A Study on Some Classes of Fractional Differential Equations with Non-instantaneous Impulses

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Dedicated

to

My Parents and Parents-in-Law

CERTIFICATE

It is certified that the work contained in the thesis titled **"A Study on Some Classes of Fractional Differential Equations with Non-instantaneous Impulses"** by **Jayanta Borah**, a student in the Department of Mathematics, Indian Institute of Technology Guwahati for the award of the degree of Doctor of Philosophy has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

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Abstract

Mathematical modeling of many time-dependent physical problems result in initial boundary value problems. These problems can be given abstract formulation in suitable Hilbert spaces, in general Banach spaces, in which the space variables are suppressed to define the domain of an operator. This kind of differential equations, known as evolution equations, are suitable in studying the invariant properties of such problems. Ordinary differential equations defined on an infinite dimensional space can be treated as evolution equations.

In the last three decades, the study of fractional calculus has become a rapidly growing area due to its relevance in diverse fields from physical science, engineering to biological sciences and finance. Although Liouville and Riemann introduced the formal definition of fractional calculus at the end of 19th century, but its idea originated at the same time of the birth of classical calculus. This appeared in the famous correspondence between G.A. de L'Hopital and G.W. Leibniz, in 1695.

Materials involving long-memory and hereditary properties are effectively described by fractional calculus. The mathematical models of systems based on the characteristics of such types of materials results in fractional differential equations and hence it is deemed appropriate to investigate the solution of such equations. In particular, solution of many applied problems in the field of viscoelasticity, biophysics, electric circuits reduce to integral equations, which can be transformed to Abel's integral equation. As a natural extension of Abel's integral equation, the study of integral equations with fractional integral operator merits its place in academic as well as applied interests. Their evolution behaves in a much more complex way than the classical integer-order case and hence is recognized as an alternative model to nonlinear differential equation. The study of fractional differential equations ranges from the abstract aspects of qualitative properties of solution to the analytical and numerical methods for finding solutions. Existence result of fractional evolution equations has been investigated by different researchers. Most of the researchers obtained some results not for the initial value problems but only for the corresponding Volterra integral equations. The continuous solution of the

corresponding Volterra integral equation, known as mild solution, has been proposed in different ways [23, 50, 114, 126].

A natural generalization of initial condition of a Cauchy type problem comes from physics; for instance, these are useful in modeling anomalous diffusion, nonlocal reactive transport in underground water flows in porous media etc. Due to the practical relevance and non-existence of analytic solution in most of the cases, the qualitative properties of solution of various differential equations are effectively examined.

Delay differential equations are differential equations in which the derivatives of some unknown functions at present time are dependent on the values of the functions at previous times. The delay term is either a fixed constant or a distributed delay, often represented by an integral. Fractional differential equation with delay is considered to be valuable in the modeling of many real world problems.

Study of dynamical systems with discontinuous trajectories, or with impulse effect, is one of the most exciting subjects due to its extensive applications in realistic mathematical modeling of a wide variety of practical situations. Fractional differential equations which already have the hereditary property, together with impulse effect, will give better results and the study of the corresponding theory is a hugely demanding task.

This thesis is devoted to the rapidly developing area of the research for the qualitative theory of fractional differential equations. In particular, our main interest lies in the existence of mild solutions of some classes of fractional differential equations with non-instantaneous impulses.

Chapter 1 contains some definitions and preliminary results from fractional calculus, techniques to examine the existence of mild solutions, some fixed point theorems and an introductory idea on semigroup theory. Relevant literature which has motivated us to study fractional differential equations is also included in this chapter.

Chapter 2 provides the existence results of a class of mixed Volterra-Fredholm integro fractional differential equation with non-instantaneous impulses. Two theorems are established in this connection. In the first theorem, the existence and uniqueness of mild solution is shown by considering the nonlinear function as well as the impulse functions of Lipschitz type. In the second theorem, we prove the result by considering a nonlinear function which is not Lipschitz but satisfies some growth condition. The existence of solution is obtained by Krasnoselskii's fixed point theorem.

Chapter 3 deals with the existence of integral solution of a class of non-instantaneous impulsive fractional evolution equations whose linear part is non densely defined on a Banach space. The theorems are proved using fractional calculus and fixed point theorems.

Chapter 4 is concerned with the existence of mild solution to an abstract non-instantaneous impulsive fractional functional evolution equation with bounded delay. The results are achieved by utilizing Burton-Kirk's fixed point theorem.

In Chapter 5, we discuss the existence of mild solution of a class of fractional functional evolution equation with infinite delay. Applying the theory of analytic semigroup and the concept of phase space, we prove the existence results with the aid of Burton-Kirk's fixed point theorem.

Chapter 6 presents the existence of mild solutions of a class of fractional Cauchy problem with non-instantaneous impulses and nonlocal initial conditions. The uniqueness of the solution is established by Banach fixed point theorem. Utilizing Krasnoselskii's fixed point, another existence result is obtained by relaxing the Lipschitz condition on the nonlinear source function.

Chapter 7 describes a fractional Cauchy problem with an almost sectorial operator. Some idea about finding results of mild solution for the problem using the a couple of classical fixed point theorem has been described.

In Chapter 8, we summarize the obtained results and present some ideas on possible future investigation on this topic.

