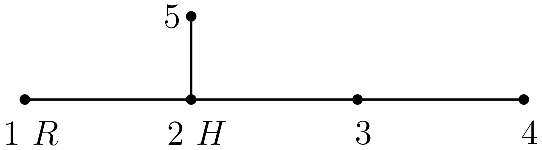

Figure 2.1: The tree T^* with the configuration C_2^1.

The result in Proposition 2.1 was first noticed by Papadimitriou et al. in [24]. We supply here a formal proof of this result.

Proposition 2.1. Let G be a bi-connected graph. Then G is complete reachable.

Proof. Let G be a bi-connected graph and u, v be two adjacent vertices in $V(G)$. Consider the configuration C_v^u of G. Let w be any other vertex in G. We claim that the robot can be taken to the vertex w. Let $[u, x_1, x_2, \ldots, x_k, w]$ be a path connecting u and w in G. Also let $x_1, z_1, z_2, \ldots, z_t, v$ be a path from x_1 to v. The robot can be moved from u to x_1 using the following sequence of moves:

$$v \overset{o}{\leftarrow} z_t \overset{o}{\leftarrow} z_{t-1} \cdots z_1 \overset{o}{\leftarrow} x_1 \overset{r}{\leftarrow} u$$

The hole will be at the vertex u after these moves. Now assume that we have taken the robot to vertex x_i in the previous move and so the hole is at vertex x_{i-1}. Let $[x_{i-1}, y_1, y_2, \ldots, y_l, x_{i+1}]$ be a path in G such that $x_i \neq y_j$, for $j \in \{1, \ldots, l\}$. Such a path always exists because G is bi-connected. Then, the following sequence of moves take the robot from vertex x_i to vertex x_{i+1}

$$x_{i-1} \overset{o}{\leftarrow} y_1 \overset{o}{\leftarrow} y_2 \cdots y_l \overset{o}{\leftarrow} x_{i+1} \overset{r}{\leftarrow} x_i$$

Thus using such sequence of moves the robot can be taken from vertex u to any vertex w. ☐

The following theorem provides an alternative way to describe complete mRJ-reachable trees.

Theorem 2.1. A connected graph G is complete mRJ-reachable if and only if every two distinct configurations in G are mRJ-reachable from each other.

Proof. ($\Rightarrow$) Pick a vertex w in G that is not a cut vertex. Notice that for each $x, y \in V(G - w)$ the configurations C_x^w and C_y^w are mRJ-reachable from each other. Now given two configurations C_v^u and $C_{v'}^{u'}$, there exists a pair of vertices $x, y \in V(G - w)$ such that C_v^u and C_x^w are mRJ-reachable from each other and $C_{v'}^{u'}$ and C_y^w are mRJ-reachable from each other. Hence C_v^u and $C_{v'}^{u'}$ are mRJ-reachable from each other.

(⇐) Assume that every pair of configurations in G are reachable from each other. Let C_v^u be a configuration of G and w be a vertex in $V(G)$. Then, we can achieve the configuration C_v^w from C_v^u. That is, the robot can be taken to the vertex w. Since w and C_v^u are chosen arbitrarily, so G is reachable from C_v^u and we can conclude that G is complete mRJ-reachable. □

Let G be a graph having a path $[u_1, u_2, \ldots, u_m]$ which does not belong to a cycle and the vertices u_i are of degree two, for $1 < i < m$. In addition, the end points u_1 and u_m have degree more than two. In [24], it is observed that such a graph, G, cannot be k-reachable for $k \in \{1, \ldots m\}$. Thus $k > m$ holes are necessary but may not be sufficient for G to be k-reachable. When mRJ moves are also allowed in addition to simple moves, it would be interesting to know whether one hole is sufficient for graphs other than bi-connected graphs.

It is easy to see that the configuration reachability in a graph is an equivalence relation and the following lemma is straight forward from this fact. But we mention it here as it will be useful to prove certain result later in this section. It gives a condition on the length of a path so that a robot can be moved from one end to the other end of the path using mRJ and simple moves.

Lemma 2.1. Given a positive integer m, the set of configurations $\mathcal{K}$ on any path P of length $m+1$ or more can partitioned into m parts $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_m$ such that any two configurations in $\mathcal{K}$ are mRJ-reachable if and only if they belong to the same $\mathcal{K}_i$ for some i.

Proof. Let $[v_0, v_1, \ldots, v_k]$ be a path of length at least m. First we note that for any $i \in \{0, 1, 2, \ldots, m-1\}$, starting from the configuration $C_{v_{i+1}}^{v_i}$ we can only reach the configurations $C_{v_k}^{v_j}$, where

$$\begin{aligned} j &\equiv i \mod m,\ k > j \text{ or} \\ j-1 &\equiv i \mod m,\ k < j. \end{aligned}$$

For $i \in \{0, 1, 2, \ldots, m-1\}$, let $\mathcal{K}_{i+1}$ be the collection of all configurations that are mRJ-reachable from the configuration $C_{v_{i+1}}^{v_i}$. Then $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_m$ forms the required partition of $\mathcal{K}$. □

Remark 2.1. Consider the path $P = [v_0, v_1, \ldots, v_k]$. Notice that, starting with configuration $C^{v_i}_{v_{i+1}}$, we can reach each of the configurations $C^{v_i}_{v_k}$ for $k > i$ and $C^{v_{i+1}}_{v_k}$ for $k \leq i$ by using simple moves only. In view of this, now onwards by saying that starting from the configuration C^u_v we can only reach the configuration $C^{v_i}_{v_{i+1}}$ we will mean that each of the configurations $C^{v_i}_{v_k}$ for $k > i$ and $C^{v_{i+1}}_{v_k}$ for $k \leq i$ are reachable from C^u_v.

Corollary 2.1. Paths of length 3 or more are not complete mRJ-reachable for $m \geq 2$.

Proof. Proof is immediate from Lemma 2.1 and Theorem 2.1. □

Note that a simple move $u \xrightarrow{r} v$ takes the configuration C^u_v to C^v_u in the path $P_1 = uv$ of length one. Thus, P_1 is complete mRJ-reachable for each m. It is easy to see that any two configuration in the path $P_2 = uvw$ of length 2 are $1RJ$-reachable from each other. For example, we can take the configuration C^u_v to the configuration C^v_w by using the sequence of moves $v \xleftarrow{o} w \xleftarrow[1]{r} u \xleftarrow{o} v \xleftarrow{r} w$. Therefore, we can conclude that P_2 is complete $1RJ$-reachable. We make the first important observation about mRJ-reachability in the following theorem.

Theorem 2.2. Let G be a connected graph. Then, G is complete $1RJ$-reachable.

Proof. Let C^u_v and C^x_y be any two configurations in a connected graph G. Without loss of generality, assume that $\{u, v\}, \{x, y\} \in E(G)$ and v, y lies on a path P connecting u to x. Let $\mathcal{K}$ be the collection of all possible configurations in P. Then by Lemma 2.1 every pair of configuration in $\mathcal{K}$ are $1RJ$-reachable from each other. In other words, C^u_v and C^x_y are $1RJ$-reachable from each other. Since the configurations C^u_v and C^x_y were chosen arbitrarily, we can conclude that G is complete $1RJ$-reachable. □

Lemma 2.2. Let T be a complete mRJ-reachable tree and v be any vertex in T. If T' is the tree obtained from T by deleting all but one pendant vertex incident with v then T' is also complete mRJ-reachable.

Proof. Without loss of generality we may assume that $m \geq 2$. Let $v_1, v_2, \ldots, v_k$ be the pendent vertices incident with v and T' be the tree obtained by deleting the vertices $v_2, v_3, \ldots, v_k$ from T. We need to show that T' is also complete mRJ-reachable. Let C_w^u and $C_{w'}^{u'}$ be any two configurations in T'. Since T is complete mRJ-reachable so there exist a sequence consisting of simple and mRJ-moves in T which takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. Let $\mathcal{S}$ be a sequence having minimum number of moves among all such sequences that takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. First we note the following:

(i) Since distance between any two pendent vertices incident with v is two, so no move in the sequence $\mathcal{S}$ can involve two pendent vertices incident with v.

(ii) No simple move in $\mathcal{S}$ can involve v and a vertex from the set of vertices $\{v_2, v_3, \cdots v_k\}$ because $u, w, u', w' \notin \{v_2, v_3, \cdots v_k\}$. Thus the only moves in $\mathcal{S}$ that may contain a vertex from the set $\{v_2, v_3, \cdots v_k\}$ must be of the type $v_i \overset{r}{\underset{m}{\longleftarrow}} x$ or $x \overset{r}{\underset{m}{\longleftarrow}} v_i$ for some $i = 2, 3, \cdots k$ and $x \notin \{v_1, v_2, \cdots v_k\}$ so in $\mathcal{S}$.

(iii) Since $u, w, u', w' \notin \{v_2, v_3, \cdots v_k\}$ so in $\mathcal{S}$ each mRJ move of the form $v_i \overset{r}{\underset{m}{\longleftarrow}} x$ for some $i = 2, 3, \cdots k$, must be followed by a move of the form $y \overset{r}{\underset{m}{\longleftarrow}} v_i$. Now if we replace a pair of moves of the form $v_i \overset{r}{\underset{m}{\longleftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{m}{\longleftarrow}} v_i$ in $\mathcal{S}$ by the pair of moves $v_1 \overset{r}{\underset{m}{\longleftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{m}{\longleftarrow}} v_1$ the modified $\mathcal{S}$ still take the configuration C_w^u to the configuration $C_{w'}^{u'}$.

Let $\mathcal{S}'$ be the sequence of move obtained from $\mathcal{S}$ by replacing each appearance of a vertex from the set $\{v_2, v_3, \cdots v_k\}$ by v_1. Then the above observations implies that the sequence $\mathcal{S}'$ takes the configuration C_w^u to the configuration $C_{w'}^{u'}$ in T'. Since C_w^u and $C_{w'}^{u'}$ are chosen arbitrarily, so any two configurations in T' are mRJ-reachable. Hence T' is complete mRJ-reachable. □

The above theorem also holds for strongly connected directed graphs for which proof is similar. We now investigate mRJ-reachability for $m \geq 2$. In the following discussion, we shall assume $m > 1$ unless otherwise stated.

Lemma 2.3. If a tree is mRJ-reachable from a given configuration, then it will also be mRJ-reachable from any other configuration in which robot is located at a pendent vertex.

Proof. Suppose that a tree, T, is mRJ-reachable from a configuration C_v^u. Let $x, y \in V(T)$ such that x is a pendent vertex. We claim that T is mRJ-reachable from C_y^x. It is enough to show that C_v^u and C_y^x are mRJ-reachable from each other. Since T is mRJ-reachable from the configuration C_v^u, so we can take the robot to the vertex x starting from the configuration C_v^u. Also, the hole will be at a vertex in the connected graph $T - x$ and so it can be taken to y without using the node x. Thus, C_v^u and C_y^x are mRJ-reachable from each other. □

However, the converse of the above lemma is not true as illustrated by Example 2.1. Observe that the tree, T^*, shown in Figure 2.1 is not complete $2RJ$-reachable. However, there are trees which are complete $2RJ$-reachable. In fact, the Theorem 2.3 shows that for every m there exists a complete mRJ-reachable graph. An example of complete $2RJ$-reachable tree is shown in Figure 2.2.

Lemma 2.4. Let $[x_0, x_1, \ldots, x_{m+1}]$ be a path of length $m + 1$ in a tree T and $C_{x_i}^{x_0}$, $i \neq 0$, be a configuration. Then the configuration $C_{x_{m+1}}^{x_m}$ is mRJ-reachable from $C_{x_i}^{x_0}$.

Proof. Starting from the configuration $C_{x_i}^{x_0}$, the configuration $C_{x_{m+1}}^{x_m}$ of T can be achieved by the following sequence of moves:

$$x_i \overset{o}{\leftarrow} x_{i+1} \overset{o}{\leftarrow} x_{i+2} \cdots \overset{o}{\leftarrow} x_m \overset{o}{\leftarrow} x_{m+1} \underset{m}{\overset{r}{\leftarrow}} x_0 \overset{o}{\leftarrow} x_1 \overset{o}{\leftarrow} x_2 \cdots \overset{o}{\leftarrow} x_m \overset{r}{\leftarrow} x_{m+1}$$

□

For simplicity, we shall denote moves of the form: $x_0 \overset{o}{\leftarrow} x_1 \overset{o}{\leftarrow} \cdots \overset{o}{\leftarrow} x_{m-1} \overset{o}{\leftarrow} x_m$ by $x_0 \overset{o^*}{\longleftarrow} x_m$, whenever their is no ambiguity. Let u, v and w be three vertices in a graph, G, such that $\{u, v\}, \{v, w\} \in E(G)$ and, u and v having a robot and a hole, respectively. Also, assume that there is a path between vertices u and w that does not go through vertex v. We use $v \overset{r}{\leftarrow} u \overset{o^*}{\longleftarrow} w \overset{r}{\leftarrow} v$ to denote the sequence of move consisting of the robot move from u to v, then the simple moves that takes the hole from u to w and then the robot move from v to w.

Theorem 2.3. Let G be a connected graph having a cycle with $m+1$ or more edges. Then G is complete mRJ-reachable.

Proof. Let Γ be a cycle in G having $m+1$ or more edges. Let C_v^u and C_y^x be any two configurations in G. We claim that the configurations C_v^u and C_y^x are reachable from each other. If $u, v, x, y \in V(\Gamma)$ then by Proposition 2.1 the configurations C_v^u and C_y^x are reachable from each other.

Let at least one of u, v, x, y does not belong to $V(\Gamma)$. Then there exists a pair of adjacent vertices w_1, w_2 in $V(\Gamma)$ such that between u and w_1 there is a path of length $sm+1$ where s is an integer and it contains the vertices v and w_2. Also, there exists adjacent vertices z_1, z_2 in $V(\Gamma)$ such that between x and z_1 there is a path of length $tm+1$ where t is an integer and it contains the vertices y and z_2. By Lemma 2.4, we can achieve the configuration $C_{w_2}^{w_1}$ starting from the configuration C_v^u. Since Γ is bi-connected, so by Proposition 2.1 we can also achieve the configuration $C_{z_2}^{z_1}$. Finally, by Lemma 2.4, we can achieve the configuration C_y^x from the configuration $C_{z_2}^{z_1}$ □

Example 2.2. The tree, T_0, shown in Figure 2.2 is complete $2RJ$-reachable.

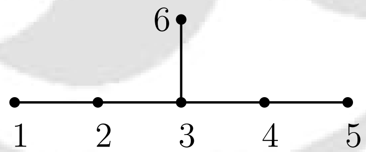

Figure 2.2: The smallest order complete $2RJ$-reachable tree.

2.2 Diameter and mRJ-Reachability

The length of a longest path in a tree, T, is referred to as the **diameter** of the tree and is denoted by $D(T)$. Clearly, if T is an mRJ-reachable tree of order three or more, then $D(G) \geq m+1$. However, the converse may not be true. Also, this bound appears to be weak for $m \geq 3$. In this section we investigate the relation between minimum diameter of mRJ-reachable and complete mRJ-reachable trees, and give strict lower bounds for the diameter of mRJ-reachable and complete mRJ-reachable trees. We begin by stating the following well known theorem about the center of a tree.

Theorem 2.4 ([15]). Every tree has a center consisting of either one vertex or two adjacent vertices.

Every internal vertex of a tree is a cut vertex. In other words, if v is an internal vertex of T then $T - v$ is disconnected. The following definition will be useful to prove the main theorem of this section.

Definition 2.1. Given two vertices u, v in a tree, T, a vertex x is said to be on the u-side of v if it is in the component of $T - v$ that contains u.

Observe that any path of length $D(T)$ of a tree T contains its central points. Also if P is a path of maximum length in a tree then it has one or two central points on P depending on whether P is of odd or even length. Now we restate the Lemma 2.1 below that will be required later.

Lemma 2.5. Let P be a path. Then starting from a configuration having robot at one end vertex we can take the robot to the other end vertex using mRJ moves in addition to simple moves if and only if $D(P) \equiv 1 \mod m$.

Proof. Let P be a path between vertices u and v and P' be the path between u and w obtained by deleting m vertices from one end of P (say, from the end with the vertex v). We first claim that if starting from one end of P' we can take the robot to the other end of P' if and only if starting from one end of P we can take the robot to the other end of P.

First assume that starting from a configuration with the robot at u we can take the robot to the vertex w in the sub-path P' of P using mRJ moves in addition to simple moves. Let x be the vertex adjacent to w in P'. Without loss of generality assume that the sequence of moves that takes the robot to w leaves the hole at x. Otherwise the hole will be on the x-side of w in P and so we can take it to the vertex x without disturbing the robot at w. Thus we can take the robot to the vertex v using the sequence of move $x \overset{r}{\leftarrow} w \overset{o^*}{\leftarrow} v \underset{m}{\overset{r}{\leftarrow}} x$.

Conversely, suppose that starting from a configuration with the robot at u we can take the robot to the vertex v in P using mRJ moves in addition to simple moves. Let x be the vertex adjacent to w in P not on the v side of w. Without loss of generality assume that the sequence of moves that takes the robot to v leaves the hole at x. Then we can take the robot to the vertex w using the sequence of moves $x \overset{r}{\underset{m}{\leftarrow}} v \overset{o^*}{\leftarrow} w \overset{r}{\leftarrow} x$. This proves our claim.

Now let P_0 be the path of length at most m obtained from P by deleting sets of m vertices repeatedly from one of its ends (say from the end with the vertex v). Now if length of P_0 is 1, clearly starting from one end we can take the robot to the other end of P_0. Hence using the moves stated above we can claim that starting from one end we can take the robot to the other end of P.

Again if length of P_0 is more than 1, then we can at most take the robot to the vertex adjacent to u. Because to apply mRJ moves in a tree it should have diameter at least $m+1$ but $D(P_0) \leq m$. Thus in this case starting from one end we cannot take the robot to the other end of P_0 and so the same is true for P. This completes the proof of the lemma. □

Theorem 2.5 gives a lower bound for the diameter of mRJ-reachable trees.

Theorem 2.5. Let T be a mRJ-reachable tree of order 3 or more and $m \geq 2$. Then $D(T) \geq 2m-1$ (respectively $D(T) \geq 2m$) if T has two (respectively one) central points.

Proof. Let $P(x,y)$ be a path of length $D(T)$ in T with x, y as the end points. Then x, y must be pendent vertices of T. Otherwise T will have a path of length more than $D(T)$, a contradiction. Since a tree has either one or two central (adjacent) points, so we consider the following cases:

Case 1: T has one central point. Let v be the central point of T. Clearly v is the middle point of P i.e., $d(x,v) = d(y,v) = t$. Also $d(u,v) \leq t$ for all $u \in V(T)$, where $d(x,y)$ denotes the distance between x and y. Now, since T is mRJ-reachable, so there exist a configuration with the robot at the pendent vertex from which the configuration C_u^v is reachable, where the vertex u adjacent to v. Equivalently, starting from the configuration C_u^v we must be able to take the robot to a pendent vertex. But to move the robot from the configuration C_u^v to a vertex other than u we have to use mRJ moves. But to make a mRJ move we need at least one vertex either at distance m from v not in the u-side of v or at a distance $m+1$ from v on the u-side of v. That is, either $t \geq m$ or $t \geq m+1$. Hence $D(T) = 2t \geq 2m$.

Case 2: T has two central points. Suppose that u and v be the two central points of T. Then by Theorem 2.4 u, v are adjacent. Also they lie on the path P. Let u be on the x-side of v. Then $d(x, u) = d(y, v) = t$. Clearly $d(u, w) \leq t+1$ for all $w \in V(T)$ in the v-side of u and $d(u, w) \leq t$ for all $w \in V(T)$ not in the v-side of u. Otherwise T will have a path of length larger than that of P, which is not possible. Similarly, $d(v, w) \leq t+1$ for all $w \in V(T)$ in the u-side of v and $d(v, w) \leq t$ for all $w \in V(T)$ not in the u-side of v. Now, since T is mRJ-reachable, so there exists a configuration with the robot at the pendent vertex from which the configuration C_w^v is reachable, where w is in the neighborhood of v. Equivalently, starting from the configuration C_w^v we must be able to take the robot to a pendent vertex. Starting from the configuration C_w^v to move the robot to a vertex other than w we have to use mRJ moves. But to make a mRJ move we need at least one vertex either at distance m from v not in the w-side of v or at a distance $m+1$ from v on the w-side of v.

The farthest vertex from v in T is at distance $t+1$. If there is a vertex at distance m from v not in the w-side of v, then $m \leq t+1$. If there is a vertex at a distance $m+1$ from v on the w-side of v then $m+1 \leq t+1$ i.e., $m \leq t$. Hence $D(T) = 2t+1 \geq 2m-1$. □

Example 2.3. The tree in the Figure 2.3 is minimal complete $4RJ$-reachable and it has diameter 8. Also, the tree in Figure 2.4 is minimal complete $5RJ$-reachable and it has diameter 11.

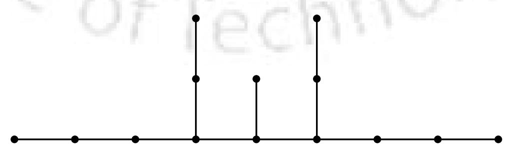

Figure 2.3: A minimal complete $4RJ$-reachable tree.

We now give the lower bound for the diameter of complete mRJ-reachable trees in Theorem 2.6.

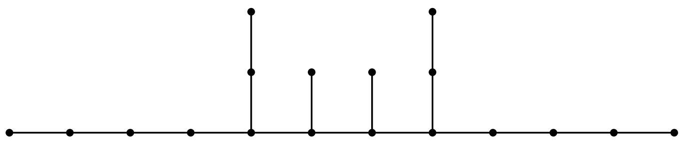

Figure 2.4: A minimal complete $5RJ$-reachable tree.

Theorem 2.6. Let T be a complete mRJ-reachable tree of order 3 or more and $m \geq 2$. Then $D(T) \geq 2m + 1$ (respectively $D(T) \geq 2m$) if T has two (respectively one) central points.

Proof. Suppose that T be a complete mRJ-reachable tree of order 3 or more and $m \geq 2$. Then T is also mRJ-reachable. So, if T has one central point by Theorem 2.5 we have $D(T) \geq 2m$. We are left with the case when T has two central points.

Suppose that u and v be the two central points of T. We know that they are adjacent. Consider the configuration C_v^u of T. Assume that $D(T) \leq 2m$. Then the maximum distance between u and any other vertex in T is m, so we can move the robot only to the vertex v. Also the maximum distance between v and any other vertex in T is m, so we can move the robot v only to the vertex u. Hence T is not reachable from the configuration C_v^u. Thus we can conclude that the tree T is not complete mRJ-reachable, which is a contradiction. Hence $D(T) \geq 2m + 1$. □

The tree T_0 in Figure 2.2 is complete $2RJ$-reachable. It has one central point and $D(T_0) = 4$.