Abbreviation and Notation

$\mathbb{N}, \mathbb{R}, \mathbb{C}$	the set of natural numbers, real numbers, complex numbers.
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$.
$\mathbb{R}^+$	the non negative real numbers.
$f^{(n)}$	the n^{th} derivative of the function f.
$Re(\lambda)$	the real part of the complex number $\lambda \in \mathbb{C}$.
$arg(\lambda)$	the argument of the complex number $\lambda \in \mathbb{C} \setminus \{0\}$.
$\mathcal{M}$	measure function on the σ algebra B.
$D(A)$	Domain of the operator A.
$\mathbb{R}^n$	the real Euclidean space, $n \geq 2$.
$\sigma(A)$	spectrum set of the operator A.
$\rho(A)$	resolvent set of the opeator A.
$L^1_{loc}(J, \mathbb{X})$	the space of all locally integrable function from J into $\mathbb{X}$.
$R(\lambda; A)$	resolvent operator of A.
a.e.	almost everywhere.
$\|.\|$	Norm.
$\Gamma(.)$	the Gamma function.
$\mathscr{L}, \mathscr{L}^{-1}$	the Laplace transform and its inverse transform.
I	the identity operator on a vector space.
Δ	the Laplacian operator.
FDE	fractional differential equation.
FEE	fractional evolution equation.
BVP	boundary value problems.

Contents

Chapter 1

Introduction

Fractional calculus is the theory of derivatives and integrals of arbitrary order, which unifies and generalizes the notion of integer order differentiation and n-fold integration. This subject has been attracting mathematicians and scientists due to its wide applications in fluid flow phenomena, seismology, rheology, electrical networks, viscoelasticity, economics, chaos and fractals, control theory of dynamical systems, nonlinear dynamics, biophysics, image processing and so forth [96]. It is as old as classical calculus, around the time 1695-1697, when Newton and Leibniz independently developed differential and integral calculus. In fact, from the time of conceptualization, fractional calculus has been fascinating the mathematicians and scientists. Subsequently, the subject has become richer by the contributions of Euler, Lagrange, Laplace, Fourier, Liouville, Reimann, Grünwald, Lentikov, Weyl, Davis etc. [71]. The first monograph on this topic was written by Oldham and Spanier [90]. Ross [99] narrated an interesting history on the development of fractional differentiation and integration and their applications from 1695 to 1974. The first book on FDE was written by Miller and Ross [82]. Under the supervision of Torvik, Bagley [16] applied FDE in modeling the behaviour of viscoelastic materials and published the first Ph.D thesis in this field. Although the subject is more than 300 years old, the applications and mathematical background surrounding fractional calculus are far from paradoxical. In recent years, there has been eloquent advancement of FDE in ordinary as well as partial fractional derivatives.

A rigorous study of existence of solution to Cauchy type FDE involving Riemann-Liouville operators started about sixty years ago. Existence of solutions to Cauchy type problems involving the Reimann-Liouville operators and some other problems attracted the attention of several mathematicians. Many partial differential equations and integro-differential equations can be exhibited in FDE and integro-differential equations in suitable Banach spaces. There are some remarkable monographs [71, 82, 96] which provide the main theoretical tools for the qualitative analysis of this research field, and at the same time, show the interconnection as well as the contrast between the classical differential and integral models and fractional differential and integral models. Although the evolution of the models described by differential equations containing fractional-order derivatives is difficult to grasp, the subject is hugely demanding due to its aforementioned applications in various fields of science and engineering such as physics, chemical sciences, biology, control theory, blood flow, aerodynamics, finance etc.

Leibniz invented the notation $\dfrac{d^n y(t)}{dt^n}, n \in \mathbb{N}$ to represent the n-th derivative of a function $y(t)$. There are many possible generalizations of $\dfrac{d^n y(t)}{dt^n}$ to the case when n is not an integer, named as Grünwald-Letnikov, Hadamard, Riemann- Liouville, Caputo's etc. Among these generalizations, Riemann-Liouville derivative and the Caputo derivative are more popular. The former concept is historically the first (developed in the works of Abel, Riemann and Liouville in the first half of the nineteenth century) and the one for which the mathematical theory has been established quite well by now. Although it has certain features that lead to difficulties when applying it to the real-world problems, yet in the success of fractional calculus, the definition of fractional derivatives due to Riemann-Liouville played a significant role. Application of fractional derivatives in real world problem requires utilization of initial conditions $y(0)$, $y'(0)$ and same as boundary conditions so that the problem can be interpreted in a meaningful way. Caputo fractional derivative satisfies these demands [26,30]. The main advantage of fractional calculus over classical calculus lies in the fact that fractional calculus has been considered as one of the finest mechanism to imitate long memory processes [119]. It is worth mentioning that the integer order differential operator is a local operator but the fractional order differential operator is nonlocal. Here the term nonlocal means that the fractional derivative of a function at current state requires not only the information at the neighbouring states but also upon all its past states, i.e., non-integer order derivatives which makes the problem global. In other words, perhaps this subject translates the reality of nature better. Existence and uniqueness results for initial value problems for various FDE and its applications to real-world problems were studied in [71, 96]. Many models are reformulated and expressed in terms of FDE so that their physical meaning can be incorporated in the mathematical models more realistically. FDE has been treated as a substitute model to the nonlinear differential equations. The theory of FDEs in comparison with the classical theory of differential equations is at infant stage of development.

Existence result of FEE has been investigated by different researchers. Most of them obtained some results not for the initial value problems but for the corresponding Volterra integral equations. The continuous solution of the corresponding Volterra integral equation, known as mild solution, has been proposed in different ways. For example, the first one was constructed in terms of probability density function and Laplace transforms, given by El-Borai [50] and later it was developed by Zhou and Jiao [126]. Bazhlekova [23] introduced a β-resolvent family under the Riemann-Liouville fractional derivatives and some constraints, and she demonstrated the existence of solution by employing the well-developed theory of resolvent operators for integral equations. Recently Wang et al. [114] initiated the new concept of piecewise continuous (PC) mild solution for impulsive Cauchy problem.

1.1 Preliminaries

In this section we state some notations, basic definitions, required function spaces and preliminary facts which will be used in developing the problems in this thesis. More details regarding the following definitions and related results can be found in text books such as [70, 71, 96, 100, 119, 126].

1.1.1 Some notations and definitions

Let $(\mathbb{X}, \|.\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|.\|_{\mathbb{Y}})$ be two Banach spaces and $J = [a, b], -\infty < a < b < \infty$, be a finite interval on $\mathbb{R}$. If there is no urgent need, we denote both the norms by the generic symbol $\|.\|$. $B(\mathbb{X}, \mathbb{Y})$ denotes the space of all bounded linear operators with the operator norm denoted by $\|.\|_{B(\mathbb{X},\mathbb{Y})}$. For $\mathbb{X} = \mathbb{Y}$, we denote the space $B(\mathbb{X}, \mathbb{X})$ by $B(\mathbb{X})$ with norm $\|.\|_{B(\mathbb{X})}$. $C(J, \mathbb{X})$ denotes the Banach space of all continuous functions from J into $\mathbb{X}$ with the norm

$$\|f\|_{C(J;X)} = \sup\{\|f(t)\| : t \in J\}.$$

We denote by $L^p(J, \mathbb{X}), 1 \le p \le \infty$, the space of all $\mathbb{X}$-valued measurable functions f on J for which

$$\|f\|_{L^p J}= \left(\int_a^b \|f(\tau)\|^p d\tau \right)^{\frac{1}{p}}, 1 \le p < \infty$$

and

$$\|f\|_{\infty}= esssup_{a \le t \le b} \|f(t)\|.$$

Here esssup$\|f(t)\|$ is the essential maximum of the function f.

$AC(J, \mathbb{X})$ is the space of functions $f : J \to \mathbb{X}$, which are absolutely continuous. It coincides with the space of primitives of summable functions

$$f \in AC(J, \mathbb{X}) \Leftrightarrow f(t) = c + \int_a^t g(\tau)d\tau, g \in L(J, \mathbb{X})$$

and therefore, an absolutely continuous function $f(t)$ has a summable derivative $f'(t) = g(t)$ almost everywhere on $J = [a, b]$. Thus

$$f'(t) = g(t) \text{ and } c = f(a).$$

We denote by $C^N(J, \mathbb{X})$ the Banach space of functions $f : J \to \mathbb{X}$ which are N times continuously differentiable on J with the norm

$$\|f\|_{C^N}= \sum_{k=0}^{N} \max_{t \in J} \|f^{(k)}(t)\|, N \in \mathbb{N}_0.$$

For $0 \le \gamma < 1$, we introduce the weighted space $C_\gamma[a,b]$ of functions f given on $[a,b]$ such that the function $(t-a)^\gamma f(t) \in C[a,b]$, and

$$\|f\|_{C_\gamma} = \|(t-a)^\gamma f(t)\|_{C(J;\mathbb{X})}, C_0[a,b] = C[a,b].$$

By $C^\gamma, 0 \le \gamma < 1$, we denote the Hölder's space defined by

$$C^\gamma = \left\{ f \in C(J,\mathbb{X}) : \sup_{s,t\in J} \frac{\|f(t)-f(s)\|_{\mathbb{X}}}{|t-s|^\gamma} < \infty \right\}.$$

For Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, we denote by $\mathbb{X} \hookrightarrow \mathbb{Y}$ a continuous embedding if

1. $u \in \mathbb{X}$ implies $u \in \mathbb{Y}$.

2. $\|u\|_{\mathbb{Y}} \le K\|u\|_{\mathbb{X}}$,

where the constant $K > 0$ does not depend on u.
Let $\mathcal{C}$ be a class of functions from J to a normed space $\mathbb{X}$.

Definition 1.1.1. *$\mathcal{C}$ is said to be equicontinuous, if for every $\epsilon > 0, \exists\ \delta > 0$ such that*

$$\|f(t_1) - f(t_2)\| < \epsilon \text{ whenever } |t_1 - t_2| < \delta, \forall f \in \mathcal{C},$$

where δ is independent of $f \in \mathcal{C}$.

Definition 1.1.2. *$\mathcal{C}$ is said to be uniformly bounded if there exists $M > 0$ independent of $f \in \mathcal{C}$ such that*

$$\|f(t)\| \le M, \forall t \in J, f \in \mathcal{C}.$$

Definition 1.1.3. *A function $f : J \to \mathbb{X}$ is said to be Hölder continuous with exponent ν, $0 < \nu < 1$ on J if there is a constant L such that*

$$\|f(t_1) - f(t_2)\| \le L|t_1 - t_2|^\nu, \ \forall\ t_1, t_2 \in J.$$

For $\nu = 1$, the function is said to be Lipschitz continuous in J.