Example 2.4. The tree shown in Figure 2.5 has two central points. It is complete $3RJ$-reachable and $D(T) = 7$. In fact no tree of diameter less than 7 is complete $3RJ$-reachable. ♯

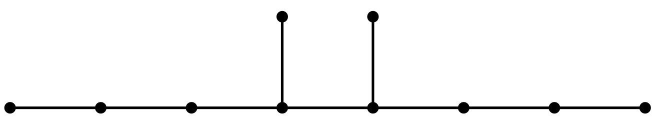

Figure 2.5: A complete $3RJ$-reachable tree.

The tree shown in Figure 2.6 is not $2RJ$-reachable as there does not exist any configuration from which robot can be taken to all vertices. Observe that it contains the $2RJ$-reachable tree T^* as subtree. That is if a tree T contains a mRJ-reachable tree then it need not be mRJ-reachable. Whereas the case is different for complete mRJ-reachable trees. The following theorem shows that every tree containing a complete mRJ-reachable tree as a subtree is complete mRJ-reachable.

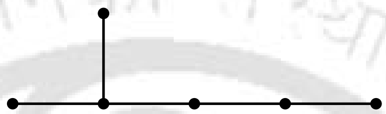

Figure 2.6: A tree which is not $2RJ$-reachable but contains a $2RJ$-reachable subtree T^*.

Theorem 2.7. If the tree T' is complete mRJ-reachable and it is a sub-tree of the tree T, then T is also complete mRJ-reachable.

Proof. Since T' is complete mRJ-reachable and it is a sub-tree of T, so by Theorem 2.6 we have $D(T) \geq 2m$. Let u, v be any two adjacent vertices in T. Consider the configuration C_v^u of T. We claim that T is mRJ-reachable from C_v^u. Let w be any vertex in T. We want to show that starting at C_v^u, the robot can reach the vertex w. To achieve this, we consider the following cases:

Case A. $u, v \in V(T')$**.** If $w \in V(T')$ then we are done. Suppose that $w \notin V(T')$ and consider a pair of adjacent vertices $x, y \in V(T')$ such that $d(w, x) \equiv 1 \mod m$ and y is on the path joining w to x. Such a pair always exist because $D(T') \geq 2m$. Now since T' is complete mRJ-reachable, the configuration C_y^x reachable from C_v^u. Finally, by the Lemma 2.5 we can take the robot to the vertex w, because $d(w, x) \equiv 1 \mod m$.

Case B. $u, v \notin V(T')$**.** Without loss of generality, assume that T' be a subtree of the component of $T - u$ containing v. Otherwise we can interchange the roles of u and v. Since $D(T') \geq 2m$, so there exists a pair of adjacent vertices $x, y \in V(T')$ such that $d(u, x) \equiv 1 \mod (m)$ and y is on the path connecting u and x. So, by Lemma 2.5, we can reach the configuration C_y^x from C_v^u. Now if $w \in V(T')$ then we are done, because T' complete mRJ-reachable. Otherwise, consider a pair of adjacent vertices $x_1, y_1 \in V(T')$ such that $d(w, x_1) \equiv 1 \mod (m)$ and y_1 is on the path joining w to x_1. Then by Theorem 2.1 we can reach the configuration $C_{y_1}^{x_1}$ from C_y^x. Finally, since $d(w, x_1) \equiv 1 \mod m$, so we can take the robot to the vertex w (see Lemma 2.5).

Case C. $u \notin V(T')$ **but** $v \in V(T')$**.** Since $D(T') \geq 2m$, so there exists a pair of adjacent vertices $x, y \in V(T')$ such that $d(u, x) = m + 1$ and y is on the path connecting u and x. So, proceeding as in case A, we can take the robot to the vertex w.

Case D. $v \notin V(T')$ **but** $u \in V(T')$**.** By a simple move we can take the robot to the vertex v and hole at u. Hence it is reduced to case B and so we can take the robot to the vertex w.

Since w were chosen arbitrarily, we can claim that T is mRJ-reachable from the configuration C_v^u. Again the pair of vertices u, v were chosen arbitrarily, so we can conclude that T is complete mRJ-reachable.

□

Theorem 2.7 describes the approach we are going to take in characterizing complete $2RJ$-reachable and complete $3RJ$-reachable trees in the sections 2.3 and 2.4 respectively. We will describe a family of trees $\mathcal{T}_m$, $m = 2, 3$ so that a tree is complete mRJ-reachable if and only if it has a subtree that is a member of $\mathcal{T}_m$.

2.3 $2RJ$ Reachable Trees

In this section we characterize $2RJ$-reachable trees and complete $2RJ$-reachable trees. We begin this section by introducing the following terms.

Definition 2.2 (Star). A tree T is **star** if $D(T) \leq 2$. In other words a star is a complete bi-graph $K_{1,n}$.

Note that $K_1 = K_{1,0}$ and $K_2 = K_{1,1}$ are the only stars of diameter 0 and 1, respectively. These two are trivial stars. All stars of order three or more has diameter 2. A non-trivial star has one central point. Since $K_{1,1}$ is the only star with two central points, so by center we will refer to any one of the two vertices of $K_{1,1}$. The following proposition is immediate from Theorem 2.5.

Proposition 2.2. A non-trivial star is not $2RJ$-reachable.

Definition 2.3 (Starlike tree). A tree T is starlike if either it is a star or it is obtained by joining the centers u and v of two stars S_1 and S_2 respectively of order 2 or more by a path $P(u, v)$.

Stars are trivial starlike trees. Paths are also starlike trees. We use $T[u, v]$ to denote a non-trivial starlike tree. A non-trivial starlike tree $T[u, v]$ is even or odd according as $P(u, v)$ is even or odd.

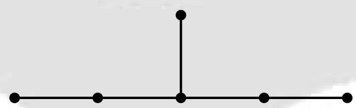

Figure 2.7: The tree T_0 is not starlike.

Proposition 2.3. The following statements are true.

(i) Starlike trees are even or odd according as its diameter is even or odd. Also if $T[u, v]$ is a non-trivial starlike tree then $D(T[u, v]) = |P(u.v)| + 2$.

(ii) Stars of diameter two are not $2RJ$-reachable (see Theorem 2.5).

(iii) For $n \geq 2$, the number of starlike trees of order n is $1 + \sum_{k=2}^{n-2} \lceil \frac{n-k}{2} \rceil$.

(iv) Any tree of order 5 or less is starlike.

Example 2.5. The tree T_0 in Figure 2.2 is not starlike. In fact T_0 is the smallest order tree that is not starlike. There are six trees of order 6 and among them five are starlike (see Proposition 2.3). Thus, T_0 is the only tree of order six that is not starlike.

Theorem 2.8. Let T be a non-trivial starlike tree. Then T is not complete mRJ-reachable for $m \geq 2$.

Proof. Proof is immediate from Corollary 2.1 and Lemma 2.2. □

Theorem 2.9. Any non starlike tree T of order six or more contains T_0 as a subtree.

Proof. Clearly T is not a path and so it contains a vertex v, of degree three or more. Let $N_2(v)$ be the set of vertices of degree two or more in the neighborhood $N(v)$ of v. Clearly $|N_2(v)| \geq 1$. Otherwise T is a star graph, which is not possible.

Case 1: $|N_2(v)| > 1$ Let $x, y, z \in N(v)$ such that $x, y \in N_2(v)$. Also let $u \in N(x) - v$ and $w \in N(y) - v$. Then the sub-tree of T generated by the set of vertices $\{x, y, z, u, v, w\}$ is T_0.

Case 2: $|N_2(v)| = 1$ Let $N_2(v) = \{x\}$. In this case T must have another vertex of degree three or more on the x-side of v. Otherwise the component of $T - v$ containing the vertex x is a path and hence T is a starlike tree, a contradiction. Let u be the nearest vertex to v in the x-side of v having degree three or more. Also, let $u_1 \in N(u)$ be the vertex on the path joining u and v and $u_2 \in N(u_1) - u$. If $N_2(u) - u_1 = \emptyset$, then T is a starlike tree with u, v as the centers of the stars, a contradiction. Thus, let $u_3 \in N_2(u) - u_1$ and $u_4 \in N(u_3) - u$. Also $N(u) - \{u_1, u_3\} \neq \emptyset$, because $deg(u) \geq 3$ and so let $u_5 \in N(u) - \{u_1, u_3\}$. Then the sub-tree of T generated by the set of vertices $\{u, u_1, u_2, u_3, u_4, u_5\}$ is T_0.

□

Corollary 2.2. Let T be a non-starlike tree having 6 or more vertices. Then T is complete $2RJ$-reachable.

Proof. By Lemma 2.9, T_0 is a sub-tree of T. Since T_0 is complete $2RJ$-reachable, by Theorem 2.7, T is complete $2RJ$-reachable. □

Thus, we can characterize the complete $2RJ$-reachable trees in terms of the following theorem.

Theorem 2.10. A tree is complete $2RJ$-reachable if and only if it is either K_2 or it contains T_0 as subtree. Also T_0 is the only minimal complete $2RJ$-reachable tree.

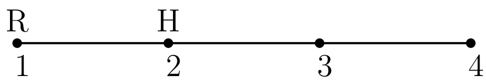

Figure 2.8: The path $P(1, 4)$.

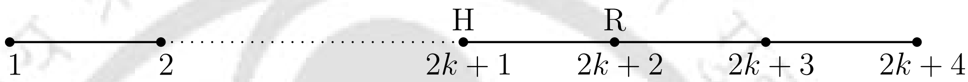

Figure 2.9: Reachability of the path $P(1, 2k + 4)$.

Lemma 2.6. A path of odd length is $2RJ$-reachable.

Proof. $P_n = v_0 v_1 v_2 \ldots v_n$ be a path of length n and n is odd. We apply induction on n to prove the result. Consider the configuration $C_{v_1}^{v_0}$.

Base step: For $n = 1$ the result is trivial. In fact, we need only simple moves to take the robot to any vertex of P. For $n = 3$, consider the path $P(1, 4)$ as shown in Figure 2.8 with the configuration C_2^1. Starting from the configuration C_2^1 we can take the robot to each of the vertices using the following sequences of moves:

(i) $2 \overset{r}{\leftarrow} 1$: to take the robot to the vertex 2.

(ii) $2 \overset{o}{\leftarrow} 3 \overset{o}{\leftarrow} 4 \underset{2}{\overset{r}{\leftarrow}} 1 \overset{o}{\leftarrow} 2 \overset{o}{\leftarrow} 3 \overset{r}{\leftarrow} 4$: to take the robot first to the vertex 4 and then to the vertex 3.

Hence the path $P(1, 4)$ is $2RJ$-reachable. This completes the base step.

Induction step: Assume that the result is true for $n = 2k + 1$. Consider a path $P(1, 2k+4)$ of length $2k+3$ with the configuration C_2^1. By induction hypothesis the sub-path $P(1, 2k+2)$ of $P(1, 2k+4)$ is $2RJ$-reachable from the configuration C_2^1. Also, by the Lemma 2.5 the configuration C_{2k+3}^{2k+4} is

$2RJ$-reachable from the configuration C_2^1. Hence, by using the base step we can claim that the sub-path $P(2k+1, 2k+4)$ of $P(1, 2k+4)$ is $2RJ$-reachable from the configuration C_{2k+3}^{2k+4}. This completes the induction step. Hence every path of odd length is $2RJ$-reachable.

□

Proposition 2.4. A path of even length is not $2RJ$-reachable.

Proof. Notice that, if a tree T is mRJ-reachable then it is mRJ-reachable from each configuration C_v^u whenever u is a pendent vertex of T. Hence the result is immediate by using the Lamma 2.5. □

Lemma 2.7. Let T be starlike and $D(T) \geq 3$. Consider a path uvw in T such that u is pendent and w is not pendent. Then the configurations C_v^u and C_w^v are not $2RJ$-reachable from each other.

Proof. Assume that C_v^u is reachable from C_w^v. We have the following two cases:

Case 1. There exists a vertex x not in the w-side of v, which is at distance 2 from v. This is not possible, because T is starlike and so all vertices in $N(v) - w$ are pendent. Thus we arrived at a contradiction. Hence our assumption is wrong.

Case 2. There exists a pair of vertices x, y on the w-side of v, such that $d(u, x)$, $d(v, x)$ are odd and $d(x, y) = 3$. This is possible only if the vertices w, x, y are not collinear. That is if T_0 is a sub-tree of T. This contradicts that T is starlike. Hence our assumption is wrong.

□

The starlike tree shown in Figure 2.6 is of even length and it is not $2RJ$-reachable. In fact, in Theorem 2.11 we have shown that odd length is a necessary and sufficient condition for a starlike tree to be $2RJ$-reachable.

Theorem 2.11. A non-trivial starlike tree is $2RJ$-reachable if and only if it is of odd length.

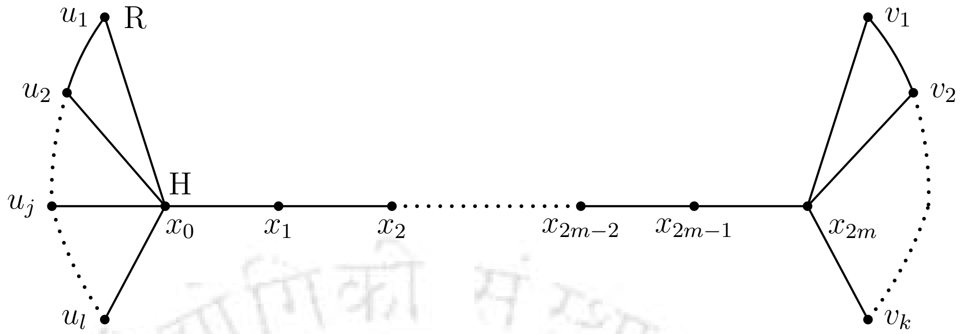

Figure 2.10: Reachability of an even starlike tree $T[u, v]$.

Proof. Let $T[u, v]$ be a starlike tree of odd length. Consider the path P : $[u = x_0, x_1, x_2, \ldots x_{2m} = v]$ of odd length connecting u and v in T. Also let $u_1, u_2, \ldots, u_l$ be the vertices adjacent to u apart from x_1 and v_1, v_2, $\ldots$ v_k be the vertices adjacent to v apart from x_{2m-1}. Consider the configuration $C_{x_0}^{u_1}$ of $T[u, v]$ (see Figure 2.10).

Now for each $j = 1, 2, \ldots, k$ the sub-graph $u_1 P v_j$ of $T[u, v]$ is a path of odd length and so by Lemma 2.6 it is reachable from the given configuration. Further the sequence of moves

$$x_0 \overset{o}{\leftarrow} x_1 \overset{o}{\leftarrow} x_2 \underset{2}{\overset{r}{\leftarrow}} u_1 \overset{o}{\leftarrow} x_0 \overset{o}{\leftarrow} u_j \underset{2}{\overset{r}{\leftarrow}} x_2$$

takes the robot from u_1 to u_j for each $j = 2, 3, \ldots k$. That is we can take the robot to each of the vertices $u_2, u_3 \ldots, u_k$. Hence $T[u, v]$ is $2RJ$-reachable.

Conversely assume that $T[u, v]$ is a starlike tree and it is $2RJ$-reachable. By assumption $D(T) \geq 3$. Let x be a pendant vertex adjacent to u and y be a pendant vertex adjacent to v. We claim that $xP(u, v)y$ is a path of odd length. Assume that $xP(u, v)y$ is of even length. Then $xP(u, v)$ is of odd length and hence $xP(u, v)$ is $2RJ$-reachable from the configuration C_u^x. That is we can obtain the configuration C_z^v, where $z \in N(v) - \{y\}$. Again $T[u, v]$ is reachable, so we can obtain the configuration C_v^y. Using transitivity, we can say that C_z^v and C_v^y are $2RJ$-reachable from each other. But by Lemma 2.7 this is not possible in T. Hence $T[u, v]$ is odd. □

Finally we conclude all the results stated in this section in the form of the following two theorems which characterize the trees in terms of $2RJ$ reachability and complete $2RJ$ reachability.

Theorem 2.12. A tree is $2RJ$-reachable if and only if it is either K_2 or odd starlike or not starlike.

2.4 $3RJ$ moves on trees

In this section we characterize the complete $3RJ$-reachable trees.

Definition 2.4. A tail in a tree T is a path $P(v_0, v_k)$ in which v_0 has degree 3 or more, v_k has degree one and all other vertices on P has degree two in T.

Let $\mathcal{T}_3$ be the collection of all trees as shown in Figure 2.11, where length of the path represented by the dotted line is not multiple of three.

Figure 2.11: Complete $3RJ$-reachable tree.

Lemma 2.8. Any tree in the collection $\mathcal{T}_3$ is complete $3RJ$-reachable.

Proof. Let T be any tree in $\mathcal{T}_3$. We label the vertices of the longest path in T by $v_0, v_1, \cdots v_k$ and the pendent vertices incident with v_3 and v_{k-3} by x, y respectively. Clearly $d(v_3, v_{k-3}) \not\equiv 0 \mod 3$. First we note the following:

(i) Starting from the configuration $C_{v_1}^{v_0}$ and moving along the path $P(v_0, v_k)$ we can only reach the configurations $C_{v_{i+1}}^{v_i}$ with $i \equiv 0 \mod 3$.

(ii) Starting from the configuration $C_{v_2}^{v_1}$ and moving along the path $P(v_0, v_k)$ we can only reach the configurations $C_{v_{i+1}}^{v_i}$ with $i \equiv 1 \mod 3$.

(iii) Starting from the configuration $C_{v_3}^{v_2}$ and moving along the path $P(v_0, v_k)$ we can only reach the configurations $C_{v_{i+1}}^{v_i}$ with $i \equiv 2 \mod 3$.

From the above observations we can conclude that any two configuration in $P(v_0, v_k)$ are reachable from each other if and only if the three configurations $C_{v_1}^{v_0}$, $C_{v_2}^{v_1}$ and $C_{v_3}^{v_2}$ are reachable from each other. The pendent vertex x is used to reach the configuration $C_{v_3}^{v_2}$ from the configuration $C_{v_1}^{v_0}$ by the following sequence of moves

$$v_1 \xleftarrow{o^*} x \underset{3}{\xleftarrow{r}} v_0 \xleftarrow{o^*} v_6 \underset{3}{\xleftarrow{r}} x \xleftarrow{o^*} v_2 \underset{3}{\xleftarrow{r}} v_6.$$

If $d(v_3, v_{k-3}) \equiv 1 \mod 3$ then the pendent vertex y is used to reach the configuration $C_{v_2}^{v_1}$ from the configuration $C_{v_1}^{v_0}$ using the sequence of moves

$$v_1 \twoheadleftarrow v_{k-4} \xleftarrow{r} v_{k-3} \xleftarrow{o^*} v_k \underset{3}{\xleftarrow{r}} v_{k-4} \xleftarrow{o^*} v_{k-1} \twoheadleftarrow v_2$$

where $v_1 \twoheadleftarrow v_{k-4}$ denotes the sequence of moves that takes the configuration $C_{v_1}^{v_0}$ to the configuration $C_{v_{k-4}}^{v_{k-3}}$ and $v_{k-1} \twoheadleftarrow v_2$ denotes the sequence of moves that takes the configuration $C_{v_{k-1}}^{v_k}$ to the configuration $C_{v_2}^{v_1}$. And if $d(v_3, v_{k-3}) \equiv 2 \mod 3$ the vertex y is used to reach the configuration $C_{v_3}^{v_2}$ from the configuration $C_{v_1}^{v_0}$ using the sequence of moves

$$v_1 \xleftarrow{o^*} x \underset{3}{\xleftarrow{r}} v_0 \xleftarrow{o^*} v_3 \twoheadleftarrow v_{k-3} \twoheadleftarrow v_3 \xleftarrow{r} v_2$$

where $v_3 \twoheadleftarrow v_{k-3}$ represents the sequence of moves that takes the configuration $C_{v_3}^{x}$ to the configuration $C_{v_{k-3}}^{y}$ and $v_{k-3} \twoheadleftarrow v_3$ represents the sequence of moves that takes the configuration $C_{v_{k-3}}^{y}$ to the configuration $C_{v_3}^{v_2}$. This completes the proof. □

Lemma 2.9. A minimal complete $3RJ$-reachable tree can have tails of length at most 3.

Proof. Let T be a minimal complete $3RJ$-reachable tree and assume that $P(v_0, v_k)$ be a tail in T of length 4 or more. Let T' be the tree obtained from T by reducing the tail to $P(v_0, v_3)$. We claim that T' is also complete $3RJ$-reachable. Let C_w^u and $C_{w'}^{u'}$ be any two configurations in T'. Since T is complete $3RJ$-reachable so there exist a sequence consisting of simple moves and $3RJ$-moves in T which takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. Let $\mathcal{S}$ be a sequence with minimum number of moves among all such sequences that takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. We note the following observations:

(i) Starting from the configuration $C^{v_0}_{v_1}$ and moving along the path $P(v_0, v_k)$ we can only reach the configurations $C^{v_i}_{v_{i+1}}$ with $i \equiv 0 \mod 3$.

(ii) Starting from the configuration $C^{v_1}_{v_2}$ and moving along the path $P(v_0, v_k)$ we can only reach the configurations $C^{v_i}_{v_{i+1}}$ with $i \equiv 1 \mod 3$.

(iii) Starting from the configuration $C^{v_2}_{v_3}$ and moving along the path $P(v_0, v_k)$ we can only reach the configurations $C^{v_i}_{v_{i+1}}$ with $i \equiv 2 \mod 3$.

Thus the three configurations $C^{v_0}_{v_1}$, $C^{v_1}_{v_2}$ and $C^{v_2}_{v_3}$ are not reachable from each other if we restrict our movements only on the tail $P(v_0, v_k)$ of T. If $\mathcal{S}$ contains a movement involving a vertex from the set $\{v_4, v_5, \cdots v_k\}$ then it must contain at least one of the three moves

$$v_4 \overset{r}{\underset{3}{\leftarrow}} v_0,\ v_5 \overset{r}{\underset{3}{\leftarrow}} v_1 \text{ and } v_6 \overset{r}{\underset{3}{\leftarrow}} v_2.$$

Since $u, w, u', w' \notin \{v_4, v_5, \cdots v_k\}$, so by the above observations we can claim that for each $i = 0, 1, 2$ if $\mathcal{S}$ contains the move $v_{i+4} \overset{r}{\underset{3}{\leftarrow}} v_i$ then it will also contain the move $v_i \overset{r}{\underset{3}{\leftarrow}} v_{i+4}$. Also all the moves in $\mathcal{S}$ starting from a move of the form $v_{i+4} \overset{r}{\underset{3}{\leftarrow}} v_i$ for some $i = 0, 1, 2$ to the nearest move of the form $v_i \overset{r}{\underset{3}{\leftarrow}} v_{i+4}$ will involve vertices only from the set $\{v_4, v_5, \cdots v_k\}$. Thus if we remove all the moves starting from a move of the form $v_{i+4} \overset{r}{\underset{3}{\leftarrow}} v_i$ for some $i = 0, 1, 2$ upto the nearest move of the form $v_i \overset{r}{\underset{3}{\leftarrow}} v_{i+4}$ from $\mathcal{S}$, the remaining moves in $\mathcal{S}$ still take the configuration C^u_w to the configuration $C^{u'}_{w'}$ which contradicts the minimality of $\mathcal{S}$. Hence $\mathcal{S}$ cannot contain a move involving vertices from the set $\{v_4, v_5, \cdots v_k\}$. That is, $\mathcal{S}$ takes the configuration C^u_w to the configuration $C^{u'}_{w'}$ in T'. Since C^u_w and $C^{u'}_{w'}$ are chosen arbitrarily so T' is also complete $3RJ$-reachable. This contradicts the minimality of T. Hence T can have tails of length at most 3. $\square$

Lemma 2.10. Let T be a complete $3RJ$-reachable tree and T' be the tree obtained from T by deleting multiple tails of length two. Then T' is also complete $3RJ$-reachable.

Proof. Let $\{v_0 v^i_1 v^i_2 |\ i = 1, 2, \ldots, k\}$ be the collection of tails of length two at the vertex v in T and T' be the tree obtained by deleting all tails of length two at v but $v_0v_1^1v_2^1$ from T. Let C_w^u and $C_{w'}^{u'}$ be any two configurations in T'. Since T is complete $3RJ$-reachable so there exist a sequence consisting of simple and $3RJ$-moves in T which takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. Let $\mathcal{S}$ be a sequence of minimum number of moves among all such sequences that takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. First we note the following:

(i) Since for $i \neq j$ we have $d(v_1^i, v_1^j) = 2$, so no $3RJ$ move in $\mathcal{S}$ can involve v_1^i and v_1^j. A $3RJ$-move involving v_1^i and v_2^j is also not possible in $\mathcal{S}$ because $d(v_1^i, v_2^j) = 3$.

(ii) Since $\mathcal{S}$ is the minimal of all such sequences and $u, w, u', w' \notin \{v_1^i, v_2^i | i = 1, 2, \ldots, k\}$ so $\mathcal{S}$ does not contain a simple move involving one or both the vertices from the set $\{v_1^i, v_2^i | i = 1, 2, \ldots, k\}$. Also $\mathcal{S}$ cannot involve a robot move between the vertices v_2^i and v_2^j.

(iii) A robot move of the form $v_2^i \overset{r}{\underset{3}{\leftarrow}} x$ in $\mathcal{S}$ must be followed by a move of the form $y \overset{r}{\underset{3}{\leftarrow}} v_2^i$, where $x, y \notin \{v_1^i, v_2^i | i = 1, 2, \ldots, k\}$. If we replaced the sequence of moves $v_2^i \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_2^i$ by the sequence of moves $v_2^1 \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_2^1$ in $\mathcal{S}$, the new sequence still take the configuration C_w^u to the configuration $C_{w'}^{u'}$. So replace all such sub-sequence of moves in $\mathcal{S}$ by the sequence moves $v_2^1 \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_2^1$.

(iv) A robot move of the form $v_1^i \overset{r}{\underset{3}{\leftarrow}} x$ in $\mathcal{S}$ must be followed by a move of the form $y \overset{r}{\underset{3}{\leftarrow}} v_1^i$, where $x, y \notin \{v_1^i, v_2^i | i = 1, 2, \ldots, k\}$. Also any move of the form $y \overset{r}{\underset{3}{\leftarrow}} v_1^i$ in $\mathcal{S}$ must be preceded by a robot move of the form $v_1^i \overset{r}{\underset{3}{\leftarrow}} x$. If we replace the sequence of moves $v_1^i \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_1^i$ by the sequence of moves $v_1^1 \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_1^1$ in $\mathcal{S}$, the new sequence still take the configuration C_w^u to the configuration $C_{w'}^{u'}$. So replace each such subsequence by the sequence of moves $v_1^1 \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_1^1$.

Let $\mathcal{S}'$ be the new sequence obtained from $\mathcal{S}$ by applying the above changes. The sequence $\mathcal{S}'$ does not contain any move involving vertices from the set

$\{v_1^i, v_2^i | i = 2, 3, \ldots, k\}$ and it takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. That is the sequence of moves $\mathcal{S}'$ takes the configuration C_w^u to the configuration $C_{w'}^{u'}$ in T'. Hence T' is complete $3RJ$-reachable. □

Lemma 2.11. Let T be a complete $3RJ$-reachable tree and let v be a vertex in T having more than one tail of length three. Let T' be the tree obtained from T by deleting all but one tail of length three and part of a tail of length three by keeping a pendent vertex at v. Then T' is also complete $3RJ$-reachable.

Proof. Let $\{[v, v_1^i, v_2^i, v_3^i] \mid i = 1, 2, \ldots, k\}$ be the collection of tails of length three at the vertex v in T and T' be the tree obtained by deleting all tails of length three at v but $[v, v_1^1, v_2^1, v_3^1$ and the vertex v_1^2 from T. Let C_w^u and $C_{w'}^{u'}$ be any two configurations in T'. Since T is complete $3RJ$-reachable so there exist a sequence consisting of simple moves and $3RJ$-moves in T which takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. Let $\mathcal{S}$ be a sequence of minimum number of moves among all such sequences that takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. First we note the following:

(i) Since T is complete $3RJ$-reachable, so by Theorem 2.6 T, has a vertex v' such that $d(v, v') = 3$ and $v' \notin \{v_3^i | i = 1, 2, \ldots, k\}$.

(ii) Since $u, w, u', w' \notin \{v_1^i, v_2^i | i = 1, 2, \ldots, k\}$ so the minimality of $\mathcal{S}$ implies that it does not contain a simple move of the robot involving one or both the vertices from the set $\{v_1^i, v_2^i | i = 1, 2, \ldots, k\}$. Also $\mathcal{S}$ cannot involve a move of the form $v_2^j \xleftarrow[3]{r} v_2^i$, because we can reach all the vertices from v_2^i to which we can reach from v_2^j by a $3RJ$ move . Also, for $i \neq j$ it cannot involve robot moves involving any one of the pair of vertices $\{v_1^i, v_1^j\}$, $\{v_1^i, v_2^j\}$, $\{v_3^i, v_3^j\}$, $\{v_2^i, v_3^j\}$ because $d(v_1^i, v_1^j) = 2$, $d(v_1^i, v_2^j) = 3$, $d(v_2^i, v_3^j) = 5$ and $d(v_3^i, v_3^j) = 6$.

(iii) For $i \neq j$, a robot move of the form $v_1^j \xleftarrow[3]{r} v_3^i$ in $\mathcal{S}$ must be part of a subsequence of moves of the form

$$v_3^i \xleftarrow[3]{r} x \xleftarrow{o^*} v_1^j \xleftarrow[3]{r} v_3^i \xleftarrow{o^*} y \xleftarrow[3]{r} v_1^j \tag{2.1}$$

in $\mathcal{S}$, where $x, y \notin \{v_1^i, v_2^i | i = 2, 3 \ldots, k\}$. If $y = v_3^1$ on replacing the subsequence (2.1) in $\mathcal{S}$ with $v' \xleftarrow[3]{r} x \xleftarrow{o^*} v_1^2 \xleftarrow[3]{r} v' \xleftarrow{o^*} y \xleftarrow[3]{r} v_1^2$, the modified $\mathcal{S}$ still take the configuration C_w^u to the configuration $C_{w'}^{u'}$.

If $y \neq v_3^1$ and $x \neq v_1^1$ on replacing the subsequence (2.1) in $\mathcal{S}$ with

$$v_3^1 \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} v_1^2 \overset{r}{\underset{3}{\leftarrow}} v_3^1 \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_1^2,$$

the modified $\mathcal{S}$ still take the configuration C_w^u to the configuration $C_{w'}^{u'}$.

If $y \neq v_3^1$ and $x = v_1^1$ on replacing the subsequence (2.1) in $\mathcal{S}$ with $v' \overset{r}{\underset{3}{\leftarrow}} x$ the modified $\mathcal{S}$ still take the configuration C_w^u to the configuration $C_{w'}^{u'}$.

(iv) For $i \neq j$ a move of the form $v_3^j \overset{r}{\underset{3}{\leftarrow}} v_1^i$ in $\mathcal{S}$ must be part of a subsequence of moves of the form

$$v_1^i \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} v_3^j \overset{r}{\underset{3}{\leftarrow}} v_1^i \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_1^i \tag{2.2}$$

in $\mathcal{S}$, where $x, y \notin \{v_1^i, v_2^i | i = 2, 3 \ldots, k\}$. Arguments similar to that in point (iii) allows us to claim that a subsequence of the form 2.2 can be replaced by a sequence of moves involving vertices on from T' and the modified sequence still still take the configuration C_w^u to the configuration $C_{w'}^{u'}$.

(v) For $x \neq y$, a subsequence of moves of the form $v_2^i \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_2^i$ in $\mathcal{S}$ can be replaced by a sequence of moves of the form $y \overset{r}{\underset{3}{\leftarrow}} x$.

(vi) Suppose $\mathcal{S}$ contains a sequence of moves of the form

$$v_3^i \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_3^i. \tag{2.3}$$

Then if $x = v_1^1$ we replace the subsequence 2.3 by the sequence of moves

$$v' \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} v_1^2 \overset{r}{\underset{3}{\leftarrow}} v' \overset{o^*}{\longleftarrow} v_3^1 \overset{r}{\underset{3}{\leftarrow}} v_1^2 \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_3^1.$$

otherwise we replace it by $v_3^1 \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_3^1$. The modified sequence also take the configuration C_w^u to the configuration $C_{w'}^{u'}$.

(vii) Suppose $\mathcal{S}$ contains a sequence of moves of the form

$$v_1^i \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_1^i.$$

We replace it by the sequence of moves

$$v_1^2 \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_1^2.$$

The modified sequence also take the configuration C_w^u to the configuration $C_{w'}^{u'}$.

Let $\mathcal{S}'$ be the sequence of moves obtained from $\mathcal{S}$ by the above changes. The sequence $\mathcal{S}'$ consists of moves involving vertices only from the tree T' and it takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. Hence the configurations C_w^u and $C_{w'}^{u'}$ are reachable from each other. Since C_w^u and $C_{w'}^{u'}$ were chosen arbitrarily, the tree T' is also $3RJ$-reachable. ☐

Theorem 2.13 characterizes the complete $3RJ$-reachable trees.

Theorem 2.13. A tree T is complete $3RJ$-reachable if and only if T contains a member of $\mathcal{T}_3$ as a subtree.

Proof. First assume that T is complete $3RJ$-reachable. Let T' be a subtree of T of smallest order such that T' is also complete $3RJ$ reachable. Let $P(v_0, v_k)$ be a path of length $D(T')$ in T'. Then $l(P) \geq 7$ because complete $3RJ$-reachable trees has diameter 7 or more. We divide the proof into the following steps:

Step 1:

Claim: Both the tails of P in T' are of length 3.

Since v_0 is a pendent vertex, so by Lemma 2.2 it is the only pendent vertex incident with v_1 in T'. Also since P is a path of length equal to $D(T')$ in T' so we cannot have a vertex other than v_2 in the neighborhood of v_1 having degree 2 or more. Hence degree of the vertex v_1 must be two.

We now show that degree of v_2 is also 2. Assume that v' be a vertex incident with v_2 apart from v_1 and v_3. Since $l(P) = D(T')$ so we cannot have vertex at a distance 3 or more from v_2 on the v'-side of v_2. Also by Lemma 2.10 we cannot have a vertex at a distance 2 from v_2 on the v'-side of v_2. So v' is a pendent vertex. Let $T^* = T' - v'$. We claim that T^* is complete $3RJ$-reachable. For, let C_w^u and $C_{w'}^{u'}$ be any two configurations in T^*. Since T' is complete $3RJ$-reachable so there exist a sequence consisting of simple moves and $3RJ$-moves in T' which takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. Let $\mathcal{S}$ be a sequence of minimum number of moves among all such sequences that takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. Since $v' \notin \{u, w, u', w'\}$ so no simple move in $\mathcal{S}$ can involve v'. Also any $3RJ$ move of the type $v' \overset{r}{\underset{3}{\leftarrow}} x$ must be followed by a move of the type $y \overset{r}{\underset{3}{\leftarrow}} v'$ with some move of the hole in between, where $x, y \notin \{v_0, v_1\}$. And so if we replace the subsequence of moves $v' \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v'$ in $\mathcal{S}$ by $v_1 \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\longleftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_1$ still the modified $\mathcal{S}$ will take the configuration C_w^u to the configuration $C_{w'}^{u'}$. Let $\mathcal{S}'$ be the sequence obtained from $\mathcal{S}$ after all possible such changes. The sequence $\mathcal{S}'$ also takes the configuration C_w^u to the configuration $C_{w'}^{u'}$ in T^*. Since C_w^u and $C_{w'}^{u'}$ were chosen arbitrarily, we can conclude that T^* is also complete $3RJ$-reachable, a contradiction. Hence degree of v_2 must be 2.

Step 2:

Claim: Degree of v_3 is three and there is a pendent vertex incident with it.

Clearly degree of v_3 is three or more (by Lemma 2.9). Let v_3' be a vertex incident with v_3 apart from v_2 and v_4. T cannot have a vertex at distance 4 or more on the v_3'-side of v_3. Let v_3'' be a vertex at distance 3 from v_3 on the v_3'-side of v_3. Then repeating the above argument with the path $P(v_3'', v_3)P(v_3, v_k)$ we can claim that $P(v_3'', v_3)$ is tail of length three, which contradicts the minimality of T' (see Lemma 2.11). Hence T' cannot have a vertex at distance 3 from v_3 on the v_3'-side of v_3.

Suppose that v_3'' is a vertex at distance 2 from v_3 in T' on the v_3'-side of v_3 and let $T^* = T' - v_3''$. We claim that T^* is also complete $3RJ$-reachable. For let C_w^u and $C_{w'}^{u'}$ be any two configurations in T^*. Since T' is complete $3RJ$-reachable so there exist a sequence consisting of simple moves and $3RJ$-moves in T' which takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. Let $\mathcal{S}$ be a sequence of minimum number of moves among all such sequences that takes the configuration C_w^u to the configuration $C_{w'}^{u'}$ in T'. Clearly $\mathcal{S}$ cannot have a simple move involving the vertex v_3''. Also any move of the form $v_3'' \overset{r}{\underset{3}{\leftarrow}} x$ (respectively $y \overset{r}{\underset{3}{\leftarrow}} v_3''$) must be followed (preceded) by a move of the form $y \overset{r}{\underset{3}{\leftarrow}} v_3''$ (respectively $v_3'' \overset{r}{\underset{3}{\leftarrow}} x$) with some move of the hole in between, where

$x, y \in V(T^*)$. Let

$$v_3'' \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\leftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_3'' \tag{2.4}$$

be a subsequence of $\mathcal{S}$. If either of x, y equal to v_1 we replace the subsequence (2.4) of moves in $\mathcal{S}$ by the move $y \overset{r}{\underset{3}{\leftarrow}} x$ and if neither of x, y equal to v_1 we replace the subsequence (2.4) of moves in $\mathcal{S}$ by the sequence of moves $v_1 \overset{r}{\underset{3}{\leftarrow}} x \overset{o^*}{\leftarrow} y \overset{r}{\underset{3}{\leftarrow}} v_1$. The modified $\mathcal{S}$ still take the configuration C_w^u to the configuration $C_{w'}^{u'}$. Let $\mathcal{S}'$ be the sequence of moves obtained by all possible changes of the above type. This new sequence $\mathcal{S}'$ also takes the configurationC_w^u to the configuration $C_{w'}^{u'}$ and it does not involve v_3''. Hence T^* is also complete $3RJ$-reachable, which is a contradiction. Hence T' cannot have a vertex at distance 2 from v_3 on the v_3'-side of v_3. Thus $[v_3, v_2, v_1, v_0]$ is a tail of length 3 in T' and degree of v_3 is 3 with a pendent vertex incident with it apart from v_2 and v_4.

Similarly $[v_{k-3}, v_{k-2}, v_{k-1}, v_k]$ is a tail of length 3 and degree of v_{k-3} is 3 with a pendent vertex incident with it apart from v_{k-4} and v_{k-2}.

Step 3:

Claim: $d(v_3, x) \not\equiv 0 \mod 3$**, for some vertex** x **on** $P(v_4, v_{k-3})$**.**

Since $l(P) \geq 7$, so $d(v_3, v_{k-3}) \geq 1$. Now, if $d(v_3, v_{k-3}) \not\equiv 0 \mod 3$ then we are done. Or, else let x be the nearest vertex to v_3 on P having degree 3 or more. If $d(v_3, x) \not\equiv 0 \mod 3$ we are done. Otherwise let T'' denote the tree $T' - v$ where v is the pendent vertex incident with v_3. Let C_w^u and $C_{w'}^{u'}$ be any two configurations in T''. Since T' is complete $3RJ$-reachable so there exist a sequence consisting of simple moves and $3RJ$-moves in T' which takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. Let $\mathcal{S}$ be a sequence of minimum number of moves among all such sequences that takes the configuration C_w^u to the configuration $C_{w'}^{u'}$ in T'. Clearly $\mathcal{S}$ cannot have a simple move involving v. Also the only moves that can involve v must be one of the following four type $v \overset{r}{\underset{3}{\leftarrow}} v_0$, $v_0 \overset{r}{\underset{3}{\leftarrow}} v$, $v \overset{r}{\underset{3}{\leftarrow}} v_6$ and $v_6 \overset{r}{\underset{3}{\leftarrow}} v$. Also the move $v \overset{r}{\underset{3}{\leftarrow}} v_0$ must be followed by $v_6 \overset{r}{\underset{3}{\leftarrow}} v$ with some moves of the hole in between and the move $v \overset{r}{\underset{3}{\leftarrow}} v_6$ must be followed by $v_0 \overset{r}{\underset{3}{\leftarrow}} v$ with some moves of the hole in between. Suppose that $\mathcal{S}$ has the sequence of moves $v \overset{r}{\underset{3}{\leftarrow}} v_0 \overset{o^*}{\leftarrow} v_6 \overset{r}{\underset{3}{\leftarrow}} v$ as a subsequence. Now let y be a vertex not on P and adjacent to x and let z be the vertex at a distance 3 from

x on P in the v_k-side of x. Since $d(v_0, y) \equiv 1 \mod 3$ so there exist a sequence of moves $\mathcal{S}_1$ which takes the configuration $C_{v_1}^{v_0}$ to the configuration C_x^y. Also $d(z, v_5) \equiv 1 \mod 3$ so there exist a sequence of moves $\mathcal{S}_2$ which takes the configuration $C_{z_1}^z$ to the configuration $C_{v_6}^{v_5}$ where z_1 is the vertex adjacent to z between x and z. Now if we replace the subsequence $v \overset{r}{\underset{3}{\leftarrow}} v_0 \overset{o^*}{\leftarrow} v_6 \overset{r}{\underset{3}{\leftarrow}} v$ by the sequence of moves $\mathcal{S}_1 x \overset{o^*}{\leftarrow} z \overset{r}{\underset{3}{\leftarrow}} y \overset{o^*}{\leftarrow} z_1 \mathcal{S}_2 v_6 \overset{r}{\leftarrow} v_5$ the modified sequence still takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. Similarly we can replace the subsequence of moves in $\mathcal{S}$ of the form $v \overset{r}{\underset{3}{\leftarrow}} v_6 \overset{o^*}{\leftarrow} v_0 \overset{r}{\underset{3}{\leftarrow}} v$by a sequence of moves that does not involve the vertex v. Now let $\mathcal{S}'$ be the sequence of moves obtained from $\mathcal{S}$ by making all possible changes as above. Then in the sequence $\mathcal{S}'$ no move involve the vertex v and still it takes the configuration C_w^u to the configuration $C_{w'}^{u'}$. Hence T'' is also complete $3RJ$-reachable, a contradiction. Hence $d(v_3, x) \not\equiv 0 \mod 3$. Thus we have a member of $\mathcal{T}_3$ as a subtree of T.

Converse part is immediate by applying the Theorem 2.7 and the Lemma 2.8. □

The following corollary characterizes the minimal complete $3RJ$-reachable trees. The proof is immediate by using Lemma 2.8.

Corollary 2.3. A tree is minimal complete $3RJ$-reachable if and only if it is in $\mathcal{T}_3$.

Chapter 3

S-Reachability in Graphs

3.1 Preamble

In Chapter 2 we have introduced complete mRJ-reachable graphs. In this context we have classified the complete mRJ-reachable trees for $m = 0, 1, 2, 3$. That is, we have investigated the trees in which any two configurations are reachable from each other using simple moves of the obstacles and mRJ-moves of the robot for $m \in \{0, m'\}$, for some positive integer m'.

Let S be a finite set of non-negative integers. A graph G is said to be S-*reachable* if there exist a configuration from which the robot can be moved to any vertex using mRJ-moves of the robot for $m \in S$ and simple moves of the obstacles. A graph G is said to be *complete* S*-reachable* if it is S-reachable from every configuration. In Chapter 2 we have classified the trees that are complete S-reachable for $S = \{0\}$, $\{0, 1\}$, $\{0, 2\}$ and $\{0, 3\}$. Thus S-reachability introduced here is a generalization of the mRJ-reachability problem introduced in Chapter 2.

Given a set of non-negative integers $S = \{m_1, m_2, \ldots m_k\}$, as a convention we assume that $m_1 < m_2 < \ldots < m_k$, unless otherwise stated.

Example 3.1. Let u, v be any two vertices in the graph G shown in Figure 3.1. It is easy to see that G is $\{4\}$-reachable from any configuration C_v^u if $u, v \in \{1, 2, \ldots 6\}$. For example, starting with the configuration C_2^1, we can take the robot to the vertex 8 using the sequence of robot moves

$$2 \xleftarrow{o^*} 9 \underset{4}{\xleftarrow{r}} 1 \xleftarrow{o^*} 8 \underset{4}{\xleftarrow{r}} 9.$$

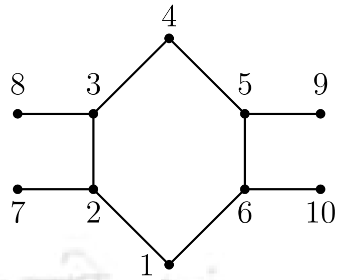

Figure 3.1: A $\{4\}$-reachable graph.

But G is not $\{4\}$-reachable from the configuration C_7^2. In fact it is not $\{4\}$-reachable from any configuration C_v^u if $v \in N(u)$ and $deg(v) = 1$.

The following proposition gives an alternative way to define complete S-reachable graphs.

Proposition 3.1. *A connected graph G is complete S-reachable if and only if every two configurations in G are S-reachable from each other.*

Proof. ($\Rightarrow$) Pick a vertex w in G that is not a cut-vertex. Notice that for each $x, y \in V(G - w)$ the configurations C_x^w and C_y^w are S-reachable from each other (using simple moves of the obstacles). Now given two configurations C_v^u and $C_{v'}^{u'}$, there exists a pair of vertices $x, y \in V(G - w)$ such that C_v^u, C_x^w are S-reachable from each other and $C_{v'}^{u'}$, C_y^w are S-reachable from each other. Hence C_v^u and $C_{v'}^{u'}$ are S-reachable from each other.