Lemma 1.1.1. *(Hölder's inequality) [100] Let p and q be conjugate exponents. Then for $1 \le p \le \infty, fg \in L(J,\mathbb{X})$ and*

$$\|fg\|_{LJ} \le \|f\|_{L^pJ}\|g\|_{L^qJ}, \text{ whenever } f \in L^p(J,\mathbb{X}), g \in L^q(J,\mathbb{X}).$$

Lemma 1.1.2. *(Arzela-Ascoli theorem) [100] If a family* $\mathcal{C}$ *in* $C(J,\mathbb{X})$ *is uniformly bounded and equicontinuous on* J *and for each* $t \in J$*, the set* $\{f(t)\}$ *is relatively compact in* $\mathbb{X}$*, then* $\mathcal{C}$ *in* $C(J,\mathbb{X})$ *is relatively compact.*

Lemma 1.1.3. *(Dominated convergence theorem) [100] Assume that the sequence of functions* $\{f_n\}$ *is measurable and* $f_n \to f$ *a.e. in a measurable set* E*. Suppose that* $\|f_n\| \leq g$ *a.e. for some integrable function* g *in* E*. Then*

$$\lim_{n\to\infty} \int_E f_n(u)du = \int_E f(u)du.$$

Definition 1.1.4. *[59] A closed linear densely defined operator* A *is said to be sectorial if there are constants* $\omega \in \mathbb{R}, \theta \in [\frac{\pi}{2}, \pi], M > 0$ *such that the following two conditions are satisfied:*

(i) $\Sigma_{(\theta,\omega)} = \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| < \theta\} \subset \rho(A)$,

(ii) $\|R(\lambda; A)\|_{B(\mathbb{X})} \leq \frac{M}{|\lambda-\omega|}, \lambda \in \Sigma_{(\theta,\omega)}$,

where $\mathbb{X}$ *is a complex Banach space.*

Definition 1.1.5. *[12] Let* $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ *be a closed linear operator. Then* A *is said to be the generator of a* β*-resolvent family if there exist* $\omega \geq 0$ *and a strongly continuous function* $T_\beta : R^+ \to B(\mathbb{X})$ *such that* $\{\lambda^\beta : \operatorname{Re}(\lambda) > \omega\} \subset \rho(A)$ *and*

$$(\lambda^\beta I - A)^{-1}u = \int_0^\infty e^{\lambda t} T_\beta(\tau) u d\tau, \operatorname{Re}(\lambda) > \omega, u \in \mathbb{X}.$$

In this case, $T_\beta(t)$ *is called* β*-resolvent family generated by* A *and defined as*

$$T_\beta(t) = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} R(\lambda^\beta; A) d\lambda,$$

where γ *is a suitable path lying on* $\Sigma_{(\theta,\omega)}$*.*

Definition 1.1.6. *[5] Let* $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ *be a closed linear operator. Then* A *is said to be the generator of a solution operator if there exist* $\omega \geq 0$ *and a strongly continuous function* $S_\beta : R^+ \to B(\mathbb{X})$ *such that* $\{\lambda^\beta : \operatorname{Re}(\lambda) > \omega\} \subset \rho(A)$ *and*

$$\lambda^{\beta-1}(\lambda^\beta I - A)^{-1}u = \int_0^\infty e^{\lambda\tau} S_\beta(\tau) u d\tau, \operatorname{Re}(\lambda) > \omega, u \in \mathbb{X}.$$

In this case, $S_\beta(t)$ is called the solution operator generated by A and defined as

$$S_\beta(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\beta-1}R(\lambda^\beta; A)d\lambda,$$

where γ is a suitable path lying on $\Sigma_{(\theta,\omega)}$.

Definition 1.1.7. *[85] A solution operator $S_\beta(t)$ is analytic if $S_\beta(t)$ admits an analytic extension to a sector $\Sigma_{\theta_0} = \{\lambda \in C \setminus \{0\} : |\arg \lambda| < \theta_0$, for some $\theta_0 \in (0, \frac{\pi}{2}]\}$.*

1.1.2 Basic fractional calculus

In this subsection we present some definitions, concepts and main properties of fractional integrals and derivatives. First, we consider some classes of functions which are the building blocks of fractional differential equations.

1.1.3 Gamma function

The continuous gamma function $\Gamma(\lambda)$, which interpolates between the factorials and allows n to take non-integer and even complex values, plays an important role in fractional calculus.

A comprehensive definition of the gamma function $\Gamma(\lambda)$ is the one provided by the Euler limit:

$$\Gamma(\lambda) = \lim_{N\to\infty} \frac{N!\,N^\lambda}{\lambda(\lambda+1)(\lambda+2)...(\lambda+N)}, \lambda \neq 0, -1, -2, \cdots$$

but the integral transform definition

$$\Gamma(\lambda) = \int_0^\infty \tau^{\lambda-1}e^{-\tau}d\tau, \mathrm{Re}(\lambda) > 0$$

is often more useful. Integration by parts leads to the property $\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$. This gives that for any $n \in \mathbb{N}, \Gamma(n+1) = n!$ and the value of gamma function at 0 and negative integers is infinite.

1.1.4 Beta function

Sometimes it is convenient to use beta function rather than the values of gamma function. It is defined as

$$\mathcal{B}(\lambda_1, \lambda_2) = \int_0^1 \tau^{\lambda_1 - 1}(1-\tau)^{\lambda_2 - 1} d\tau, \mathrm{Re}(\lambda_1) > 0, \mathrm{Re}(\lambda_2) > 0.$$

The relationship between gamma and beta functions is given by

$$B(\lambda_1, \lambda_2) = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1 + \lambda_2)}.$$

1.1.5 Mittag-Leffler function

In the theory of integer order differential equations, the exponential function plays an important role. The function $E_\beta(\lambda)$ defined as a power-series expansion of the form

$$E_\beta(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\beta k + 1)}$$

plays an analogous role in fractional order differential equations. This one-parameter generalization of exponential function is known as the Mittag-Leffler function. As a pioneering work, Hille and Tamarkin [63] used Mittag-Leffler function to represent the solution of Abel's integral equation of second kind. Baret [22] used it to express the general solution of constant coefficient linear differential equation of non-integer order. Elementary functions are recovered from Mittag-Leffler functions as $E_1(\lambda) = e^\lambda, E_2(\lambda^2) = \cosh(\lambda), E_2(-\lambda^2) = \cos(\lambda)$.

The two-parameter function

$$E_{\beta,\beta_1}(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\beta k + \beta_1)} = \frac{1}{2\pi} \int_{Ha} \frac{\mu^{\beta - \beta_1} e^\mu}{\mu^\beta - \lambda} d\mu, \beta > 0, \beta_1 > 0, \lambda \in \mathbb{C},$$

introduced by Agarwal [3], is known as Mittag-Leffler type function. Here Ha,the Hankel path starts and ends at $-\infty$, after crossing the origin.

The Laplace transform of Mittag-Leffler function of order $\beta > 0$ is

$$\mathscr{L}(E_\beta(at^\beta)) = \frac{s^{\beta-1}}{s^\beta - a}, Re(s) > |a|^{\frac{1}{\beta}}.$$

The most frequently used property of the Mittag-Leffler type functions associated with their Laplace integral is

$$\int_0^\infty e^{-\lambda t} t^{\beta_1 - 1} E_{\beta,\beta_1}(\omega t^\beta) dt = \frac{s^{\beta-\beta_1}}{s^\beta - \omega}, \text{Re}(s) > \omega^{\frac{1}{\beta}}, \omega > 0.$$

Definition 1.1.8. *[9] A real function $f(t)$ is said to be in the space $C_\beta, \beta \in \mathbb{R}$ if there exists a real number $p > \beta$ such that $f(t) = t^p g(t)$ where $g \in C[0, \infty)$ and it is said to be in the space C_β^N iff $f^{(N)} \in C_\beta, N \in \mathbb{N}$.*

A frequently encountered definition of an integral of fractional order is via an integral transform called the Riemann-Liouville integral. To motivate this definition, we recall Cauchy's formula for repeated integration:

$$D_t^{-n} f(t) = \int_a^t \int_a^{t_{n-1}} ... \int_a^{t_1} f(t_0) dt_0 ... dt_{n-2} dt_{n-1} = \frac{1}{(n-1)!} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-n}} d\tau, n = 1, 2, ...,$$

with $D^0 f(t) = f(t)$. Replacing the integer number n by the real number β and the discrete factorial $(n-1)!$ by the continuous gamma function $\Gamma(n)$, the Riemann-Liouville fractional integral is obtained.

Definition 1.1.9. *[119] The Riemann-Liouville fractional integral of order $\beta > 0$ for a function f is defined as*

$${}_aD_t^{-\beta} f(t) = {}_a I_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-\tau)^{(\beta-1)} f(\tau) d\tau, t > a$$

provided the right side is point-wise defined on $[a, b]$.

Definition 1.1.10. *[119] The Riemann-Liouville derivative of order $\beta > 0$ for a function $f \in C_{-1}^n, n \in \mathbb{N}$ is defined as*

$${}_aD_t^\beta f(t) = D^n {}_aD_t^{\beta-n} f(t) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\beta-1} f(\tau) d\tau, t > a, n-1 < \beta < n.$$

Definition 1.1.11. *[119] The Caputo fractional derivative of order $\beta > 0$ for a function $f \in C_{-1}^n, n \in \mathbb{N}$ is defined as*

$${}_a^C D_t^\beta f(t) = {}_aD_t^{\beta-n} D^n f(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t (t-\tau)^{n-\beta-1} f^n(\tau) d\tau, t > a, n-1 < \beta < n.$$

Remark 1.1.1. *(i) The Caputo derivative of constant k is equal to zero whereas the Riemann-Liouville derivative of constant k may not be equal to zero. In fact for a constant k, we have*

$$ {}_aD_t^{\beta}k = \frac{k(t-a)^{-\beta}}{\Gamma(1-\beta)}, 0 < \beta < 1. $$

(ii) The fractional operators are defined through convolution integrals and are, therefore, unlike integer order derivatives, non local operators. They depend on all function values from its lower limit $s = a$ upto the evolution point $s = t$.

(iii) The Caputo fractional derivative of order $\beta > 0$ for a function $f \in L^1([a,b], \mathbb{R}^+)$ in terms of Riemann-Liouvile derivative is defined as

$$ {}_a^CD_t^{\beta}f(t) = {}_aD_t^{\beta}[f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{(k)!} f^{(k)}(a)], t > a, n-1 < \beta < n, $$

which reflects that for zero initial conditions, both the derivatives are same.

Definition 1.1.12. *[119] The Laplace transform of Riemann-Liouville derivative and Caputo derivative are, respectively, given by*

$$ \mathscr{L}[{}_aD_t^{\beta}f(t)](s) = s^{\beta}\mathscr{L}[f(t)](s) - \sum_{k=0}^{n-1} s^k (D^{\beta-k-1}f)(a), $$

$$ \mathscr{L}[{}_a^CD_t^{\beta}f(t)](s) = s^{\beta}\mathscr{L}[f(t)](s) - \sum_{k=0}^{n-1} s^{\beta-k-1} (D^k f)(a). $$

Here the last terms reveal the initial conditions. By comparing both transformations, we see that the initial conditions for Caputo's approach is of the same form as classical analysis and ensures the advantage over the Riemann-Liouvile's approach.

Here we list some important properties of fractional integral and derivative operators which will be useful throughout our thesis.