($\Leftarrow$) Assume that every pair of configurations in G are S-reachable from each other. Let C_v^u be any configuration of G and $w \in V(G)$. By assumption C_v^w is S-reachable from C_v^u. That is, the robot can be taken to the vertex w. Since w and C_v^u were chosen arbitrarily, so G is S-reachable from C_v^u and hence G is complete S-reachable.

□

Proposition 3.2. *Given a set $S = \{m_1, m_2, \ldots m_k\}$ of non-negative integers with $m_1 \geq 1$. If a connected graph G is complete S-reachable then $\delta(G) \geq 2$.*

Proof. Assume that G has a pendent vertex v and let u be the parent vertex of v. Recall that an mRJ move from a vertex x to a vertex y in G possible if there exist a path of length $m+1$ connecting x and y in G. Consider the configuration C_v^u of G. Since $m_1 \geq 1$, so we cannot take the robot to the vertex u from v directly. Also we cannot move the hole to any other vertex of the graph without disturbing the robot at v. Thus starting with the configuration C_v^u of G we can't take the robot to any other vertex of G. That is the graph G is not $\{m\}$-reachable from the configuration C_u^v, a contradiction. Therefore $\delta(G) \geq 2$. □

Given a block B in a graph G and a cut-vertex $v \in B$. Any vertex u in the component of $G - v$ that intersects with the block B is said to be on the *B-side* of v. For example, in the Figure 3.2 the vertices x, y, z etc. are on the B-side of v but the vertex w is not on the B-side of v.

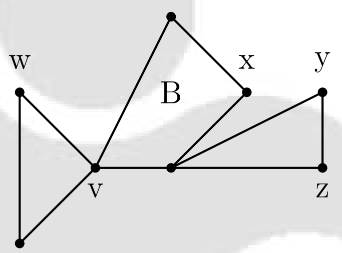

Figure 3.2: The vertex w is not on the B-side of v.

Proposition 3.3. *Given a set $S = \{m_1, m_2, \ldots m_k\}$ of positive integers with $m_1 \geq 1$. Let B be a block in a the graph G and let $v \in B$ be a cut-vertex. If G is complete S-reachable then there exist a path $P(u,v)$ of length m_1+1 connecting v to a vertex u on the B-side of v. Also all the vertices on $P(u,v)$ except v are from the B-side of v.*

Proof. Consider the configuration C_u^v of the graph G, where the vertex u is on the B-side of v. Then to move the robot to any other vertex of G we have to move it first to a vertex on the B-side of v. And it is possible only when there exist a path in G of length m_1+1 connecting v to a vertex x on the B-side v. The second part is immediate from the fact that v is a cut-vertex in G. □

A block in a graph is said to be a *pendent block* if it has only one cut vertex. For example, the graph in Figure 3.2 has exactly two pendent blocks whereas the graph in Figure 3.1 has four pendent blocks. The following corollary gives the lower bound for the number of vertices a pendent block should have in a complete S-reachable graph with $m_1 \geq 1$.

Corollary 3.1. *If G is complete S-reachable and $m_1 \geq 1$, then each pendent block in G must be of order at least $m_1 + 2$.*

In the following section we explore the class of graphs that are complete $\{m\}$-reachable, for some positive integer m.

3.2 Complete $\{m\}$-Reachability

In Chapter 2 we observed that biconnected graphs are complete $\{0\}$-reachable and conversely. But for $m \geq 1$, there exist biconnected graphs that are not complete $\{m\}$-reachable. For example, C_6 is not complete $\{2\}$-reachable. For each $m \geq 1$ there exist biconnected graphs that are not complete S-reachable. Also, there exist complete $\{m\}$-reachable graphs that are not biconnected. For example, the graph in Figure 3.3 is complete $\{4\}$-reachable. So, it would be interesting to know the class of graphs that are complete $\{m\}$-reachable, for some positive integer m. Throughout this section m is a positive integer.

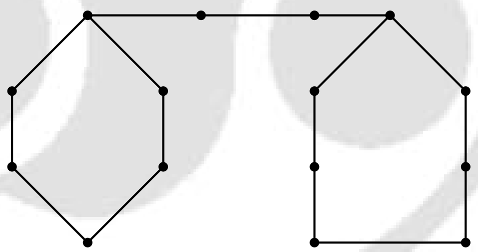

Figure 3.3: A complete $\{4\}$-reachable graph.

Given two positive integers m, n. Let $\mathbb{Z}_n = \{0, 1, 2, \ldots n-1\}$. Define a relation $\sim$ on $\mathbb{Z}_n$ as follows: for any $x, y \in \mathbb{Z}_n$,

$$x \sim y \text{ iff } x \equiv y + t(m+1) \mod (n)$$

for some positive integer t.

It is easy to see that $\sim$ is an equivalence relation on $\mathbb{Z}_n$. So, this relation partitions the set $\mathbb{Z}_n$ into equivalence classes. We have the following result about the partition of $\mathbb{Z}_n$ generated by $\sim$.

Lemma 3.1. *Given two positive integers m, n. If $\mathcal{P}$ is the partition on $\mathbb{Z}_n$ generated by the relation '$\sim$', then $|\mathcal{P}| = gcd(m+1, n)$.*

Proof. Let $k = gcd(m+1, n)$. For $i \in \{0, 1, \cdots k-1\}$, define

$$S_i = \{i + s(m+1) \mod (n) \mid s \in \mathbb{Z}\}.$$

We claim that $\{S_0, S_1, \cdots S_{k-1}\}$ is the required partition on $\mathbb{Z}_n$. Since $\mathbb{Z}_n$ is a group of order n under addition modulo(n) and $k \in \mathbb{Z}_n$, so let $G_k =< k >$ be the subgroup of $\mathbb{Z}_n$ generated by k. Since $k = gcd\{m+1, n\}$, so $G_k =< m+1 >$. If $n = kt$, then G_k is of order t and it has k distinct co-sets in $\mathbb{Z}_n$. Also, these k co-sets are nothing but the sets $S_0, S_1, \cdots S_{k-1}$. Hence $\mathcal{P} = \{S_0, S_1, \cdots S_{k-1}\}$ is a partition of $\mathbb{Z}_n$.

Let $x, y \in S_i$. Then there exists $s_1, s_2 \in \mathbb{Z}$ such that

$$x = i + s_1(m+1) \mod (n), \text{ and } y = i + s_2(m+1) \mod (n).$$

Therefore

$$\begin{aligned} y - x &= (s_2 - s_1)(m+1) \mod (n) \\ \Rightarrow \quad y &= x + s(m+1) \mod (n), \text{ where } s = s_2 - s_1 \\ \Rightarrow \quad x &\sim y \end{aligned}$$

Next assume that $x, y \in \mathbb{Z}_n$ and $x \sim y$. Then there exists $t \in \mathbb{Z}$ such that

$$y = x + t(m+1) \mod (n). \tag{3.1}$$

Since P is a partition of $\mathbb{Z}_n$, so $x \in S_i$ for some $i \in \{0, 1, \cdots k-1\}$. That is

$$x = i + s(m+1) \mod (n) \text{ for some } s \in \mathbb{Z} \tag{3.2}$$

From 3.1 and 3.2 we can conclude that

$$y = i + s'(m+1) \mod (n) \text{ where } s' = s + t \in \mathbb{Z}.$$

which in turn implies that $y \in S_i$. This completes the proof. $\square$

Notice that if u is a non cut-vertex in the graph G then for any set of non-negative integers S, the configurations C_x^u and C_y^u are S-reachable from each other (by means of obstacle moves only). So the vertex set of a bi-connected graph can be partitioned in such a way that C_x^u and C_y^v are S-reachable from each other if and only if u, v belongs to the same set of the partition. Also if this partition has cardinality one then G is complete S-reachable. All these observations leads us to the following corollary and it characterizes the complete $\{m\}$-reachable cycles.

Corollary 3.2. *Given two positive integers m, n with $m < n$. If $k = gcd(m+1, n)$, then the the set of vertices in C_n can be partitioned into k parts such that two configuration C_x^u and C_y^v in C_n are $\{m\}$-reachable if and only if u, v belong to the same set of the partition. Also C_n is complete $\{m\}$-reachable if and only if $k = 1$.*

Proof. Label the vertices of C_n as $0, 1, \cdots n-1$ in clockwise order. Since each mRJ move of the robot covers a distance of $m+1$, so for any $x, y \in V(C_n)$, we can take the robot from x to y if and only if $y = x + t(m+1) \mod (n)$ for some $t \in \mathbb{Z}$. Hence the result follows by using the Lemma 3.1. ☐

Thus, based on the Corollary 3.2 we can say that, a cycle C_n is not complete $\{m\}$-reachable when $gcd(m+1, n) \neq 1$. That is, there are plenty of biconnected graphs which are not complete $\{m\}$-reachable.

Proposition 3.4. *Given a connected graph G and a positive integer m. If G is complete $\{m\}$-reachable, then G satisfies the following*

(a.) $\delta(G) \geq 2$;

(b.) Each pendent block of G must have a path of length $m+1$ with the cut vertex in the block as the initial vertex.

(c.) A pendent block in G has order $m+2$ or more.

(d.) Length of the longest critical path in G is at most $m-1$.

Proof.

(a.) It is immediate from the Proposition 3.2.

(b.) Let B be a pendent block of G and let u be the cut-vertex in B. Consider the configuration C_v^u, where v is any other vertex in B. Since u is the only cut-vertex in B so without disturbing the robot, we can move the hole only to the vertices in the component B of G. So to move the robot to a vertex in $V(G) - V(B)$ first we have to move it to a vertex in $B - u$. But this is possible only when B has a vertex w such that there is a path of length $m + 1$ connecting u and w in G. This completes the proof of (b).

(c.) It is immediate from (b.).

(d.) Assume that $P(u, v) : [u_0, u_1, u_2, \ldots u_k(= v)]$ be a critical path in G with $k \geq m$. Let V_u be the collection of all vertices of G that are not on the v-side of u and let V_v be the collection of all vertices of G that are not on the u-side of v. Then starting with the configuration C_x^u for some $x \in V_u$ we can take it at most to the vertices $u_0, u_1, \ldots u_m$ on the v-side of u, and in all these cases the hole will remain at a vertex in V_u. Let $y \in V_u$. For some $i = 0, 1, \ldots m$, consider the configuration $C_y^{u_i}$. Suppose we want to take the robot to a vertex z on the v-side of u_m. In order to realize this, before it, the hole must move from y to z. But this not possible without disturbing the robot at u_i, because u_i is a cut-vertex. Hence G is not $\{m\}$-reachable from each of the configuration C_x^u, a contradiction. Thus the length of the longest critical path in G cannot exceed $m - 1$.

□

Suppose that C_n is a cycle of order $m+2$ or more and let $k = gcd(m+1, n)$. Starting from a vertex $u \in V(C_n)$, label the vertices of C_n with $1, 2, \cdots k$ periodically in clockwise order. Here we note that for $u, v \in \{1, 2, \ldots k\}$ the configurations C_x^u and C_y^v are $\{m\}$-reachable if and only if $u = v$.

A graph G is said to be a *chordless* graph if every non-trivial block in G is a cycle. For example, trees are chordless graphs. Also the graphs in Figure 3.1, Figure 3.2, and Figure 3.3 are chordless. Given two cycles C_{n_1} and C_{n_2}, we denote by $C(n_1, n_2; r)$ the graph obtained by joining a vertex $u \in$

$V(C_{n_1})$ to a vertex $v \in V(C_{n_2})$ by a path $P(u,v)$ of length r. Thus $C(6,7;3)$ represents the graph in Figure 3.2. Clearly, $C(n_1,n_2;r)$ is a chordless graph. We now characterize the graphs of the form $C(n_1,n_2;r)$ that are complete $\{m\}$-reachable.

Remark 3.1. In view of Proposition 3.4, if the graph $C(n_1,n_2;r)$ is complete $\{m\}$-reachable then $n_1 \geq m+2$, $n_2 \geq m+2$, and $r \leq m-1$.

The following example suggests that the conditions listed in the Remark 3.1 are not sufficient for the graph $C(n_1,n_2;r)$ to be complete $\{m\}$-reachable.

Example 3.2. It is easy to see that the graph $C(15,16;r)$ is not complete $\{11\}$-reachable for $r \geq 7$. But it is complete $\{11\}$-reachable for $r \leq 6$.

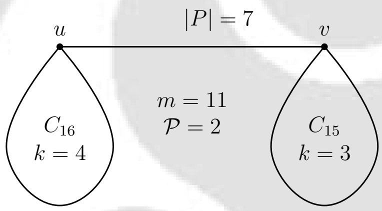

Figure 3.4: A graph that is not complete $\{11\}$-reachable.

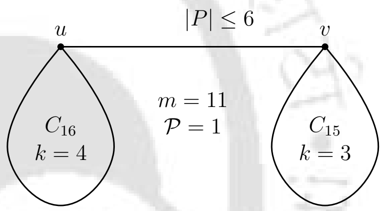

Figure 3.5: A complete $\{11\}$-reachable graph.

Given the graph $C(n_1,n_2;r)$ and a positive integer m satisfying the inequalities of the Remark 3.1, we use the following as the standard labeling of $C(n_1,n_2;r)$: Let $k_i = gcd(m+1,n_i)$, for $i=1,2$. Label the vertices of C_{n_1} (respectively C_{n_2}) beginning with the vertex u (respectively v), using $1,2,\cdots k_1$ (respectively with $1,2,\cdots k_2$) periodically in clockwise order. Also label the internal vertices of the path $P(u,v)$ using $2,3,\ldots r$ starting from the vertex in $P(u,v)$ adjacent to u. We refer this as the m-*labeling* of $C(n_1,n_2;r)$. For example, a 5-labeling of the graph $C(9,8;3)$ is shown in figure 3.6.

Remark 3.2. In a m-labeling of $C(n_1,n_2;r)$ the vertices in either side of u (respectively v) in C_1 (respectively C_2) at distances $k_1,\ 2k_1,\cdots$ (respectively at distances $k_2,\ 2k_2,\cdots$) are labeled with 1. Also, by Corollary 3.2, if we restrict the robot moves only to the vertices in C_{n_1} (respectively C_{n_2}), the

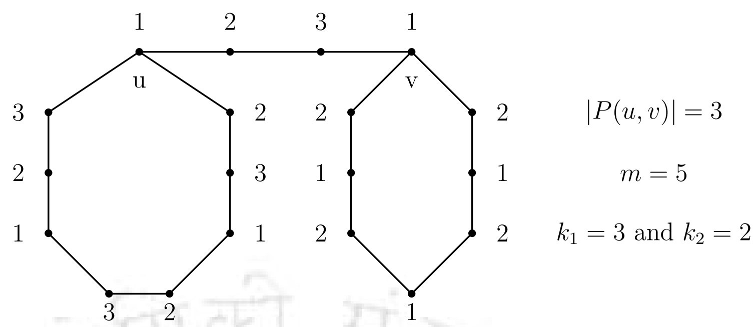

Figure 3.6: A 5-labeling of $C(9, 8; 2)$.

reachability relation will partition the vertices of C_{n_1} (respectively C_{n_2}) into k_1 (respectively k_2) sets with the vertices having identical label are in the same set of the partition.

Lemma 3.2. Given the graph $C(n_1, n_2; r)$ and a positive integer m such that $m \leq min\{n_1, n_2\} - 2$. If $C(n_1, n_2; r)$ is complete $\{m\}$-reachable then $r \leq m - k_{max}$.

Proof. Consider the m-labeling of the graph $C(n_1, n_2; r)$. Also, consider the configuration C_y^x of $C(n_1, n_2; r)$, where x is a vertex with label 1 in $C_{n_1} - u$. Then to move the robot to a vertex in C_{n_1} with a label different from 1 we must first take it to a vertex in $C_{n_2} - v$ (directly from a vertex with label 1 in $C_{n_1} - u$) and it is possible only when $|P| \leq m - k_1$. Otherwise all vertices in C_{n_2} will be at a distance of $m + 2$ or more from any of the vertices having label 1 in $C_{n_1} - u$. In similar fashion we can show that $|P| \leq m - k_2$. The proof is complete. □

Lemma 3.3. Given the graph $C(n_1, n_2; r)$ and an even positive integer m such that $m \leq min\{n_1, n_2\} - 2$. If $C(n_1, n_2; r)$ is complete $\{m\}$-reachable then at least one of n_1, n_2 must be odd.

Proof. Assume that both n_1 and n_2 are even. Let $k_1 = gcd(m + 1, n_1)$ and $k_2 = gcd(m + 1, n_2)$. Then both k_1 and k_2 are even. Consider the m-labeling of the graph $C(n_1, n_2; r)$. Since $C(n_1, n_2; r)$ is complete $\{m\}$-reachable, so $r \leq m - k_{max}$. Let $\mathfrak{C}$ be the collection of all configurations in $C(n_1, n_2; r)$. Notice that, each mRJ move of the robot covers a path of length $m + 1$, i.e. a path of even length. We now consider the following cases:

Case 1. r is odd. In this case, let

$$\begin{aligned}\mathfrak{C}_1 &= \{C_y^x \mid x \in V(C_{n_1}) \cup V(C_{n_2}) \text{ and label of } x \text{ is even, or}\\ &\quad x \in P(u,v), d(x,u) \text{ is odd and } y \text{ is on the } u\text{-side of } x, \text{ or}\\ &\quad x \in P(u,v), d(x,u) \text{ is even and } y \text{ is on the } v\text{-side of } x\}\end{aligned}$$

and $\mathfrak{C}_2 = \mathfrak{C} - \mathfrak{C}_1$.

Case 2. r is even. In this case, let

$$\begin{aligned}\mathfrak{C}_2 &= \{C_y^x \mid x \in V(C_{n_1}) \cup V(C_{n_2}) \text{ and label of } x \text{ is even, or}\\ &\quad x \in P(u,v), d(x,u) \text{ is odd and } y \text{ is on the } u\text{-side of } x, \text{ or}\\ &\quad x \in P(u,v), d(x,u) \text{ is odd and } y \text{ is on the } v\text{-side of } x\}\end{aligned}$$

and $\mathfrak{C}_2 = \mathfrak{C} - \mathfrak{C}_1$.

Since both k_1 and k_2 are even, so in either case $\mathfrak{C}_1 \neq \Phi$ and $\mathfrak{C}_2 \neq \Phi$. Notice that, in either case if

$$C_y^x \;:\; y \overset{r}{\underset{m}{\leftarrow}} x \;:\; C_x^y$$

then either $C_y^x, C_x^y \in \mathfrak{C}_1$ or $C_y^x, C_x^y \in \mathfrak{C}_2$. Also, if

$$C_y^x \;:\; y \overset{o^*}{\leftarrow} x \;:\; C_z^x$$

then either $C_y^x, C_z^x \in \mathfrak{C}_1$ or $C_y^x, C_z^x \in \mathfrak{C}_2$. So, two configurations $C_{y_1}^{x_1}$ and $C_{y_2}^{x_2}$ in $C(n_1,n_2;r)$ are $\{m\}$-reachable from each other if and only if either $C_{y_1}^{x_1}$, $C_{y_2}^{x_2} \in \mathfrak{C}_1$ or $C_{y_1}^{x_1}$, $C_{y_2}^{x_2} \in \mathfrak{C}_2$, a contradiction. Hence, at least one of n_1 and n_2 must be odd. □

Given a graph $C(n_1,n_2;r)$ and a positive integer $m \leq min\{n_1,n_2\}-2$. Let $V_1 = \{a_1, a_2, \cdots a_{k_1}\}$ and $V_2 = \{b_1, b_2, \cdots b_{k_2}\}$, where $k_i = gcd(n_i, m+1)$ for $i = 1,2$. We construct a graph $mC(n_1,n_2;r)$ with vertex set $V(mC(n_1,n_2;r)) = V_1 \cup V_2$. In $mC(n_1,n_2;r)$ any edge joins a vertex $a_i \in V_1$ to a vertex $b_j \in V_2$. Also $a_i \in V_1$ adjacent to $b_j \in V_2$ if one of the following holds:

(a) $\exists\ s_i, s_j \in \mathbb{W}$ satisfying $s_ik_1 + s_jk_2 = m-i-j-r+3$ such that $s_t > 0$ when $t = 1$.

(b) a_{k_1-i+2} adjacent to b_j, when $i \neq 1$.

(c) a_i is adjacent to b_{k_2-j+2} when $j \neq 1$.

Example 3.3. *The graph $17C(27,30;4)$ is shown in the Figure 3.7. Notice that this graph is connected and also the graph $C(27,30;4)$ is complete $\{17\}$-reachable.*

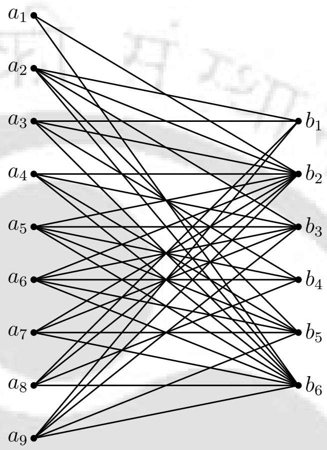

Figure 3.7: The graph $17C(27,30;4)$.

Clearly $mC(n_1,n_2;r)$ is bipartite. We refer the graph $mC(n_1,n_2;r)$ as the *m-configuration graph* of $C(n_1,n_2;r)$.

Lemma 3.4. *Given the vertices a_i and b_j in the graph $mC(n_1,n_2;r)$. If a_i and b_j are adjacent then there exists vertices $\alpha \in C_{n_1} - u$ and $\beta \in C_{n_2} - v$ with labels i and j respectively, such that $d(\alpha,\beta) = m+1$.*

Proof. We consider the following cases
Case 1: $\exists$ $s_i, s_j \in \mathbb{W}$ satisfying $s_ik_1 + s_jk_2 = m - i - j - r + 3$ such that $s_t > 0$ when $t = 1$. In this case, let α be the vertex at a distance $s_ik_1 + i - 1$ while traversing C_{n_1} in clockwise direction starting from u and β be the vertex at a distance $s_jk_2 + j - 1$ while traversing C_{n_2} in clockwise direction starting from v. Clearly, label of α is i and that of β is j. Also,

$$\begin{aligned} d(\alpha,\beta) &= d(\alpha,u) + |P(u,v)| + d(v,\beta) \\ &= s_ik_1 + i - 1 + r + s_jk_2 + j - 1 \\ &= m+1 \end{aligned}$$

Case 2: a_{k_1-i+2} **adjacent to** b_j**, when** $i \neq 1$**.** That is, $\exists\ s_i, s_j \in \mathbb{W}$ satisfying $s_i k_1 + s_j k_2 = m - (k_1 - i + 2) - j - r + 3$ such that $s_t > 0$ when $t = 1$. In this case, let α be the vertex at distance $s_i k_1 + k_1 - i + 1$ while traversing C_{n_1} in anti-clockwise direction starting from u and β be the vertex at distance $s_j k_2 + j - 1$ while traversing C_{n_2} in clockwise direction starting from v. Clearly, label of α is i and that of β is j. Also,

$$\begin{aligned} d(\alpha,\beta) &= d(\alpha,u) + |P(u,v)| + d(v,\beta) \\ &= s_i k_1 + k_1 - i + 1 + r + s_j k_2 + j - 1 \\ &= m + 1 \end{aligned}$$

Case 3: a_i **is adjacent to** b_{k_2-j+2}**, when** $j \neq 1$**.** This is similar to case 2. Hence the proof is complete. □

The following proposition gives a necessary and sufficient condition for the graph $C(n_1, n_2; r)$ to be complete $\{m\}$-reachable.