Lemma 1.1.4. *[71] The fractional integral operator ${}_aI_t^{\beta}$ with $\beta > 0$ is bounded in $L^p([a,b], \mathbb{R}^n), 1 \le p \le \infty :$*

$$ \|{}_aI_t^{\beta}f(t)\|_p \le K\|f\|_p, K = \frac{(b-a)^{\beta}}{\beta\Gamma(\beta)}. $$

Lemma 1.1.5. *[119] If $0 < \beta < 1$ and $1 < p < \frac{1}{\beta,}$ then the operator ${}_aI_t^{\beta}$ is bounded from $L^p([a,b],\mathbb{R}^n)$ into $L^q([a,b],\mathbb{R}^n)$, where $q = \frac{p}{1-\beta p}$.*

Lemma 1.1.6. *[119] If $\beta > 0$ and $\beta_1 > 0$, then*

$$ {}_aD_t^{-\beta}(t-a)^{\beta_1-1} = \frac{\Gamma(\beta_1)}{\Gamma(\beta+\beta_1)}(t-a)^{\beta+\beta_1-1}. $$

Lemma 1.1.7. *[71] (Semigroup property of fractional integration operators) If $\beta > 0$ and $\beta_1 > 0$, then the equations*

$$ {}_aD_t^{-\beta}\,{}_aD_t^{-\beta_1}f(t) = {}_a D_t^{-(\beta+\beta_1)}f(t) $$

are satisfied at almost every point $t \in [a,b]$ for $f \in L^p([a,b],\mathbb{R}^N), 1 \leq p < \infty$. If $\beta+\beta_1 > 1$, then the relationship holds at any point of $[a,b]$.

Lemma 1.1.8. *[119] If $\beta > 0$, $f \in L^\infty([a,b],\mathbb{R}^N), 1 \leq p \leq \infty$, then*

$$ {}_a^CD_t^{\beta}({}_a^CD_t^{-\beta}f(t)) = f(t). $$

Lemma 1.1.9. *[119]. For $0 < \beta \leq 1$ and $f \in AC([a,b],\mathbb{R}^N)$,*

$$ {}_a^CD_t^{-\beta}({}_a^CD_t^{\beta}f(t)) = f(t) - f(a). $$

Note:

(i) For an abstract space valued function, the integrals appearing in the above definitions are taken in Bochner's sense.

(ii) If the lower limit of fractional integration and differentiation is fixed for a particular problem, we will not write it in the expression.

1.2 Impulsive differential equations

Many physical evolutionary processes undergo abrupt changes of state at fixed moments (e.g., shocks, harvesting etc.) or at time intervals (e.g., natural disasters). Over the years, the theory of impulsive differential equations has borne swift development and

played a very paramount role in modeling of modern applied real processes arising in phenomena studied in physics, population dynamics, control theory and economics. The duration of changes is often negligible in comparison with that of the entire evolution process and thus the abrupt changes can be well-approximated in terms of instantaneous changes of state, i.e., impulses. The existence of solutions to impulsive differential equations is an attractive topic in the qualitative theory of impulsive differential equations. The impulsive conditions can be used to model more number of physical phenomena than the traditional initial value problems with $y(0) = x_0$. In general, ordinary differential equations are used to capture the continuous evolution, while the jump conditions represent the moments and the magnitude of the jumps.
Let $\Omega \subset \mathbb{X}$ be the set of all possible states of a process and $y(t)$ be the state at a fixed time t. The set $\Omega \times \mathbb{R}$ is called the phase space of the evolution process. Depending on the moments of the change by jumps, the impulsive differential equations are broadly classified into two categories.

The equations with fixed moments of impulse effect, where the moments of jump are previously fixed, are of the form (for first order impulsive differential equation)

$$y'(t) = f(t, y(t)), t \neq t_k, \tag{1.1}$$

$$\Delta y(t) = I(t, y(t)) = y(t^+) - y(t^-), t = t_k. \tag{1.2}$$

The moments t_k are called moments of impulse response.

In the second impulsive class, the moments of jump occur when certain space-time relations $h(t, y(t)) = 0$ are satisfied. That is, its general form is

$$y'(t) = f(t, y(t)),\ t \in \Omega,\ h(t, y(t)) \neq 0, \tag{1.3}$$

$$\Delta y(t) = I(t, y(t)) = y(t^+) - y(t^-),\ h(t, y(t)) = 0. \tag{1.4}$$

Here $y(t^+) = \lim_{h \to 0} y(t + h), y(t^-) = \lim_{h \to 0} y(t - h)$ represent the right limit and left limit, respectively, of the state at time t. Details on this topic can be found in [28, 76, 102, 106].

1.3 Delay differential equations

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. Delay differential equations are differential equations in which the derivatives of some unknown functions at present time are dependent on the values of the function at previous times. Mathematically, a general delay differential equation with finitely many argument deviations for $y(t) \in \mathbb{R}^n$ takes the form

$$y^{(n)}(t) = f(t, y^{(m_1)}(t - f_1(t)), y^{(m_2)}(t - f_2(t)), ..., y^{(m_l)}(t - f_l(t))). \tag{1.5}$$

Here the functional f, delays $f_i(t) \geq 0, i = 1, 2, ..., l$ and $m_i \geq 0, i = 1, 2, ..., l$ are known. The functional equation (1.5) is called functional differential equation of retarded type (RDE), functional differential equation of neutral type (NDE), functional differential equation of advanced type (ADE), respectively, according as $\max_{1 \leq i \leq l} m_i$ is $<, =, > n$. The RDE is called an equation with time lag and equation with after-effect according to the discrete and continuous behavior of the arguments ν_i.
In application, the first order RDE with discrete retardation

$$y'(t) = f(t, , y(t - f_1(t)), ..., y(t - f_l(t))) \tag{1.6}$$

and continuous retardation

$$y'(t) = f(t, y_t) \tag{1.7}$$

is widely used. Here y_t represents the history (fragment) of the state from y from left to the present point t.