Theorem 3.1. *Given a graph $C(n_1, n_2; r)$ and a positive integer m such that $m \leq min\{n_1, n_2\} - 2$. The graph $C(n_1, n_2; r)$ is complete $\{m\}$-reachable if and only if $mC(n_1, n_2; r)$ is connected.*

Proof. Consider the graph $C(n_1, n_2; r)$ with its m-labeling and let $r \leq m - k_{max}$. First assume that $mC(n_1, n_2; r)$ is connected. Let C_y^x and $C_{y'}^{x'}$ be any two configurations in $C(n_1, n_2; r)$. We claim that C_y^x and $C_{y'}^{x'}$ are $\{m\}$-reachable from each other. Without loss of generality assume that $x \in C_{n_1} - u$, $x' \in C_{n_2} - v$. Also, let i, j be the labels of x and x' respectively. Since $mC(n_1, n_2; r)$ is connected, so let $P(a_i, b_j)$ be a path connecting a_i and b_j in $mC(n_1, n_2; r)$. Let $|P(a_i, b_j)| = 2l + 1$. We apply induction on l.

Base step : Let $l = 0$. Then, by the Lemma 3.4 there exist vertices $\alpha \in C_{n_1} - u$ and $\beta \in C_{n_1} - v$ with labels i and j respectively, such that $d(\alpha, \beta) = m+1$. But, by the Corollary 3.2 we can claim that, C_y^x, C_y^α are $\{m\}$-reachable from each other, and $C_{y'}^{x'}$, $C_{y'}^\beta$ are $\{m\}$-reachable from each other. Finally, since $d(\alpha, \beta) = m + 1$, so the configuration $C_{y'}^\beta$ is $\{m\}$-reachable from C_y^α by using the sequence of moves

$$y \overset{o^*}{\longleftarrow} \beta \underset{m}{\overset{r}{\longleftarrow}} \alpha \overset{o^*}{\longleftarrow} y'.$$

This proves our claim.

Induction step : As an induction hypothesis, assume that the result holds for $l = p - 1$. Let $|P(a_i, b_j)| = 2p + 1$ and $[a_i b_{j_1} a_{i_1} b_{j_2} a_{i_2} \dots b_{j_p} a_{i_p} b_j]$ be a path of lenght $2p + 1$ connecting a_i and b_j. Let β be a vertex in $C_{n_2} - v$ with label j_p. The inductive hypothesis implies that $C_{y'}^{\beta}$ is $\{m\}$-reachable from C_y^x. Let $\alpha \in C_{n_1} - u$ and $\beta' \in C_{n_2} - v$ be vertices with labels i_p and j respectively such that $d(\beta, \alpha) = d(\alpha, \beta') = m + 1$. Then $C_{y'}^{\beta'}$ is $\{m\}$-reachable $C_{y'}^{\beta}$ by using the sequence of moves

$$y' \overset{o^*}{\longleftarrow} \alpha \underset{r}{\overset{m}{\longleftarrow}} \beta \overset{o^*}{\longleftarrow} \beta' \underset{r}{\overset{m}{\longleftarrow}} \alpha \overset{o^*}{\longleftarrow} y'.$$

Finally, by the Corollary 3.2 we can claim that, C_y^x and $C_{y'}^{x'}$ are $\{m\}$-reachable from each other. This completes the induction step.

Conversely, assume that the graph $C(n_1, n_2; r)$ is complete $\{m\}$-reachable. Consider the vertices a_i, b_j in the graph $mC(n_1, n_2; r)$. It is enough to show that a_i and b_j are connected. Conside the vertices $\alpha \in C_{n_1} - u$ and $\beta \in C_{n_1} - v$ with labels i and j respectively. Then for any vertex γ, the configurations C_γ^α and C_γ^β are $\{m\}$-reachable. Let S be a minimal sequence of moves that takes C_γ^α to C_γ^β. We apply induction on the number of robot moves in S. Let $r(S)$ denotes the number of robot moves in S.

Base Step : Let $r(S) = 1$. Then there exist a path $P(\alpha, \beta)$ of length $m+1$ connecting α and β in $C(n_1, n_2; r)$. Let $P(\alpha, u)$ be the sub-path of $P(\alpha, \beta)$ connecting α to u and $P(v, \beta)$ be the sub-path of $P(\alpha, \beta)$ connecting β to v. Then, the path $P(\alpha, \beta)$ consist of the paths $P(\alpha, u)$, $P(u, v)$ and $P(v, \beta)$.

Now if $P(\alpha, u)$ is the segment of C_{n_1} traversed in clockwise (respectively, anti-clockwise) direction starting from u, then there exist $s_i \in \mathbb{W}$ such that $|P(\alpha, u)| = s_i k_1 + i - 1$ (respectively, $|P(\alpha, u)| = s_i k_1 + k_1 - i + 1$). Similarly, there exist $s_j \in \mathbb{W}$ such that $|P(\alpha, u)| = s_j k_2 + j - 1$ or $s_j k_2 + k_2 - j + 1$. We consider the following cases:

Case 1 : $|P(\alpha, u)| = s_i k_1 + i - 1$ **and** $|P(\alpha, u)| = s_j k_2 + j - 1$**.** In this case $s_i k_1 + s_j k_2 = m - i - j - r + 3$, because $|P(\alpha, \beta)| = m + 1$. Therefore, a_i and b_j are adjacent.

Case 2 : $|P(\alpha, u)| = s_i k_1 + i - 1$ **and** $|P(\alpha, u)| = s_j k_2 + k_2 - j + 1$**.** In this case $s_i k_1 + s_j k_2 = m - i - (k_2 - j + 2) - r + 3$, because $|P(\alpha, \beta)| = m + 1$. Therefore, a_i, $b_{k_2 - j + 2}$ are adjacent and this in turn implies that a_i, b_j are adjacent, because $k_2 - (k_2 - j + 2) + 2 = j$. The proof for the remaining cases can also be argued in similar way.

Induction step : As an induction hypothesis, assume that the result holds for a sequence consisting of t robot moves. Let $r(S) = t + 1$. Let $\beta \xleftarrow[m]{r} \beta'$ be the last robot move in S and S' be the subsequence of S consisting of all moves in S prior to the move $\beta \xleftarrow[m]{r} \beta'$. Without loss of generality, assume that $\beta' \in C_{n_1} - u$. Let i' be the label of β' in $C(n_1, n_2; r)$. Then by induction hypothesis a_i and $a_{i'}$ are connected in $mC(n_1, n_2; r)$. Also, the sequence of moves S' takes the configuration C^{α}_{γ} to $C^{\beta'}_{\beta}$. Further, notice that the C^{β}_{γ} can be achived from $C^{\beta'}_{\beta}$ by a sequence of moves that contains only one robot move. So, arguments similar to the base step can be used to conclude that $a_{i'}$ and b_j are connected in $mC(n_1, n_2; r)$. Hence a_i and b_j are connected.

□

3.3 {1, 2}-Reachability

In this section we characterize the graphs that are complete {1, 2}-reachable. We have also obtained necessary and sufficient conditions for a tree to be {1, 2}-reachable.

Lemma 3.5. *Given a path $P(u, v) : [u = u_0, u_1, \ldots u_k = v]$ of length 2 or more. The {1, 2}-reachability relation partitions the collection $\mathfrak{C}$ of all configurations on P into the following three sets $\{C^{u_1}_{u}\}$, $\{C^{u_{k-1}}_{v}\}$ and $\mathfrak{C}_1 = \mathfrak{C} - \{C^{u_1}_{u}, C^{u_{k-1}}_{v}\}$*

Proof. It is enough to show that any two configurations in $\mathfrak{C}_1$ are {1, 2}-reachable. Let C^{x}_{y} and $C^{x'}_{y'}$ be any two configurations in $\mathfrak{C}_1$. We apply induction on k to prove our claim.

Base step: $k = 2$.

In this case $P(u, v) : [u = u_0, u_1, u_2 = v]$. Therefore, $\mathfrak{C}_1 = \{C^{u}_{u_1}, C^{u}_{v}, C^{v}_{u_1}, C^{v}_{u}, \}$. Since

$$C^{u}_{u_1} \;:\; u_1 \xleftarrow{o} v \;:\; C^{u}_{v} \;:\; v \xleftarrow[1]{r} u \;:\; C^{v}_{u} \;:\; u \xleftarrow{o} u_1 \;:\; C^{v}_{u_1},$$

so any two configurations in $\mathfrak{C}_1$ are {1, 2}-reachable. This completes the base step.

Induction Step :

As an induction hypothesis, assume that the result is true for $k = r \geq 3$. We prove that it is true for $k = r+1$.

Consider the path $P(u,v) : [u = u_0 u_1 \dots u_{r+1} = v]$. Let $\mathfrak{C}$ and $\mathfrak{C}'$ be the collection of all configurations on the paths P and $P - u_{r+1}$. It is enough to show that any two configurations in $\mathfrak{C}_1 = \mathfrak{C} - \{C_{u_0}^{u_1}, C_{u_{r+1}}^{u_r}\}$ are $\{1,2\}$-reachable.

By induction hypothesis any two configurations in $\mathfrak{C}'_1 = \mathfrak{C}' - \{C_{u_0}^{u_1}, C_{u_r}^{u_{r-1}}\}$ are $\{1,2\}$-reachable. Also,

$$\begin{aligned}\mathfrak{C}_1 &= \mathfrak{C}'_1 \cup \{C_{u_i}^{u_{r+1}} \mid i = 0,1,2\dots r\} \cup \\ &\quad \{C_{u_{r+1}}^{u_i} \mid i = 0,1,2\dots r-1\} \cup \{C_{u_r}^{u_{r-1}}\}.\end{aligned}$$

We note the following observations

(i) Any two configurations in the collection $\{C_{u_i}^{u_{r+1}} \mid i = 0,1,2\dots r\}$ are $\{1,2\}$-reachable by a sequence of obstacle moves only.

(ii) $C_{u_{r-1}}^{u_r} \in \mathfrak{C}'_1$ and $C_{u_{r-1}}^{u_r}$ is $\{1,2\}$-reachable from $C_{u_r}^{u_{r+1}}$ by using the following sequence of moves

$$C_{u_{r-1}}^{u_r} \;:\; u_{r-1} \overset{o}{\leftarrow} u_{r-2} \underset{1}{\overset{r}{\leftarrow}} u_r \overset{o}{\leftarrow} u_{r+1} \underset{2}{\overset{r}{\leftarrow}} u_{r-2} \overset{o}{\leftarrow} u_{r-1} \overset{o}{\leftarrow} u_r \;:\; C_{u_r}^{u_{r+1}}.$$

(iii) $C_{u_r}^{u_{r-2}} \in \mathfrak{C}'_1$ and $C_{u_r}^{u_{r-1}}$ is $\{1,2\}$-reachable from $C_{u_r}^{u_{r-2}}$ by using the sequence of moves

$$C_{u_r}^{u_{r-2}} \;:\; u_r \overset{o}{\leftarrow} u_{r+1} \underset{2}{\overset{r}{\leftarrow}} u_{r-2} \overset{o}{\leftarrow} u_{r-1} \underset{1}{\overset{r}{\leftarrow}} u_{r+1} \overset{o}{\leftarrow} u_r \;:\; C_{u_r}^{u_{r-1}}.$$

(iv) For $0 \leq i \leq r-2$, the configuration $C_{u_{r+1}}^{u_i}$ is $\{1,2\}$-reachable from $C_{u_{r+1}}^{u_{i+1}}$ by using the sequence of move

$$C_{u_{r+1}}^{u_{i+1}} \;:\; u_{r+1} \overset{o^*}{\longleftarrow} u_{i+3} \underset{1}{\overset{r}{\leftarrow}} u_{i+1} \overset{o}{\leftarrow} u_i \underset{2}{\overset{r}{\leftarrow}} u_{i+3} \overset{o^*}{\longleftarrow} u_{r+1} \;:\; C_{u_{r+1}}^{u_i}.$$

So, any two configurations in $\{C_{u_{r+1}}^{u_i} \mid i = 0,1,2\dots r-1\}$ are $\{1,2\}$-reachable from each other.

(v) $C_{u_{r+1}}^{u_{r-1}}$ is $\{1,2\}$-reachable from $C_{u_{r-1}}^{u_{r+1}}$ by using the move $u_{r-1} \underset{1}{\overset{r}{\leftarrow}} u_{r+1}$ and $C_{u_{r+1}}^{u_{r-1}} \in \{C_{u_{r+1}}^{u_i} \mid i = 0,1,2\dots r-1\}$.

From (i) and (ii) we can infer that any two configurations in $\mathfrak{C}'_1 \cup \{C_{u_i}^{u_{r+1}} \mid i = 0, 1, 2 \ldots r\}$ are $\{1,2\}$-reachable. Also, from (iii) we can conclude that any two configurations in $\mathfrak{C}'_1 \cup \{C_{u_r}^{u_{r-1}}\}$ are $\{1,2\}$-reachable. Further, from (iv) and (v) we can infer that any two configurations in $\{C_{u_{r+1}}^{u_i} \mid i = 0, 1, 2 \ldots r-1\} \cup \{C_{u_i}^{u_{r+1}} \mid i = 0, 1, 2 \ldots r\}$ are $\{1,2\}$-reachable. Thus, from all these observations we can conclude that any two configuration in $\mathfrak{C}_1$ are $\{1,2\}$-reachable from each other. This completes the induction step.

□

It is easy to see that a graph with a pendent vertex cannot be complete $\{1,2\}$-reachable. Thus it will be interesting to know the trees that are $\{1,2\}$-reachable. The following corollary fulfills this aspect.

Corollary 3.3. *A tree T is $\{1,2\}$-reachable from a configuration C_v^u if and only if the following condition holds.*

(i). $D(T) \geq 3$.

(ii). $deg(v) \neq 1$ or $d(u,v) \neq 1$.

Lemma 3.6. *A non-separable graph of order 3 or more is complete $\{1,2\}$-reachable.*

Proof. Let G be a non-separable graph of order 3 or more. Notice that for any vertex $u, x, y \in V(G)$, the configurations C_x^u and C_y^u are $\{1,2\}$-reachable by a sequence of moves consisting of obstacle moves only.

Let C_v^u and C_y^x be any two configurations on G. If $u = x$ then there is nothing to prove. So, let $u \neq x$. Also, without loss of generality assume that $d(u,v) = 1$. Let $P(u,x) : [u = u_0, u_1, \ldots u_r = x]$ be a path connecting u and x in G. If v is on the path connecting $P(u,x)$, then by the Lemma 3.5, the configuration C_v^x is $\{1,2\}$-reachable from C_v^u and this in turn implies that C_y^x is $\{1,2\}$-reachable from C_v^u.

Otherwise, let $w \in N(v) - \{u\}$. We consider the following cases.

Case 1. $w \neq u_1$. In this case, by using the sequence of moves

$$v \overset{o}{\leftarrow} w \underset{1}{\overset{r}{\leftarrow}} u \overset{o}{\leftarrow} u_1 \underset{2}{\overset{r}{\leftarrow}} w \overset{o}{\leftarrow} v \underset{1}{\overset{r}{\leftarrow}} u_1$$

we can achieve the configuration $C^v_{u_1}$. Hence, by the Lemma 3.5, we can achieve the configuration C^x_y.

Case 2. $w = u_1$.

In this case, we can achieve the configuration $C^u_{u_1}$ by the single move $v \overset{o}{\leftarrow} u_1$. Hence by the Lemma 3.5, the configuration C^x_y is $\{1,2\}$-reachable.

Hence the proof is complete. □

Theorem 3.2. *A connected graph G is complete $\{1,2\}$-reachable if and only if $\delta(G) \geq 2$.*

Proof. If G is complete $\{1,2\}$-reachable then obviously $\delta(G) \geq 2$. Otherwise, if $u, v \in V(G)$ be such that $deg(u) = 1$ and $v \in N(u)$, then G is not $\{1,2\}$-reachable from the configuration C^v_u.

Conversely, suppose that $\delta(G) \geq 2$ and G is connected. Then each pendent block of G must be of order 3 or more. Hence, each pendent block of G is complete $\{1,2\}$-reachable.

Now, let C^u_v and C^x_y be any two configurations in G. Without loss of generality, assume that $d(u,v) = d(x,y) = 1$. If u, v, x, y are in the same pendent block of G then there is nothing to prove. Let $P(u,x)$ be a path connecting u and x in G. If $v, y \in P(u,x)$ then by the Lemma 3.5, C^u_v and C^x_y are $\{1,2\}$-reachable from each other. Otherwise, assume that $v \notin P(u,x)$. We consider the following two cases.

Case 1. u **is a cut-vertex.** In this case, there must be a vertex, say w, on the v-side of u such that $d(u,w) = 2$ because $\delta(G) \geq 2$. So, the configuration C^w_v is $\{1,2\}$-reachable from C^u_v by using the sequence of moves

$$v \overset{o}{\leftarrow} w \underset{1}{\overset{r}{\leftarrow}} u \overset{o}{\leftarrow} v.$$

Now, the path $P' : wvuP(u,x)$ contains the vertex v and so by the Lemma 3.5, C^w_v and C^x_y are $\{1,2\}$-reachable from each other. Thus, C^u_v and C^x_y are also $\{1,2\}$-reachable from each other.

Case 2. u **is not a cut-vertex.** In this case u and v must belong to a block of order three or more. And so by the Lemma 3.6 the C^u_v and C^v_u are $\{1,2\}$-reachable from each other. Now the path $P' : vuP(u,x)$ contains the vertex v and so by the Lemma 3.5, C^v_u and C^x_y are $\{1,2\}$-reachable from each other. Which in turn implies that C_v^u and C_y^x are $\{1, 2\}$-reachable from each other.

Hence the proof is complete.

□

Chapter 4

Motion Planning in Product Graphs

4.1 Preamble

Let G and H be two graphs with vertex set $\{u_1, u_2, \ldots u_{n_1}\}$ and $\{v_1, v_2, \ldots v_{n_2}\}$, respectively. A *product graph* of G and H is a new graph, denoted by $G * H$, whose vertex set is $V(G) \times V(H)$, the Cartesian product of $V(G)$ and $V(H)$. However, adjacency rule for each product is different. Given two vertices (u_i, v_j) and (u_p, v_q), their adjacency in $G * H$ is determined entirely by the adjacency (or equality or non adjacency) of u_i and u_p in G and that of v_j and v_q in H. Consider an edge $e = \{(u_i, v_j), (u_p, v_q)\}$ in $G * H$. The edge e is said to be a H-*edge* if $\{u_i = u_p$ and $\{v_j, v_q\} \in E(H)\}$ or a G-*edge* if $\{v_j = v_q$ and $\{u_i, u_p\} \in E(G)\}$ or a N-*edge* otherwise. Throughout this chapter all our graphs are simple, finite, and non-trivial. Also, we use v^u to denote the vertex (u, v) in $G * H$.

Definition 4.1 (G-length and H-length)**.** Given a path P in $G * H$. By G-*length* of P we mean the number of G-edges in P. We define H-*length* and N-*length* of P in similar way. We use $l_G(P)$, $l_H(P)$ and $l_N(P)$ to denote the G-length, H-length and N-length of P, respectively. Also, we use $l(P)$ to denote the length of P.

Definition 4.2. By a G-*move* of the robot in $G * H$ we mean a move of the robot along a G-edge. We define H-*move* and N-*move* of the robot in a similar way.

Definition 4.3. Consider the vertices u^i, v^j in $G * H$. By *G-distance* between u^i and v^j in $G * H$, denoted by $d_G(u^i, v^j)$, we mean the distance between i and j in G and by *H-distance* between u^i and v^j in $G * H$, denoted by $d_H(u^i, v^j)$, we mean the distance between u and v in H.

The four standard product graphs are: the Cartesian product, the direct product, the lexicographic product and the strong product. These products have been widely investigated in the context of network design [10] and have many significant applications. Imrich et. al., [13] have studied the graph products and their structural properties more deeply and noticed that the four standard products are the most relevant ones. For example, one of the important common property of the four standard products is that if we take the product of two simple graphs, then we will get a simple graph. Also, all the four products are associative and except the lexicographic product all the other three standard products are commutative.

Like the graph operations, graph products are also used in constructing many important classes of graphs. For example the grid of order n^2 can also be viewed as the Cartesian product of P_n with itself. Figure 4.1 shows the Cartesian product of P_6 with itself. For clarity, P_6 is displayed to the right of the product graph.

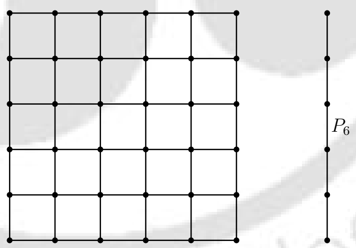

Figure 4.1: The graph P_6 and the grid graph of order 6^2.

In Chapter 1, we explored the problem of robot motion planning on graphs. It is known that, computing the minimum number of moves for robot motion planning problem in a graph is NP-complete in general and also when restricted to planar graphs [24]. In this chapter, we focus on the four standard products and show that the motion planning problem on graphs can be solved efficiently when restricted to these product graphs. We give the minimum number of moves required for the motion planning problem in Graph Cartesian product, Graph Strong product and Graph Lexicographic Product of two graphs.

Definition 4.4. Given an initial configuration of a graph G with the robot at u, a sequence of moves that takes the robot from u to a vertex v in G is called *minimum* if it is a sequence with minimum number of moves among all sequences of moves that takes the robot from u to v in G.

Example 4.1. Consider the configuration C_v^u as shown in Figure 4.2. It easy to see that the minimum number of moves required to take the robot from u to v is 21 and the path traced by the robot moves in any minimum sequence of moves is not the shortest path from u to v. In fact, it is easy to see that to

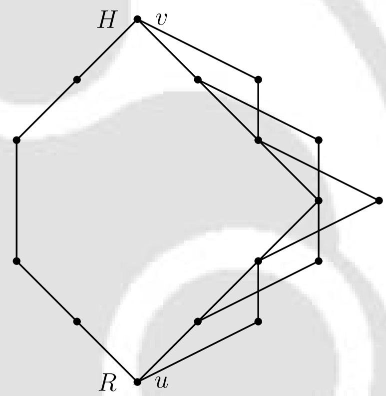

Figure 4.2: Path traced by robot moves is not a shortest path.

move the robot from u to v along the shortest path requires at least 45 moves.

Remark 4.1. In case of each of the product graphs discussed here, the path traced by the robot moves in a minimum sequence of moves that takes the robot from a source vertex to a target vertex is a shortest path.

Grid graphs on n^2 vertices has been extensively studied in the context of (n^2-1)-puzzle [19, 25, 28, 29, 34]. Ratner and Warmuth [28] have shown that finding a solution with minimum number of moves for the (n^2-1)-puzzle is NP-hard. They give an approximation algorithm that produces a solution of length at most constant times the length of the optimum solution. Ian Parberry, in [25], gives an algorithm that solves the (n^2-1)-puzzle in at most

$5n^3 + O(n^2)$ moves. We show that some of the ideas presented in [25,28] can be generalized to all Cartesian products (see Propositions 4.2, 4.3, 4.4 and 4.5). We apply these generalized results to compute the minimum number of moves required in motion planning problem for the Cartesian product of two given graphs.

4.2 Graph Cartesian Product

The Cartesian product of two graphs G and H, denoted as $G\Box H$, is a graph with vertex set $V(G) \times V(H)$ in which (u_i, v_j) and (u_p, v_q) are adjacent if one of the following conditions holds:

(i) $u_i = u_p$ and $\{v_j, v_q\} \in E(H)$.

(ii) $v_j = v_q$ and $\{u_i, u_p\} \in E(G)$.

The graphs G and H are known as the factors of $G\Box H$. Notice that each edge in $G\Box H$ is either a H-edge or a G-edge. Cartesian product has been widely investigated and is arguably the most natural one among all products. It is associative, commutative and distributes over disjoint union. Now onwards, G and H are finite simple graphs with $V(G) = \{1, 2, \ldots m\}$.

Remark 4.2. One well known fact about Cartesian product is that, we may also view $G\Box H$ as the graph obtained from G by replacing each vertex $i \in V(G)$ by a copy H^i (say) of H and each of its edges $\{i, j\}$ with $|V(H)|$ edges joining corresponding vertices of H^i and H^j.

In view of Remark 4.2, for any vertex $i \in V(G)$ we refer the copy of H in $G\Box H$ corresponding to the vertex i as the i^{th} *copy* of H in $G\Box H$. Also for any $i \in V(G)$, notice that u^i denotes the vertex in H^i corresponding to the vertex $u \in V(H)$. Further, commutativity of the Cartesian product allows us to view $G\Box H$ as the graph obtained from H by replacing each of its vertices by a copy of G and each of its edges $\{u, v\}$ with $|V(G)|$ edges joining corresponding vertices of G in the two copies G^u and G^v.

Example 4.2. Consider the Cartesian product of P_2 with C_4. The graph $P_2\Box C_4$ is shown in Figure 4.3. For clarity, P_2 and C_4 are displayed above and to the right of the product graph, respectively.

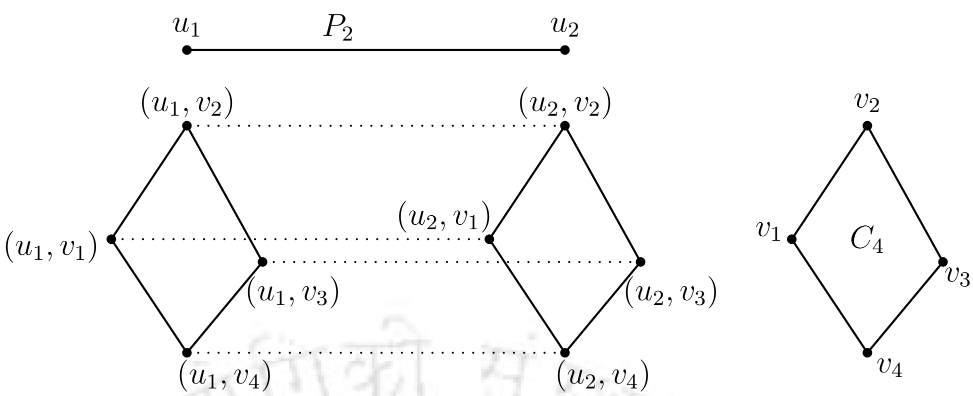

Figure 4.3: The Cartesian Product graph $P_2 \Box C_4$.

In view of Definitions 4.1 and 4.3 we have the following remark.

Remark 4.3. Let P be a shortest path connecting u^i and v^j in $G \Box H$. Then

(i) $l_G(P) = d_G(u^i, v^j) = d_G(i, j)$

(ii) $l_H(P) = d_H(u^i, v^j) = d_H(u, v)$

Where $d_G(i, j)$ denotes the distance between i and j in G.

Lemma 4.1. Given two graphs G and H. Let P be a shortest path connecting u^i and v^j in $G \Box H$. If P' is any other path connecting u^i and v^j in $G \Box H$, then

(i) $l_H(P) \leq l_H(P')$, and

(ii) $l_G(P) \leq l_G(P')$.

Proof. Let $l_G(P) = p, l_H(P) = q$. Then there exist a path of length p connecting i and j in G and a path of length q connecting u and v in H. Let $[i = i_0, i_1 \ldots i_p = j]$ be a path of length p in G and $[u = u_0, u_1 \ldots u_q = v]$ be a path of length q in H. Assume that $l_H(P) > l_H(P')$. Now , if $l_H(P') = r$, then there exist a path, say $[i = k_0, k_1 \ldots k_l = j]$, of length $l \leq r$ connecting i and j in H. Also, $[u^i = u^{k_0}, u^{k_1}, \ldots, u^{k_l} = u_0^{k_l}, \ldots, u_q^{k_l} = v^j]$ is a path connecting u^i and v^j in $G \Box H$ of length smaller than that of P, a contradiction. Hence $l_H(P) \leq l_H(P')$. Similarly, we can show that $l_G(P) \leq l_G(P')$. □

We next turn our attention to distance in the Cartesian product of two graphs. For the sake of completeness we list some properties about the distance function in Cartesian product in terms of the following proposition. We refer [13] for proofs of these properties.

Proposition 4.1. [13] Given two graphs G and H. Then for any $u^i, v^j \in V(G\Box H)$ the following holds:

(i) $d_G(u^i, v^j) = 0$ if and only if $u = v$.

(ii) $d_H(u^i, v^j) = 0$ if and only if $i = j$.

(iii) $d_{G\Box H}(u^i, v^j) = 0$ if and only if $u = v$ and $i = j$.

(iv) $d_{G\Box H}(u^i, v^j) = d_G(u^i, v^j) + d_H(u^i, v^j)$.

The following corollary gives a necessary and sufficient condition for connectedness of $G\Box H$.

Corollary 4.1. [13] Given two graphs G and H. The graph $G\Box H$ is connected if and only if both G and H are connected.

In view of Corollary 4.1, now onwards in this section we consider only connected graphs.

Local Moves of The Hole

Proposition 4.2. Given two graphs G and H. Let $\{u, v\}, \{v, w\} \in E(H)$ and $i \in V(G)$. Then $d_{G\Box H - v^i}(u^i, w^i) = \min\{d_{H-v}(u, w), 4\}$.

Proof. Let P be a shortest path connecting u^i and w^i in $G\Box H - v^i$. We need to show that $|P| = min\{d_{H-v}(u, w), 4\}$. We consider the following cases:

Case I. $V(P) \cap V(H^i) = V(P)$**.** In this case $V(P) \subseteq V(H^i - v^i)$ and so $|P| = d_{H-v}(u, w)$.

Case II. $V(P) \cap V(H^i) \neq V(P)$**.** We claim that $|P| = 4$. Notice that for any $\{i, j\} \in E(G)$, the vertices x^i, y^j are adjacent in $G\Box H$ if and only if $x = y$. Therefore if we are moving away from the copy H^i using the path P we must also come back to the copy H^i. Hence G-distance covered along the path P must be at least two. Also $d(u, w) = 2$, otherwise $\{u, w\} \in E(H)$ and this implies $|P| = 1$, which is not possible. So H-distance traveled along the path P must be at least two. Hence $|P| \geq 4$. Now for any $\{i, j\} \in E(G)$ the path $[u^i, u^j, v^j, w^j, w^i]$ connects u^i and w^i in $G\Box H$. This proves our claim.

□

Corollary 4.2. Given two graphs G and H. Let $\{u,v\},\{v,w\} \in E(H)$ and $i \in V(G)$, where u,v,w are distinct. Then starting from the configuration $C_{u^i}^{v^i}$ of $G\Box H$, we require at least $\min\{1+d_{H-v}(u,w),5\}$ moves to take the robot to w^i. In particular, if $g(H) \geq 6$ then we need at least five moves to take the robot to w^i.

Proof. Notice that, $\{u^i,v^i\},\{v^i,w^i\} \in E(G\Box H)$. In order to move the robot from the vertex v^i to w^i, before it, the hole must be moved from u^i to w^i. This needs $min\{d_{H-v^i}(u^i,w^i),4\}$ moves, because

$$d_{G\Box H-v^i}(u^i,w^i) = min\{d_{H-v}(u,w),4\}.$$

Then the move $v^i \overset{r}{\leftarrow} w^i$ takes the robot from v^i to w^i. Hence the result follows.

If $g(H) \geq 6$ then $d_{H-v}(u,w) \geq 4$ and so $\min\{1+d_{H-v}(u,w),5\} = 5$. Thus, at least five moves are required to take the robot from v^i to w^i. □

As Cartesian product of graphs is commutative, so the proof of the following proposition can be drawn in the same line as that of Proposition 4.2.

Proposition 4.3. Given two graphs G and H. Let $\{i,j\},\{j,k\} \in E(G)$ and $u \in V(H)$. Then $d_{G\Box H-u^j}(u^i,u^k) = \min\{d_{G-j}(i,k),4\}$.

Corollary 4.3. Given two graphs G and H. Let $\{i,j\},\{j,k\} \in E(G)$ and $u \in V(H)$. Then starting from the configuration $C_{u^i}^{u^j}$ of $G\Box H$, we need at least $\min\{1+d_{G-j}(i,k),5\}$ moves to take the robot to u^k. In particular, if $g(G) \geq 6$, then we need at least 5 moves to take the robot to u^k.

Proposition 4.4. Given two graphs G and H. Let $\{i,j\} \in E(G)$ and $\{u,v\} \in E(H)$. Then, starting from the configuration $C_{u^i}^{u^j}$ of $G\Box H$ we need at least three moves to take the robot to v^j.

Proof. To move the robot from u^j to v^j, before it, the hole must be moved from u^i to v^j. This takes two steps, since $d_{G\Box H-u^j}(u^i,v^j) = 2$. Then the move $v^j \overset{r}{\leftarrow} u^j$ takes the robot to v^j. Hence the result follows. □

As Cartesian product of graphs is commutative, so the proof of the following proposition can be drawn in the same line as that of Proposition 4.4.

Proposition 4.5. Given two graphs G and H. Let $\{i, j\} \in E(G)$ and $\{u, v\} \in E(H)$. Then, starting from the configuration $C_{u^i}^{v^i}$ we need at least three moves to take the robot to v^j.

Definition 4.5. A robot move in $G\Box H$ is said to be a *G-move* (respectively, *H-move*) if the edge along which the move took place is a G-edge (respectively, H-edge).

Definition 4.6. Let S be a sequence of moves that take the robot from u^i to v^j in $G\Box H$. A H-move (respectively G-move) in S of the robot preceded by another H-move (respectively G-move) of the robot is said to be a secondary H-move (respectively secondary G-move). A H-move (respectively G-move) of the robot preceded by a G-move (respectively H-move) of the robot is said to be a primary H-move (respectively G-move). Also the edges corresponding to a primary G-move (respectively primary H-move) in S is said to be a primary G-edge (respectively primary H-edge).

For example, consider the configuration $C_{x^2}^{x^1}$ of the graph in Figure 4.4 with $G = P_3$ and $H = C_4$. The following sequence of moves, say S, takes the robot from x^1 to v^3 along the path $[x^1, x^2, u^2, v^2, v^3]$.

$$C_{x^2}^{x^1} : x^2 \xleftarrow{r} x^1 \xleftarrow{o^*} u^2 \xleftarrow{r} x^2 \xleftarrow{o^*} v^2 \xleftarrow{r} u^2 \xleftarrow{o^*} v^3 \xleftarrow{r} v^2 : C_{v^2}^{v^3}$$

In S, $x^2 \xleftarrow{r} x^1$ is neither secondary nor primary, $u^2 \xleftarrow{r} x^2$ is a primary H-move, $v^2 \xleftarrow{r} u^2$ is a secondary H-move and $v^3 \xleftarrow{r} v^2$ is a primary G-move

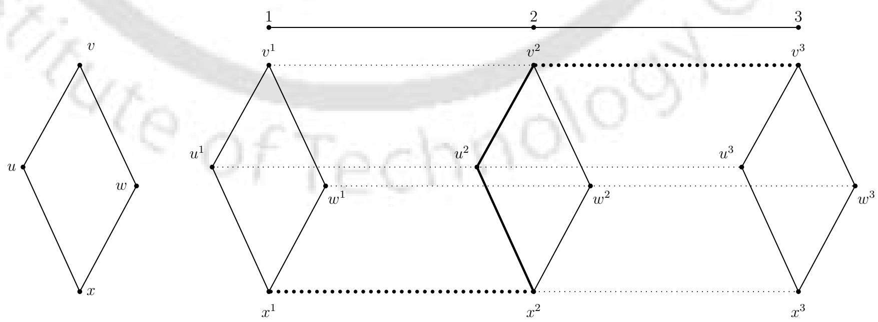

Figure 4.4: The Cartesian Product graph $P_3\Box C_4$.

In view of the definition 4.6, we summarize the results of this section in terms of the following remarks.

Remark 4.4. Given two graphs G and H, each having girth six or more.

(i) In view of corollary 4.2 and corollary 4.3, to perform each secondary G-move (or H-move) of the robot we require at least five moves.

(ii) In view of the propositions 4.4 and 4.5, to perform each primary G-move (or H-move) of the robot we require at least three moves.

(iii) In a minimum sequence of moves, the robot should take as many primary moves as possible.

Trace of The Robot

Proposition 4.6. Let G and H be two graphs such that $i, j \in V(G)$ and $u, v \in V(H)$. Further, let S be a sequence of moves that take the robot from u^i to v^j in $G\Box H$. Then

(i) the number of H-moves of the robot in S is at least $d_H(u, v)$.

(ii) the number of G-moves of the robot in S is at least $d_G(i, j)$.

Proof. Let $x \in V(H)$ and $\{r, s\} \in E(G)$. Then we observe that

$$d_H(x^r, v^j) = d_H(x^s, v^j) = d_H(x, v) \quad \text{and} \quad \{x^r, x^s\} \in E(G\Box H).$$

Thus a G-move of the robot from x^r to x^s does not alter the H-distance of the robot from v^j.

Also if $r \in V(G)$ and $\{x, y\} \in E(H)$ then,

$$d_H(x^r, v^j) = \begin{cases} d_H(y^r, v^j) - 1 & \text{if } y^r \text{ is on the shortest path from } x^r \text{ to } v^j. \\ d_H(y^r, v^j) + 1 & \text{if } x^r \text{ is on the shortest path from } y^r \text{ to } v^j. \\ d_H(y^r, v^j) & \text{otherwise.} \end{cases}$$

Therefore any H-move of the robot can reduce its H-distance from v^j at most by one. Hence to take the robot from u^i to v^j we need at least $d_H(u, v)$ number of H-moves. We can argue the second statement in similar manner because the Cartesian product is commutative. □

Lemma 4.2. Consider the graphs G and H each having girth six or more. Let $i, j \in V(G)$ and $\{u, v\}, \{u, w\} \in E(H)$. Then, each robot move in a minimum sequence of moves that takes $C_{v^i}^{u^i}$ to $C_{w^j}^{u^j}$ in $G\Box H$ is a G-move. Also, such a minimum sequence involves exactly k number of G-moves and $5k$ moves in total, where $k = d(i, j) \geq 1$.

Proof. Let S be a sequence of moves that takes $C_{v^i}^{u^i}$ to $C_{w^j}^{u^j}$ in $G\Box H$. First assume that the number of robot moves in S is t and each of these robot moves in S is a G-move. By Proposition 4.5, we need at least three moves to accomplish the first G-move of the robot. Notice that each of the remaining $t-1$ robot moves in S is a secondary G-moves. So by Remark 4.4, we need minimum $5(t-1)$ moves to accomplish these $t-1$ secondary G-moves. Now, if $u^j \overset{r}{\leftarrow} u^k$ is the t^{th} robot move in S, it will leave the graph $G\Box H$ with the configuration $C_{u^k}^{u^j}$. Since $d_{G\Box H - u^j}(u^k, w^j) = 2$, so we need minimum two more moves to take the hole from u^k to w^j. Hence S involves minimum $5t$ moves. Notice that, the expression $5t$ takes the minimum value when t is minimum.

Next, let $d(i, j) = k$ and $[i = i_0, i_1, \ldots, i_k = j]$ be a path of length k connecting i and j in G. Then $[u^i = u^{i_0}, u^{i_1}, \ldots, u^{i_k} = u^j]$ is a path of length k in $G\Box H$ joining u^i to u^j. So, the sequence of moves

$$v^i \overset{o^*}{\leftarrow} u^{i_1} \overset{r}{\leftarrow} u^{i_0} \overset{o^*}{\leftarrow} u^{i_2} \overset{r}{\leftarrow} u^{i_1} \overset{o^*}{\leftarrow} u^{i_3} \ldots u^{i_{k-2}} \overset{o^*}{\leftarrow} u^{i_k} \overset{r}{\leftarrow} u^{i_{k-1}} \overset{o^*}{\leftarrow} w^j$$

takes the robot from u^i to u^j along this path and each move in this sequence is a G-move. Also it involves exactly k number of G-moves of the robot. Therefore by Proposition 2.1, a minimum sequence of moves S (not involving H-moves of the robot) that takes the configuration $C_{v^i}^{u^i}$ to $C_{w^j}^{u^j}$ involves exactly $5k$ moves.

Finally, assume that the sequence S involves H-moves also and let p be the number of primary H-moves in S. It is enough to show that the sequence S involves more than $5k$ moves. We consider the following cases:

Case I. The first move of the robot is a H-move Note that to make the first move of the robot requires at least one move. Since the first move of the robot in S takes it away from G^u, and G-moves always keep the robot in the same copy of G, so to bring the robot back to G^u we need one more H-move. Therefore, minimality of S implies that S involves at least one primary H-move. That is $p \geq 1$. Also the maximum number of primary G-move possible in S is $p+1$ or p according as the last move of the robot is a G-move or H-move. If the last move of the robot is a G-move then we will be required two more moves to take the hole to w^j. So the number of moves involved in S is at least $1+3p+3(p+1)+5(k-p-1)+2$ or $1+3p+3p+5(k-p)$ according as the last move of the robot is a G-move or H-move. That is, S involves at least $5k+p+1$ moves if the first move of the robot is a H-move.

Case II. The first move of the robot is a G-move In this case the first move of the robot requires at least three moves and the first H-move of the robot is primary and so $p \geq 2$. Also the maximum number of primary G-move possible in S is p or $p-1$ according as the last move is a G-move or a H-move. If the last move is a G-move then we will be required two more moves to take the hole to w^j. So the number of moves required is at least $3+3p+3p+5(k-p-1)+2$ or $3+3p+3(p-1)+5(k-p)$ according as the last move is a G-move or a H-move. That is, S involves at least $5k+p$ moves if the first move of the robot is a G-move.

Thus the number of moves in S is at least $5k+2$, if it involves H-moves of the robot.

This completes the proof. □

Since the Cartesian product of graphs is commutative, so proof of the following lemma can be drawn in the same line as that of Lemma 4.2.

Lemma 4.3. Consider the graphs G and H each having girth six or more. Let $\{i,j\}$, $\{i,k\} \in E(G)$ and $u,v \in V(H)$. Then, each robot move in a minimum sequence of moves that takes $C_{u^j}^{u^i}$ to $C_{v^k}^{v^i}$ in $G \square H$ is a H-move. Also, such a minimum sequence involves exactly p number of H-moves of the robot and $5p$ moves in total, where $p = d(u,v) \geq 1$.

The Lemma 4.4 gives the minimum number of moves required to take the robot from a give copy to another copy of the same factor of $G \square H$. The proof of this lemma is immediate from Lemma 4.2 and Lemma 4.3.

Lemma 4.4. Consider the graph $G\Box H$ with the initial configuration $C_{v^i}^{u^i}$, where G and H are connected and have girth six or more. Then

(i) to move the robot from H^i to H^j we require $l + 2 + 5(k-1)$ moves.

(ii) to move the robot from the G^u to G^v we require at least $l + 5(l-1)$ moves.

where $k = d(i,j)$ and $l = d(u,v)$.

Definition 4.7. Let $\{u,v\} \in E(H)$ and $i,j \in V(G)$. Then a pair of robot moves of the form $v^i \overset{r}{\leftarrow} u^i$ and $u^j \overset{r}{\leftarrow} v^j$ in $G\Box H$ is said to be a pair of *opposing H-move* of the robot. Similarly, we can define a pair of *opposing G-move* of the robot in $G\Box H$.

Lemma 4.5. Given the graphs G and H each having girth six or more. Let S be a sequence of moves that takes the robot from u^i to v^j in $G\Box H$. If S is minimum then there is no opposing moves of the robot in S.

Proof. Suppose that S involves a pair of opposing H-move of the robot. Let this pair of moves be $y^k \overset{r}{\leftarrow} x^k$ and $x^l \overset{r}{\leftarrow} y^l$. Let S_1 be the subsequence of S consisting of all moves starting from the move $y^k \overset{r}{\leftarrow} x^k$ upto the move $x^l \overset{r}{\leftarrow} y^l$. Clearly S_1 takes the configuration $C_{y^k}^{x^k}$ to the configuration $C_{y^l}^{x^l}$ and it involves the H-moves $y^k \overset{r}{\leftarrow} x^k$ and $x^l \overset{r}{\leftarrow} y^l$ of the robot, a contradiction (see the Lemma 4.2). Therefore S cannot involve a pair of opposing H-move of the robot. Similarly using the Lemma 4.3, we can conclude that S cannot involve a pair of opposing G-move of the robot.

□

Lemma 4.6. Consider the graphs G and H each having girth six or more. Let S be a sequence of moves that takes the robot from u^i to v^j in $G\Box H$. Then the H-moves (respectively G-moves) of the robot in S trace a walk in H (respectively in G). If S is minimum then these walks are paths and the number of H-moves (respectively G-moves) of the robot in S is $l_H(\mathcal{P})$ (respectively $l_G(\mathcal{P})$), where $\mathcal{P}(u^i, v^j)$ is the path in $G\Box H$ traced by the robot moves in S.

Proof. Let P be the sub-graph of H induced by the edges in H corresponding to the H-moves of the robot in S. We claim that P is a walk in H connecting u and v. Let $x^i \xleftarrow{r} w^i$ and $z^j \xleftarrow{r} y^j$ be two consecutive H-moves in S, i.e., all other robot moves in S taken place between these two moves are G-moves. Clearly $\{w^i, x^i\}$, $\{y^j, z^j\} \in E(G\Box H)$ and hence $\{w, x\}$, $\{y, z\} \in E(H)$. Notice that, $C_{z^j}^{y^j}$ is reachable from $C_{w^i}^{x^i}$ by means of H-moves and obstacle moves only, so $x = y$. So the edges in P corresponding to two consecutive H-moves $x^i \xleftarrow{r} w^i$ and $z^j \xleftarrow{r} y^j$ are incident with each other at x in H. It follows by similar argument that if $y^p \xleftarrow{r} x^p$ and $w^q \xleftarrow{r} z^q$ are the first and the last H-moves in S respectively, then $x = u$ and $w = v$. Therefore, P is a walk in H connecting u and v.