1.4 Calculus of Banach space valued function

Here we present some basic definitions and results of vector-valued functions that are necessary for the study of differential equations in Banach spaces. For more details readers are refered to the book [75].

Let $(\mathbb{X}, \|.\|)$ denote a complex Banach space.

Definition 1.4.1. *Let J be any interval of the real line $\mathbb{R}$. A function $f : J \to \mathbb{X}$ is said to be strongly continuous at $t_1 \in J$ if* $\lim_{t \to t_1} \| f(t) - f(t_1) \| = 0$.

We say that f is strongly differentiable at $t_1 \in J$ if $\lim_{\delta t \to 0} \frac{f(t_1 + \delta t) - f(t_1)}{\delta t}$ exists, where the limit is taken in the sense of the norm of $\mathbb{X}$. We say that f is strongly continuously differentiable at t_1 if the derivative f' of f is strongly continuous at t_1.

Let $(J, \Sigma, \mathcal{M})$ be a measure space. A countable valued function $f : J \to \mathbb{X}$ is said to be Bochner integrable if $\|f\|$ is Lebesgue integrable. A function $f : J \to \mathbb{X}$ is said to strongly measurable in J if there exists a sequence $\{f_n(t)\}_{n=1}^{\infty}$ of countably-valued functions (strongly) converging almost everywhere in J to $f(t)$.

For any $E \in \Sigma$, by definition, we have

$$\int_E f(\tau) d\mathcal{M} = \sum_{k=1}^{\infty} u_k \mathcal{M}(E_k \cap E),$$

where $f(\tau) = u_k$ on $E_k \in \Sigma, k = 1, 2, 3, ...$

A characterization of Bochner integrable function is given by the following theorem:

Theorem 1.4.1. *(Bochner's theorem) A function $f : J \to \mathbb{X}$ is Bochner-integrable if and only if $f(t)$ is strongly measurable and $\|f(t)\|$ is Lebesgue measurable. Further we have the estimate*

$$\| \int_J f(t) dt \| \leq \int_J \|f(t)\| dt.$$

1.5 Methods

1.5.1 Semigroup of bounded linear operators

Semigroup theory of bounded linear operators plays a vital role in establishing solution of nonlinear differential equation in abstract spaces. This theory of bounded linear operators developed rapidly after the generation theorem by Hille and Yosida in 1948, as detailed in Pazy [92], and by now it is an interesting and expanding area in applied mathematics. For more details on the theory of semi-group of bounded linear operators, readers are referred to the books [62, 75, 92].

A one-parameter family $\{Q(t)\}_{t \geq 0}$ of bounded linear operators from a Banach space $\mathbb{X}$ to itself is said to be a semigroup of bounded linear operators on $\mathbb{X}$ if

(i) $Q(0) = I$, where I is the identity operator on $\mathbb{X}$ and

(ii) $Q(t+s) = Q(t)Q(s)$ for every $t, s \geq 0$ (the semigroup property).

If the semigroup $\{Q(t)\}_{t\geq 0}$ is strongly continuous, i.e., $\lim_{t\to 0^+} Q(t)u = u, \forall u \in \mathbb{X}$, then it is said to be a C_0 semigroup.

Let $\{Q(t)\}_{t\geq 0}$ be a C_0 semigroup of bounded linear operators on $\mathbb{X}$. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|Q(t)\| \leq Me^{\omega t} \quad \text{for} \quad 0 \leq t < \infty.$$

If $\omega = 0$, the semigroup $\{Q(t)\}_{t\geq 0}$ is said to be uniformly bounded. If $\omega = 0$, and $M = 1$, the semigroup $\{Q(t)\}_{t\geq 0}$ is said to be a contraction semigroup. A semigroup of bounded linear operators $\{Q(t)\}_{t\geq 0}$ is said to be uniformly continuous if

$$\lim_{t\to 0^+} \|Q(t) - I\| = 0.$$

The infinitesimal generator of the semigroup $\{Q(t)\}_{t\geq 0}$ is the linear operator A and is defined as $D(A) = \left\{ u \in \mathbb{X} : \lim_{t\to 0} \dfrac{Q(t)u - u}{t} \text{ exists} \right\}$ and

$$Au = \lim_{t\to 0} \frac{Q(t)u - u}{t} \quad \text{for } u \in D(A).$$

Let A be the infinitesimal generator of a C_0 semigroup $\{Q(t)\}_{t\geq 0}$. Then the following hold:

(i) For $u \in \mathbb{X}$,
$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} Q(s)ud\tau = Q(t)u.$$

(ii) For $u \in \mathbb{X}$, $\int_0^t Q(\tau)ud\tau \in D(A)$ and
$$A(\int_0^t Q(\tau)ud\tau) = Q(t)u - u.$$

(iii) For $u \in D(A), Q(t)u \in D(A)$ and
$$\frac{d}{dt} Q(t)u = AQ(t)u = Q(t)Au.$$

(iv) For $u \in D(A)$,
$$Q(t)u - Q(s)u = \int_s^t Q(\tau)Au \, d\tau = \int_s^t AQ(\tau)ud\tau.$$

The set of all complex numbers λ such that $\lambda I - A$ is invertible is called the resolvent set $\rho(A)$ of a linear operator A.