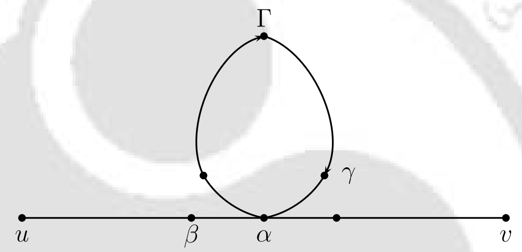

Figure 4.5: The walk $P(u, v)$ in H.

Now assume that S is minimum. We claim that P is a path. On the contrary, assume that P contains a cycle, say Γ. While moving along P from u to v in H, let α be the vertex on P at which it enters Γ, β be the vertex on P just before entering Γ and γ be the vertex on Γ adjacent to α at which it reaches just before re-entering α after moving along the cycle Γ (see Figure 4.5). Then there exist a sub-sequence S_1 of S such that for some $r, s \in V(G)$, the sequence S_1 takes the configuration $C_{\beta^r}^{\alpha^r}$ to $C_{\gamma^s}^{\alpha^s}$. Since Γ has at least three vertices, so the sub-sequence S_1 must involve at least one H-move, a contradiction (see Lemma 4.2). Therefore we can conclude that P is a path.

Since the graph Cartesian product is commutative so the proof of the remaining part of the lemma can be argued as above. □

In view of the results obtained in this section so far we have the following concluding remarks.

Remark 4.5. Given two graphs G and H each having girth six or more. Let S be a minimum sequence of moves that takes the robot from u^i to v^j in $G\Box H$. Then, by Lemma 4.5 and 4.6, the robot moves in S traces a path $P(u^i, v^j)$ in $G\Box H$ such that

(i) $l_H(P) =$ the length of the uv-path in H traced by the H-moves in S.

(ii) $l_G(P) =$ the length of the ij-path in G traced by the G-moves in S.

Minimum Number of moves

Definition 4.8. Given a path P connecting u^i and v^j in $G\Box H$. By a *minimal* sequence of moves with *trace* P we mean a sequence with minimum number of moves that takes the robot from u^i to v^j along the path P in $G\Box H$.

Definition 4.9. By a *minimal* u^iv^j-path in $G\Box H$ we mean a u^iv^j-path P such that the G-edges in P induces a ij-path in G and the H-edges in P induces a uv-path in H.

In view of the above definitions, we have the following remark.

Remark 4.6. Given two graphs G and H each having girth six or more.

(i) A shortest path in $G\Box H$ is a minimal path but a minimal path in $G\Box H$ need not be a shortest path.

(ii) By remark 4.5, the path traced by the robot moves in a minimum sequence of moves in $G\Box H$ is a minimal path.

Definition 4.10. Give two graphs G, H and a path P in $G\Box H$. By a *primary edge* in P we mean a H-edge that is preceded by a G-edge or a G-edge that is preceded by a H-edge. By a *secondary edge* in P we mean a H-edge that is preceded by a H-edge or a G -edge that is preceded by a G-edge.

We now state the following lemma without proof. This lemma gives the maximum number of primary edges that a path can have in $G\Box H$ with given H-length and G-length.

Lemma 4.7. Given two graphs G and H. Let P be a path connecting u^i and v^j in $G\Box H$ such that $l_G(P) = a$ and $l_H(P) = b$. Then, the maximum number of primary edges P can have is

(i) $2a - 1$ if $a = b$

(ii) $2b$ if $a > b$ and first edge in P is a G-edge.

(iii) $2b - 1$ if $a > b$ and first edge in P is a H-edge.

(iv) $2a$ if $b > a$ and first edge in P is a H-edge.

(v) $2a - 1$ if $b > a$ and first edge in P is a G-edge.

Theorem 4.1. Given two graphs G and H each having girth six or more. Consider the configuration $C_{v^i}^{u^i}$ of $G\Box H$. For some $j \in G\Box H$, let P be a minimal path connecting u^i and v^j in $G\Box H$. Let S be a minimal sequence with trace P. If $l_G(P) = a$ and $l_H(P) = b$, then S involves at least

(i) $k + 5a + 5b - 2m - 5$ moves if the first move of the robot is a H-move.

(ii) $k + 5a + 5b - 2m - 3$ moves if the first move of the robot is a G-move.

where m is the number of primary moves of the robot in S and $k = d(u, v)$.

Proof. Since S is minimal so it involves exactly $a + b$ robot moves.

Case I. The first edge in P is a H-edge. In this case the first robot move is a H-move, say $w^i \overset{r}{\leftarrow} u^i$. In order to realize this move, before it, the hole must move from v^i to w^i. Therefore, we require k moves to realize the first robot move, since $d_{G\Box H-u^i}(v^i, w^i) = k - 1$ ($k - 1$ moves to bring the hole at w^i plus the robot move $w^i \overset{r}{\leftarrow} u^i$). Since m is the number of primary moves in S, so the number of secondary robot moves in S is $a + b - m - 1$. Hence, by the Remark 4.4, the number of moves in S is $k + 3m + 5(a + b - m - 1)$ i.e., $k + 5a + 5b - 2m - 5$.

Case II : The first edge in P is a G-edge. In this case the first robot move is a G-move. Let this move be $u^k \overset{r}{\leftarrow} u^i$. So, to perform this move we must first take the hole from v^i to u^k. Clearly $d_{G\Box H-u^i}(v^i, u^k) = k+1$

and so we require $k+2$ moves to perform the first robot move ($k+1$ moves to bring the hole at u^k plus the robot move $u^k \overset{r}{\leftarrow} u^i$). Since m is the number of primary moves in S, so the number of secondary robot moves in S is $a+b-m-1$. Hence, by the Remark 4.4, the number of moves in S is $k+2+3m+5(a+b-m-1)$ i.e., $k+5a+5b-2m-3$.

□

Notice that, among all minimal paths P with $l_G(P)=a$ and $l_H(P)=b$, the two expressions $k+5a+5b-2m-5$ and $k+5a+5b-2m-3$ in the above theorem attains the minimum when P has the maximum number of primary edges. Thus, we have the following corollary.

Corollary 4.4. Given two graphs G and H each having girth six or more. Consider the configuration $C_{v^i}^{u^i}$ of $G\Box H$. For some $j \in G\Box H$, let S be a minimum sequence of moves that takes the robot from u^i to v^j. Let P be the trace of S, with $l_G(P)=a$ and $l_H(P)=b$. Then among all minimal paths P' with $l_G(P')=a$ and $l_H(P')=b$, the path P has maximum number of primary edges. Also, the number of moves involved in S is

(i) $k+5a+b-3$, if $a \geq b$.

(ii) $k+a+5b-5$, if $a<b$.

where $k=d(u,v)$. Further, if $a \geq b$ then the first edge in P must be a H-edge.

Proof. We consider the following cases:

Case I. $\mathbf{a=b}$ In this case maximum number of primary moves possible is $2a-1$ and it is independent of the type of the first edge in P (see Lemma 4.7). So, by Theorem 4.1, the first move of the robot in S must be a H-move and the number of moves involved in S is $k+5a+5b-2(2a-1)-5$ i.e., $k+6a-3$ moves.

Case II. $\mathbf{a>b}$ In this case maximum number of primary moves S can have is $2b$ or $2b-1$ according as the first edge in P is G-edge or a H-edge (see Lemma 4.7). And, by Theorem 4.1, in either case the number of moves involved in S is $k+5a+b-3$.

Case III. a < b In this case maximum number of primary moves S can have is $2a$ or $2a-1$ according as the first edge in P is a H-edge or a G-edge. So, by Theorem 4.1, the first move in S must be a H-move and the number of moves involved in S is $k+a+5b-5$.

Hence the proof is complete.

□

Finally, the two expressions $k+5a+b-3$ and $k+a+5b-5$ attains the minimum when a and b are minimum. That is, when P is a shortest path with maximum number of primary edges. Thus, we have the following corollary.

Corollary 4.5. Given two graphs G and H each having girth six or more. Consider the configuration $C_{v^i}^{u^i}$ of $G\Box H$. For some $j \in G\Box H$, let S be a minimum sequence of moves that takes the robot from u^i to v^j. Let P be the trace of S, with $l_G(P)=a$ and $l_H(P)=b$. Then P is a shortest path with maximum number of primary edges. Also, the number of moves involved in S is

(i) $5a+2b-3$, if $a \geq b$.

(ii) $a+6b-5$, otherwise.

Further, if $a \geq b$ then the first edge in P must be a H-edge.

Thus we can summarize all the results in this section in terms of the following theorems.

Theorem 4.2. Consider the graph $G\Box H$ with the configuration $C_{v^i}^{u^i}$, where G and H are connected and have girth six or more. Let S be a minimum sequence of moves that takes the robot from u^i to v^j. Let P be the trace of S. Then

(i) P is a shortest path in $G\Box H$ connecting u^i and v^j.

(ii) Among all shortest paths connecting u^i and v^j, P has the maximum number of primary edges.

Also the minimum number of moves required is $2d(u,v) + 5d(i,j) - 3$ or $6d(u,v) + d(i,j) - 5$ according as $d(i,j) \geq d(u,v)$ or $d(i,j) < d(u,v)$. Further, if $d(i,j) \geq d(u,v)$, the first edge in P must be a H-edge.

Theorem 4.3. Given two graphs G and H each having girth six or more. Consider the configuration $C_{v^i}^{u^i}$ in $G \square H$. Let S be a minimum sequence of move that takes the robot from u^i to v^j in $G \square H$. Then each robot move in S reduces its distance from v^j by one.

Theorem 4.4. Given two graphs G and H each having girth six or more. Consider the configuration $C_{v^i}^{u^i}$ in $G \square H$. Let $d(i,j) = k$ and $d(u,v) = l$. Then any arrangement of k number of G-moves and l number of H-moves of the robot takes the robot from u^i to v^j if each move of the robot reduces its distance from v^j by one. Also any such arrangement with maximum number of primary robot moves gives a minimum sequence of moves that takes the robot from u^i to v^j.

Example 4.3. Consider the vertices u^1, v^6 in the graph $P_6 \square P_6$ shown in the Figure 4.6. Starting with the configuration $C_{v^1}^{u^1}$, the path traced by the robot moves corresponding to a minimum sequence of moves that takes the robot to v^6 is shown by the dotted line.

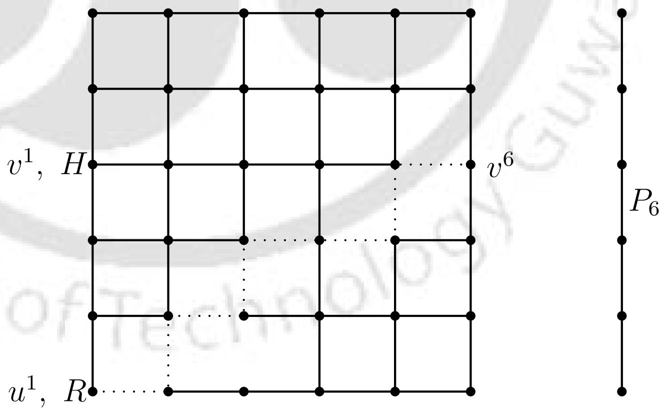

Figure 4.6: The graph $P_6 \square P_6$

Also, the minimum number of moves required to take the robot to the vertex v^6 is

$$= 2d(u,v) + 5d(1,6) - 3 = 28.$$

4.3 Graph Strong Product

Given two graphs G and H on vertex sets $\{u_1, u_2, \ldots u_m\}$ and $\{v_1, v_2, \ldots v_n\}$ respectively. The *strong product* $G \boxtimes H$ of G and H is a graph with vertex set $V(G) \times V(H)$ where the adjacency of vertices is determined by the following rule: (u_i, v_j) and (u_p, v_q) adjacent in $G \boxtimes H$ if one of the following conditions holds:

(i) $u_i = u_p$ and $\{v_j, v_q\} \in E(G)$.

(ii) $\{u_i, u_p\} \in E(G)$ and $v_j = v_q$.

(iii) $\{u_i, u_p\} \in E(G)$ and $\{v_j, v_q\} \in E(G)$

Like graph Cartesian product, strong product is also associative, commutative and distributes over disjoint union. We label the vertices of G in $G \boxtimes H$ as $\{1, 2, \ldots m\}$, unless otherwise stated.

Example 4.4. Let $G = P_3$, the path of order 3 and $H = C_4$, the cycle of order 4. The graph $G \boxtimes H$, obtained by taking strong product of G and H, is shown in Figure 4.7.

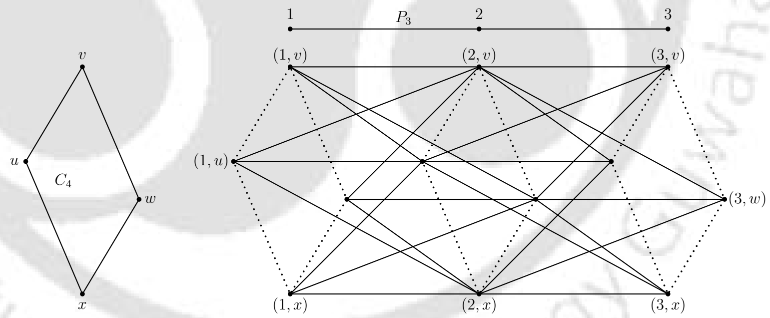

Figure 4.7: The Strong Product $P_3 \boxtimes C_4$.

Remark 4.7. Given two graphs G and H, we can also view $G \boxtimes H$ as the graph obtained from G by replacing:

(i) each vertex $i \in V(G)$ by a copy H^i (say) of H, and

(ii) each edge $\{i, j\} \in E(G)$ by the collection of edges

$$\{\{(i, u), (j, v)\} \mid u \in V(H) \text{ and } v \in \overline{N(u)}\},$$

where (i, u) denotes the vertex in H^i corresponding to the vertex $u \in V(H)$.

In view of remark 4.7, we use H^i to denote the copy of H in $G \boxtimes H$ corresponding to the vertex $i \in V(G)$. Also for any $i \in V(G)$, notice that u^i denotes the vertex in H^i corresponding to the vertex $u \in V(H)$. Further, commutativity of the strong product allows us to view $G \boxtimes H$ as the graph obtained from H by replacing each vertex $u \in V(H)$ by a copy G^u (say) of G, and each edge $\{u, v\} \in E(H)$ by the collection of edges

$$\{\{i^u, j^v\} \mid i \in V(G) \text{ and } j \in \overline{N(i)}\}.$$

Example 4.5. Let $G = P_4$, the path of order 4 and $H = P_6$, the path of order 6. The graph $G \boxtimes H$, obtained by taking strong product of G and H, is shown in Figure 4.8.

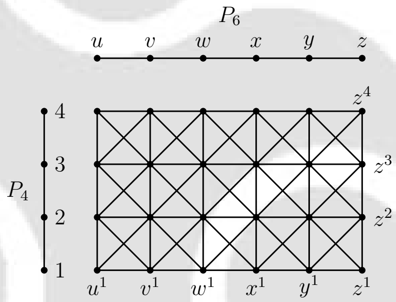

Figure 4.8: The graph $P_4 \boxtimes P_6$.

The following proposition gives the distance between two vertices u^i and v^j in $G \boxtimes H$.

Proposition 4.7. [13] Suppose that u^i and v^j are two vertices of a strong product $G \boxtimes H$. Then

$$d_{G \boxtimes H}(u^i, v^j) = max\{d_G(i, j), d_H(u, v)\}$$

The following corollary gives a necessary and sufficient condition for connectedness of $G \boxtimes H$.

Corollary 4.6. [13] The strong product $G \boxtimes H$ is connected if and only if both G and H are connected.

In view of Corollary 4.6, now onwards in this section we consider only connected graphs.

Proposition 4.8. Given two graphs G and H each with girth 5 or more. Let $u \in N_H(v)$ and $i, k \in N_G(j)$ be such that $i \neq k$. Consider the configuration $C_{u^i}^{v^j}$ of $G \boxtimes H$. Then for some $w \in \overline{N(v)}$, the minimum number of moves required to take the robot from v^j to w^k is 3 or 4 according as $w \in \overline{N(u)}$ or $w \notin \overline{N(u)}$.

Proof. Notice that $\{v^j, w^k\} \in E(G \boxtimes H)$. So if we can take the hole to w^k without displacing the robot from v^j, then the robot move $w^k \overset{r}{\leftarrow} v^j$ will take the robot to w^k. Also notice that any other sequence of moves that takes the robot to w^k without using the robot move $w^k \overset{r}{\leftarrow} v^j$ must involve at least two robot moves each of which is preceded by at least one obstacle move. Now, if $w \in \overline{N(u)}$ then $[u^i, u^j, w^k]$ is a shortest path connecting u^i and w^k in $G \boxtimes H - v^j$ and so we need at least two moves to take the hole at w^k. Also the sequence $u^i \overset{o}{\leftarrow} u^j \overset{o}{\leftarrow} w^k$ takes the hole to w^k. Again, if $w \notin \overline{N(u)}$ then $[u^i, u^j, v^k, w^k]$ is a shortest $u^i - w^k$ path in $G \boxtimes H - v^j$ and so we require at least 3 moves to take the hole to w^k. Also the sequence of move $u^i \overset{o}{\leftarrow} u^j \overset{o}{\leftarrow} v^k \overset{o}{\leftarrow} w^k$ takes the hole to w^k. The proof is complete. □

The proof of the following proposition is similar to that of the Proposition 4.8.

Proposition 4.9. Given two graphs G and H each with girth 4 or more. Let $i, k \in N_G(j)$ be such that $i \neq k$ and $u \in V(H)$. Consider the configuration $C_{u^i}^{u^j}$ of $G \boxtimes H$ for some $u \in V(H)$. Then for some $v \in \overline{N(u)}$, the minimum number of moves required to take the robot from u^j to v^k is 3.

Proof. Notice that $\{u^j, v^k\} \in E(G \boxtimes H)$. So, if we can take the hole to v^k without displacing the robot from u^j, then the robot move $v^k \overset{r}{\leftarrow} u^j$ will take the robot to v^k. Also notice that $d_{G \boxtimes H - u^j}(u^i, v^k) \geq 2$. So any other sequence of moves that takes the robot to v^k without using the robot move $v^k \overset{r}{\leftarrow} u^j$ must involve at least two robot moves each of which is preceded by at least one obstacle move. Now $[u^i, u^j, v^k]$ is a shortest path connecting u^i and v^k in $G \boxtimes H - u^j$. Also the sequence $u^i \overset{o}{\leftarrow} u^j \overset{o}{\leftarrow} v^k$ takes the hole to v^k. The proof is complete. □

Proposition 4.10. Given two graphs G and H. Let $\{u, v\} \in E(H)$ and $\{i, j\} \in E(G)$. Consider the configuration $C_{u^i}^{v^i}$ of $G \boxtimes H$. Then for some $w \in N(v)$, the minimum number of moves required to take the robot from v^i to w^j is 2 or 3 according as $w \in \overline{N(u)}$ or $w \notin \overline{N(u)}$.

Proof. The proof is immediate from the fact that $d_{G\boxtimes H - v^i}(u^i, w^j) = 1$ or 2 according as $w \in \overline{N(u)}$ or $w \notin \overline{N(u)}$. □

Proposition 4.11. Given two graphs G and H. Let $u, w \in N_H(v)$ and $\{i, j\} \in E(G)$. Consider the configuration $C_{u^i}^{v^j}$ of $G \boxtimes H$. Then the minimum number of moves required to take the robot from v^j to w^j is 2 or 3 according as $w \in \overline{N(u)}$ or $w \notin \overline{N(u)}$.

Proof. The proof is immediate from the fact that $d_{G\boxtimes H - v^j}(u^i, w^j) = 1$ or 2 according as $w \in \overline{N(u)}$ or $w \notin \overline{N(u)}$. □

Proposition 4.12. Given two graphs G and H each with girth 4 or more. For some $u, v \in V(H)$ and $i \in V(G)$, consider the configuration $C_{v^i}^{u^i}$ of $G \boxtimes H$. For some $j \in V(G)$, let S be a minimum sequence of moves that takes the robot from u^i to u^j. Then

$$|S| \leq \begin{cases} d(u, v) + 1, & \text{if } d(i, j) = 1; \\ d(u, v) + 3[d(i, j) - 1], & \text{otherwise.} \end{cases} \tag{4.1}$$

Also, there exist a sequence of moves satisfying 4.1 in which each robot move is either a G-move or a N-move.

Proof. Let $d(i, j) = p$ and $d(u, v) = q$. Let $[i = i_0 i_1 \dots i_p = j]$ be a shortest $i - j$ path in G and let $[u = u_0 u_1 \dots u_q = v]$ be a shortest $u - v$ path in H. If $d(i, j) = 1$, then $d_{G\boxtimes H - u^i}(v^i, u^j) = q$ and $u^i \in N(u^j)$. So $q + 1$ moves are enough to take the robot from u^i to u^j (q moves to take the hole from v^i to u^j without disturbing the robot at u^i plus the robot move $u^j \overset{r}{\leftarrow} u^i$). Hence the result is true for $i = j$.

Next assume that $d(i, j) \geq 2$. Consider the $u^i - u^j$ path

$$P = [u^i = u_0^{i_0} u_1^{i_1} u_0^{i_2} u_0^{i_3} \dots u_0^{i_p} = u^j]$$

in $G \boxtimes H$. Clearly P is a shortest $u^i - u^j$ path in $G \boxtimes H$. We claim that the robot can be taken from u^i to u^j along the path P using $d(u, v) + 3[d(i, j) - 1]$

moves. Notice that $d(u^j, u^{i_1}) = q - 1$ and so we can take the hole from u^j to u^{i_1} by $q - 1$ moves. Therefore to accomplish the first robot move along the path P we require at least q moves. Also, by Propositions 4.8 and 4.9, each of the subsequent robot moves along the path P can be accomplished by three moves. Hence the result follows. Further notice that each robot move in the sequence that takes the robot from u^i to v^j along the path P is either a G move or a N-move. □

Proposition 4.13. Given two graphs G and H each with girth 4 or more. For some $u, v \in V(H)$ and $i \in V(G)$, consider the configuration $C_{v^i}^{u^i}$ of $G \boxtimes H$. For some $j \in V(G)$, let S be a minimum sequence of moves that takes the robot from u^i to u^j. Then

$$|S| \leq d(u, v) + 3[d(i, j) - 1]. \tag{4.2}$$

Also, there is a sequence of moves involving $d(u, v) + 3[d(u, v) - 1]$ moves in which each robot move is either a H-move or a N-move.

Proof. Let $d(u, v) = q$. Let $[u = u_0u_1 \ldots u_q = v]$ be a shortest $u - v$ path in H. Then $P : [u^i = u_0^i, u_1^i, \ldots u_q^i = v^i]$ is a shortest $u^i - v^i$ path in $G \boxtimes H$. Let S be a sequence of moves that takes the robot from u^i to v^i along the path P. Since $d(u_q^i, u_1^i) = q - 1$, so we require q moves to accomplish the first robot move in S. Also by Proposition 4.9, each of the subsequent robot moves in S requires three moves. Hence the result follows. Also notice that, each of the robot moves in S is a H-move.

Similarly if $q \geq 2$ then for some $j \in N(i)$, it can be shown that there is a sequence of moves involving $d(u, v) + 3[d(u, v) - 1]$ moves that takes the robot to v^i along the path $P : [u^i = u_0^i, u_1^j, u_2^i, \ldots u_q^i = v^i]$. And each of the robot moves in such a sequence is either a H-move or a N-move. □

Proposition 4.14. Given two graphs G and H each with girth 5 or more. For some $u, v \in V(H)$ and $i \in V(G)$, consider the configuration $C_{v^i}^{u^i}$ of $G \boxtimes H$. For $j \in V(G) - \{i\}$, let S be a minimum sequence of moves that takes the robot from u^i to w^j. Then

$$|S| \leq \begin{cases} 2+3[d(u^i,v^j)-1], & \text{if } d(u,v)=1; \\ d(u,v)+3[d(u^i,v^j)-1], & \text{if } d(u^i,v^j)\geq 2t \text{ and } d(u,v)>1; \\ d(u,v)+2[d(u^i,v^j)+t-2], & \text{otherwise.} \end{cases} \tag{4.3}$$

where $t = \min\{d(i,j), d(u,v)\}$.

Proof. Let $[i = i_0 i_1 \ldots i_p = j]$ be a shortest $i-j$ path in G and let $[u = u_0 u_1 \ldots u_q = v]$ be a shortest $u-v$ path in H.

Case 1. $d(u,v)=1$**.** In this case $q=1$. So $P : [u^i = u_0^{i_0}, u_1^{i_1}, u_1^{i_2}, u_1^{i_3}, \ldots u_1^{i_p} = v^j]$ is a shortest $u^i - v^j$ path in $G \boxtimes H$. Let S' be a sequence with minimum number of moves that takes the robot from u^i to v^j along the path P. Since $u_1 = v$, so the sequence of moves

$$u_1^{i_0} \overset{o}{\leftarrow} u_1^{i_1} \overset{r}{\leftarrow} u_0^{i_0}$$

takes the robot to $u_1^{i_1}$, leaving the hole at $u_0^{i_0}$. Now by Propositions 4.8 the second robot move in S' can be accomplished by three moves Also by 4.9 each of the subsequent robot moves in S' can be accomplished by three moves. Hence $|S'| = 2 + 3[d(u^i, v^j) - 1]$.

Case 2. $d(u,v) > 1$ **and** $d(u^i,v^j) \geq 2t$**.** First assume that $t = d(u,v)$. So,

$$P : [u^i = u_0^{i_0}, u_1^{i_1}, u_1^{i_2}, u_2^{i_3}, u_2^{i_4}, u_3^{i_5}, \ldots u_q^{i_{2q-1}}, u_q^{i_{2q}}, u_q^{i_{2q+1}}, \ldots u_q^{i_p} = v^j]$$

is a shortest $u^i - v^j$ path in $G \boxtimes H$. Let S' be a sequence with minimum number of moves that takes the robot from u^i to v^j along the path P. Since $d(v^i, u_1^{i_1}) = q-1$, so the first robot move in S' can be accomplished by q moves. Also by Propositions 4.8 and 4.9 each of the subsequent robot moves in S' can be accomplished by three moves. So, $|S'| = q + 3[d(u^i, v^j) - 1]$.

Similarly, if $t = d(i,j)$ then taking the shortest $u^i - v^j$ path

$$P : [u^i = u_0^{i_0}, u_1^{i_1}, , u_2^{i_1}, u_3^{i_2}, u_4^{i_2}, u_5^{i_3}, \ldots u_{2p-1}^{i_p} u_{2p}^{i_p} u_{2p+1}^{i_p} \ldots u_q^{i_p} = v^j]$$

we can show that $|S'| = q + 3[d(u^i, v^j) - 1]$.

Case 2. $d(u, v) > 1$ **and** $d(u^i, v^j) < 2t$**.** Let $s = |d(i, j) - d(u, v)|$. Clearly $d(u^i, v^j) = s + t$ and $s < t$. First assume that $t = d(u, v)$. Then

$$P \ : \ [u^i = u_0^{i_0}, u_1^{i_1}, u_1^{i_2}, u_2^{i_3}, u_2^{i_4}, u_3^{i_5}, \ldots u_s^{i_{2s-1}}, u_s^{i_{2s}}, u_{s+1}^{i_{2s+1}}, u_{s+2}^{i_{2s+2}}, \ldots u_q^{i_p} = v^j]$$

is a shortest $u^i - v^j$ path in $G \boxtimes H$. Let S' be a sequence with minimum number of moves that takes the robot from u^i to v^j along the path P. Since $d(u_1^{i_1}, v^i) = q - 1$, so the first robot move in S requires q moves. Also by Proposition 4.8 and 4.9 each of the subsequent $2s$ robot moves requires three moves. Finally by Proposition 4.8, each of the remaining $t - s - 1$ robot moves in S' requires four moves. So,

$$\begin{aligned} |S'| &= q + 3 \times 2s + 4[t - s - 1] \\ &= d(u, v) + 2[d(u^i, v^j) + t - 2]. \end{aligned}$$

Since S is a minimum sequence of moves that takes the robot from u^i to v^j, so $|S| \leq |S'|$. This completes the proof. □

Conjecture 4.1. Each of the inequalities in the in-equations 4.1, 4.2 and 4.3 can be replaced by equality.

4.4 Graph Lexicographic Product

Let G and H be two graphs on vertex sets $\{u_1, u_2, \ldots u_m\}$ and $\{v_1, v_2, \ldots v_n\}$ respectively. The *lexicographic product* $G \circ H$ of G and H is the graph with vertex set $V(G) \times V(H)$ where the adjacency of vertices is determined by the following rule: (u_i, v_j) and (u_p, v_q) are adjacent if one of the following conditions holds:

(i) $\{u_i, u_p\} \in E(G)$

(ii) $u_i = u_p$ and $\{v_j, v_q\} \in E(H)$.

This product was first introduced as the *composition* of graphs by Harary [14]. Note that, the lexicographic product of G and H can also be viewed as the graph obtained from G by substituting a copy H^i (say), of H for every vertex

i of G and by joining all vertices of H^i with all vertices of H^j if $\{i,j\} \in E(G)$. Thus it is also known as *substitution* (see [15]). Also for any $i \in V(G)$, u^i denotes the vertex in H^i corresponding to the vertex $u \in V(H)$.

Example 4.6. Let $G = P_2$ and $H = K_3$. The graph $G \circ H$, obtained by taking lexicographic product of G and H, is shown in Figure 4.9. Notice that, the

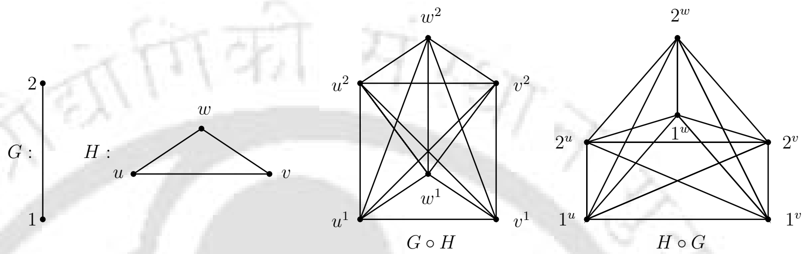

Figure 4.9: Lexicographic Product $P_2 \circ K_3$

graphs $P_2 \circ K_3 = K_3 \circ P_2$. In general it is not true. That is, the lexicographic product is not commutative.

The following proposition gives the distance formula for the lexicographic product $G \circ H$. For the proof of this proposition we refer the book [13].

Proposition 4.15. [13] Suppose that u^i and v^j are two vertices in $G \circ H$. Then

$$d_{G \circ H}(u^i, v^j) = \begin{cases} d_H(u,v), & \text{if } i = j \text{ and } d_G(i) = 0; \\ min\{d_H(u,v), 2\}, & \text{if } i = j \text{ and } d_G(i) \neq 0; \\ d_G(i,j), & \text{if } i \neq j; \end{cases} \tag{4.4}$$

Corollary 4.7. Let G and H be two non-trivial graphs. Then $G \circ H$ is connected if and only if G is connected.

In view of Corollary 4.7, throughout this section the graph G in $G \circ H$ is connected.

Lemma 4.8. *Given two graphs G and H. Let $\{i,j\}, \{j,k\} \in E(G)$ such that $\{i,k\} \notin E(G)$ and $u, v, w \in V(H)$. Consider the configuration $C_{u^i}^{v^j}$ of $G \circ H$. Then we require at least 3 moves to take the robot from v^j to w^k.*

Proof. Notice that, $\{v^j, w^k\} \in E(G \circ H)$ and $\{u^i, w^k\} \notin E(G[H])$. So to take the robot from v^j to w^k we have to first take the hole to w^k without the robot. Since $d_{G[H]-v^j}(u^i, w^k) = 2$, so we need at least 2 moves to take the hole from u^i to w^k. Also we need one more move of the robot from v^j to w^k. Hence at least three moves are required to take the robot from v^j to w^k. □

Lemma 4.9. Given two graphs G and H. Let $u, v, w \in V(H)$ such that $\{u, v\}, \{v, w\} \in E(H)$ and $\{u, w\} \notin V(H)$. For some $i \in V(G)$, consider the configuration $C_{u^i}^{v^i}$ of $G \circ H$. Then minimum three moves are required to take the robot from v^i to w^i.

Proof. To move the robot from v^i to w^i first we have to take the hole to w^i. Since $d_{G \circ H-v^i}(u^i, w^i) = min\{d_{H-v}(u, w), 2\}$ and $\{u^i, w^i\} \notin E(G \circ H)$, so at least three moves are required to take the robot from v^i to w^i (two moves to take the hole from u^i to w^i plus the robot move $w^i \overset{r}{\leftarrow} v^i$). The proof is complete. □

Theorem 4.5. Given two graphs G and H. Consider the configuration $C_{v^i}^{u^i}$ of $G \circ H$. Let S be a minimum sequence of moves that takes the robot from u^i to v^j for some $j \in V(G)$. Suppose that either $i \neq j$ or $d(u, v) \geq 3$. Then each robot move in S is either a G-move or a N-move.

Proof. Assume that S involves a H-move. We consider the following cases.

Case 1. $i \neq j$. Since H-moves of the robot always keep the robot in the same copy of H, so S involves at least one robot move that is not a H-move.

Case 1a. The first robot move in S is not a H-move: Let $y^p \overset{r}{\leftarrow} x^p$ be the first H-move of the robot in S. So each of the robot moves in S that appears before the move $y^p \overset{r}{\leftarrow} x^p$ is either a G-move or a N-move. Let $x^p \overset{r}{\leftarrow} z^q$ be the robot move preceding $y^p \overset{r}{\leftarrow} x^p$ in S. Since $d(z^q, y^p) = 1$ so $z^q \overset{o}{\leftarrow} y^p$ is the only move between $x^p \overset{r}{\leftarrow} z^q$ and $y^p \overset{r}{\leftarrow} x^p$ in S. Now, if $u^i = z^q$, then

$$v^i \overset{o}{\leftarrow} x^p \overset{r}{\leftarrow} z^q \overset{o}{\leftarrow} y^p \overset{r}{\leftarrow} x^p$$

constitute the first four moves in S and these four moves takes $C_{v^i}^{u^i}$ to $C_{x^q}^{y^p}$. But the sequence $v^i \overset{o}{\leftarrow} y^p \overset{r}{\leftarrow} z^q \overset{o}{\leftarrow} x^p$ also takes $C_{v^i}^{u^i}$ to $C_{x^q}^{y^p}$, a contradiction. Otherwise $x^p \overset{r}{\leftarrow} z^q$ must be preceded by another robot move, say $z^q \overset{r}{\leftarrow} w^t$. Since, $d(w^t, x^p) = 2$ so two obstacle moves are involved between $z^q \overset{r}{\leftarrow} w^t$ and $x^p \overset{r}{\leftarrow} z^q$ in S. Thus, S involves total 5 moves to take $C_{w^t}^{z^q}$ to $C_{x^q}^{y^p}$. But, for some $w_1 \neq z$, the following sequence of moves

$$w^t \overset{o}{\leftarrow} w_1^q \overset{o}{\leftarrow} y^p \overset{r}{\leftarrow} z^q \overset{o}{\leftarrow} x^p$$

also takes $C_{w^t}^{z^q}$ to $C_{x^q}^{y^p}$, a contradiction.

Case 1b. The first move of the robot in S is a H-move: Let $y^p \overset{r}{\leftarrow} x^i$ be the first non H-move of the robot in S. Let $x^i \overset{r}{\leftarrow} z^i$ be the robot move preceding $y^p \overset{r}{\leftarrow} x^i$ in S. Now if $x^i \overset{r}{\leftarrow} z^i$ is not the only robot move preceding $y^p \overset{r}{\leftarrow} x^i$ in S, then S involves at least three moves to reach the configuration $C_{x^i}^{z^i}$ from $C_{v^i}^{u^i}$ and then the two moves

$$z^i \overset{o}{\leftarrow} y^p \overset{r}{\leftarrow} x^i$$

to reach $C_{x^i}^{y^p}$. Thus the sequence S involves at least five moves to reach $C_{x^i}^{y^p}$ from $C_{v^i}^{u^i}$ in $G \circ H$. But the sequence of moves

$$v^i \overset{o}{\leftarrow} y^p \overset{r}{\leftarrow} u^i \overset{o}{\leftarrow} x^p \overset{o}{\leftarrow} x^i$$

also takes the configuration $C_{v^i}^{u^i}$ to $C_{x^i}^{y^p}$ in $G \circ H$, a contradiction.

Otherwise $x^i = u^i$. And so S involves at least three moves to reach $C_{x^i}^{y^p}$ from $C_{v^i}^{u^i}$ in $G \circ H$. Now, if $p = j$, then the following sequence

$$v^i \overset{o}{\leftarrow} v^j \overset{r}{\leftarrow} u^i$$

takes the robot from u^i to v^j, a contradiction. Otherwise, let $z^q \overset{r}{\leftarrow} y^p$ be the robot move that follows $y^p \overset{r}{\leftarrow} x^i$ in S. Clearly $p \neq q$ (since S is minimum) and so $z^q \overset{r}{\leftarrow} y^p$ is not a H-move. Therefore by Lemma 4.8, S involves three moves to reach $C_{y^p}^{z^q}$ from $C_{x^i}^{y^p}$ and hence at least six moves to reach $C_{y^p}^{z^q}$ from $C_{v^i}^{u^i}$ in $G \circ H$. But for some vertex $w \neq y$ in H, the sequence

$$v^i \overset{o}{\leftarrow} y^p \overset{r}{\leftarrow} u^i \overset{o}{\leftarrow} w^p \overset{o}{\leftarrow} z^q \overset{r}{\leftarrow} y^p$$

also takes $C_{v^i}^{u^i}$ to $C_{y^p}^{z^q}$ in $G \circ H$, a contradiction. Hence S cannot involve a H-move.

Case 2. $i = j$ **and** $d(u, v) \geq 3$**:** First we note that the sequence of moves

$$v^i \overset{o}{\leftarrow} v^k \overset{r}{\leftarrow} u^i \overset{o}{\leftarrow} u^k \overset{o}{\leftarrow} v^i \overset{r}{\leftarrow} v^k$$

takes the robot to v^i for some vertex $k \in V(G)$ adjacent to i. So, by Lamma 4.9, the sequence S involves at least two robot that are not H-move. Which in turn implies that, S involves at least three robot moves. But each of these robot moves is preceded by an obstacle move (to place the hole in the appropriate vertex). Thus the number of moves involved in S is at least six, a contradiction.

Thus, each robot move in S, is either a G move or a N-move. □

Corollary 4.8. Given two graphs G and H. Consider the configuration $C_{v^i}^{u^i}$ of $G \circ H$. Suppose that either $i \neq j$ or $d(u, v) \geq 3$. Let S be a minimal sequence of moves that takes the robot from u^i to v^j for some $j \in V(G)$. Then the robot moves in S traces a shortest path connecting u^i to v^j in $G \circ H$. Also the number of moves involved in S is $2 + 3[d(i, j) - 1]$.

Proof. Let $d(i, j) = k$. Notice that, if $[i = i_0, i_1 \cdots i_k = j]$ is a path connecting i and j in G, then $[u^i = u^{i_0}, u^{i_1}, \cdots u^{i_k} = v^j]$ connecting u^i and v^j in $g \circ H$. Also, the sequence of move

$$C_{v^j}^{u^i} : v^{i_0} \overset{o}{\leftarrow} u^{i_1} \overset{r}{\leftarrow} u^{i_0} \overset{o}{\leftarrow} v^{i_1} \overset{o}{\leftarrow} u^{i_2} \overset{r}{\leftarrow} u^{i_1} \cdots v^{i_{k-1}} \overset{o}{\leftarrow} u^{i_k} \overset{r}{\leftarrow} u^{i_{k-1}} : C_{u^{i_{k-1}}}^{v^j}$$

takes the robot from u^i to v^j and it involves exactly $2 + 3(k - 1)$ move. Now, let W be the $u^i - v^j$ walk in $G \circ H$ traced by the robot moves in S. Then each edge in W is either a G-edge or a H-edge (see Theorem 4.5). The first robot move in S must be preceded by at least one obstacle move and by Lemma 4.8, to perform each of the remaining robot moves we require at least three moves. So, W involves at most k edges. But $d(u^i, v^j) = k$, so W involves exactly k edges that is, W is a shortest path connecting u^i and v^j. Hence the proof is complete. □

Theorem 4.6. Given the graphs G and H. Consider the configuration $C_{v^i}^{u^i}$. Let S be a minimum sequence of moves that takes the robot from u^i to v^j for some $j \in V(G)$. Then

(i) $|S| = 1$, if $i = j$ and $\{u, v\} \in E(H)$.

(ii) $|S| = 2 + 3[d(i, j) - 1]$, otherwise.

Proof. We consider the following cases

Case 1. $i = j$ **and** $\{u, v\} \in E(H)$**.** Clearly $\{u^i, v^i\} \in E(G \circ H)$ and so the single move $v^i \overset{r}{\leftarrow} u^i$ takes the robot to v^i. Notice that this move of the robot is a H-move. Thus, S involves a H-move of the robot when $\{u, v\} \in E(H)$ and $i = j$.

Case 2a. $i \neq j$ **or** $d(u, v) \geq 3$**:** Same as the proof of Corollary 4.8.

Case 2b. $i = j$ **and** $d(u, v) = 2$**:** Let $[u, w, v]$ be a path connecting u and v in G. Notice that each of the following two sequences

$$C_{v^i}^{u^i} : v^i \overset{o}{\leftarrow} w^i \overset{r}{\leftarrow} u^i \overset{o}{\leftarrow} w^k \overset{o}{\leftarrow} v^i \overset{r}{\leftarrow} w^i : C_{w^i}^{v^i}$$

$$C_{v^i}^{u^i} : v^i \overset{o}{\leftarrow} w^k \overset{r}{\leftarrow} u^i \overset{o}{\leftarrow} w^i \overset{o}{\leftarrow} v^i \overset{r}{\leftarrow} w^k : C_{w^k}^{v^i}$$

takes the robot u^i to v^i, for some $k \in V(G)$. Since $d(u^i, v^j) = 2$, so S involves at least two robot moves. Also, the number of robot moves in S cannot be more than three, otherwise $|S| \geq 6$. So, S involves exactly two robot moves. Now to perform the first robot move we require at least two moves and the first robot move leaves the hole at u^i. Since, $d(u^i, v^j) = 2$, so to perform the second robot move we require at least three moves.

Hence the the proof is complete. ☐

Chapter 5

Conclusions and Future Work

In this thesis, we have restricted ourselves to the study of the combinatorial aspects of the RMPG (see Chapter 1) problem by introducing more generalized robot moves, called mRJ moves of the robot. It started with the objective to classify the graphs in which any two configuration are reachable by a sequence of moves that consists of simple moves of the obstacles and mRJ moves of the robot for some fixed positive integer m. We referred such graphs as complete mRJ-reachable graphs. In this context, we have obtained lower bound for the diameter of the trees that are complete mRJ-reachable. Also, we have classified the complete mRJ-reachable trees for $m = 1, 2, 3$. Thus, the class of complete mRJ-reachable trees remain unexplored for $m \geq 3$. Notice that, for $m = 0$ our problem simply reduces to the RMPG problem that was introduced in [24].

Moreover, we have also addressed the issue of identifying the minimal complete mRJ-reachable trees for $m = 1, 2, 3$. The tree T_0 in the Figure 2.2 is the only minimal complete $2RJ$-rechable tree. Also, we have shown that the collection $\mathcal{T}$, introduced in the section 2.4, is the complete class of minimal complete $3RJ$-reachable. Thus, it would be also interesting to identify the complete class of minimal complete mRJ-reachable trees for $m \geq 4$.

To add more flavor to the story, we have also introduced the concept of complete S-reachability, for a given set S of positive integers. This is a more generalized version of the RMPG. In this context, we have characterized the cycles and some classes of graphs obtained from cycles by simple graph operation that are complete $\{m\}$-reachable, for some non-negative integer m. We have also obtained some necessary conditions for a graph to be complete $\{m\}$-reachable. But sufficient condition for such graphs is remain as an open problem. It is worth mentioning here that, a necessary and sufficient condition for a graph G to be complete $\{0\}$-reachable is that the graph G is bi-connected. We have also obtained necessary and sufficient condition for a graph to be complete $\{1, 2\}$-reachable. So, another direction of this work may be to characterize the set of positive integers for which complete S-reachability classes are identical. For example, to identify the sets of positive integers S such that a graph G is complete S-reachable if and only if G is complete $\{1, 2\}$-reachable.

This thesis also gives the minimum number of moves for the RMPG problem with a single hole in case of the product of two given graphs. It would be nice to know the minimum number of moves for the RMPG problem with a single hole for the classes of graphs obtained by other well known graph operations. We have observed that, the path traced by the robot moves in such a minimum sequence of moves for each of the product graphs discussed in this thesis are shortest paths, which is not true in general (see example 4.1). Thus, it would be interesting to explore the class of graphs for which such paths are the shortest.

List of published/communicated papers

Based on the work in this thesis, the following research articles are published/ communicated.

1. **B. Deb**, K. Kapoor and S. Pati, *On mRJ reachability in trees*, Discrete Mathematics, Algorithms and Applications, volume 4(4),2012.
2. **B. Deb**, K. Kapoor, *On S-reachability in graphs.* In Indo-Slovenia Conference on Graph Theory and Applications, February 22-24, 2013, Thiruvananthapuram, India.
3. **B. Deb**, K. Kapoor, *Motion planning in Cartesian product graphs.* Discussiones Mathematicae Graph Theory(accepted), 2013.
4. **B. Deb**, K. Kapoor, *Motion planning in product graphs*, to be communicated.

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