Theorem (Hille-Yosida) [92] A linear (unbounded) operator A on $\mathbb{X}$ is the infinitesimal generator of a C_0 semigroup of contraction $\{Q(t)\}_{t\geq 0}$ if and only if

(i) A is closed and $D(A)$ is dense in $\mathbb{X}$,

(ii) the resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$, and for every $\lambda > 0$,

$$\|R(\lambda; A)\| \leq \frac{1}{\lambda}.$$

Theorem (Hille-Yosida-Phillips) [92] A linear operator A on $\mathbb{X}$ is the infinitesimal generator of a C_0 semigroup $\{Q(t)\}_{t\geq 0}$ with $\|Q(t)\| \leq M, (M \geq 1)$ if and only if

(i) A is closed and $D(A)$ is dense in $\mathbb{X}$,

(ii) the resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$, and

$$\|R(\lambda; A)^n\| \leq \frac{M}{\lambda^n} \quad \text{for} \quad \lambda > 0, n = 1, 2, ...$$

Pazy [92] discussed the existence and uniqueness of evolution equations using semigroup theory and fixed point theorems. By using the abstract approach, it is possible to extend the analysis developed in finite dimensional linear systems to infinite dimensional linear systems.

Since fractional derivative does not satisfy the Leibnitz rule and the corresponding fractional operator equation does not satisfy the properties of semigroups, $T_\beta(t) = t^{\beta-1}E_{\beta,\beta}(At^\beta)$ does not satisfy the semigroup relations, that is, $T_\beta(t+s) \neq T_\beta(t)T_\beta(s)$, indicating the properties of corresponding resolvent operator. It is difficult to define the mild solutions of fractional partial differential evolution equations.

1.5.2 Measure of noncompactness

Measure of noncomapctness is an important tool to study nonlinear analysis. The measure of noncompactness $\mathscr{M}$ is a function from the set $\mathbb{B}$ of bounded sets of a Banach space to the set of non-negative real numbers. For details on this topic, readers can refer to [20, 47, 77, 119].

It is said to be

(i) Monotone if for all sets B_1, B_2 of $\mathbb{B}$, $B_1 \subset B_2$ implies $\mathscr{M}(B_1) \leq \mathscr{M}(B_2)$;

(ii) Non-singular if $\mathscr{M}(\{u\} \cup B) = \mathscr{M}(B)$ for every $u \in \mathbb{X}$ and nonempty member of B of $\mathbb{B}$;

(iii) Regular if $\mathscr{M}(B) = 0$ for relatively compact set $B \in \mathbb{B}$ in $\mathbb{X}$.

One of the most important measures of noncompactness is Hausdorff measure of noncompactness (χ) and defined as, for each bounded set B of $\mathbb{X}$,

$$\chi(B) = \inf\{\epsilon > 0 : B \subset \bigcup_{i=1}^{m} B_\epsilon(u_j) \text{ where } u_j \in \mathbb{X}\},$$

where $B_\epsilon(u_j)$ is a ball of radius $\leq \epsilon$ centered at u_j, $j = 1, 2, ..., m$.

Lemma 1.5.1. *Let $\mathbb{X}$ be a Banach space, and let $B, B_1, B_2 \subset \mathbb{X}$ be bounded. Then*

(i) $\chi(B_1) = 0$ *if and only if B is pre-compact;*

(ii) $B_1 \subset B_2$ *implies* $\chi(B_1) \leq \chi(B_2)$;

(iii) $\chi(B_1 \cup B_2) \leq \max\{\chi(B_1), \chi(B_2)\}$;

(iv) $\chi(B_1 + B_2) \leq \chi(B_1) + \chi(B_2)$, *where* $B_1 + B_2 = \{u + v : u \in B_1, v \in B_2\}$;

(v) $\chi(cB_1) \leq |c|\chi(B_1)$ *for any* $c \in \mathbb{R}$.

Definition 1.5.1. *A continuous map $\mathcal{T} : B \subset \mathbb{X} \to \mathbb{X}$ is said to be a χ-contraction if there exists a positive constant $\nu < 1$ such that $\chi(\mathcal{T}U) \leq \nu\chi(U)$ for any bounded closed subset $U \subseteq B$.*

Lemma 1.5.2. *If $B \subset C(J, \mathbb{X})$ is bounded and equicontinuous, then the closure of the convex hull $\overline{coB} \subset C(J, \mathbb{X})$ is also bounded and equicontinuous.*

Lemma 1.5.3. *If $B \subset C(J, \mathbb{X})$ is bounded and equicontinuous, then $t \to \chi(B(t))$ is continuous on J, and*

$$\chi(B) = \max_{t \in J} \chi(B(t)).$$

Lemma 1.5.4. *If B is bounded, then for each $\epsilon > 0$, there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset B$ such that*

$$\chi(B) \leq 2\chi(\{y_n\}_{n=1}^{\infty}) + \epsilon.$$

Lemma 1.5.5. *[83] Let $\{y_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from J into $\mathbb{X}$ with $\|y_n(t)\| \leq m(t)$ for almost all $t \in J$ and every $n \in \mathbb{N}$, where $m \in L^1(J, \mathbb{R}^+)$, then the function $\phi(t) = \chi(\{y_n\}_{n=1}^{\infty})$ belongs to $L^1(J, \mathbb{R}^+)$ and satisfies*

$$\chi\Big(\Big\{\int_0^t y_n(\tau) d\tau : n \in \mathbb{N}\Big\}\Big) \leq 2\int_0^t \phi(\tau) d\tau.$$

1.5.3 Fixed point theorems

Fixed point theorems are considered to be most effective tools to study the existence and uniqueness of nonlinear system of differential equations. Due to its importance, several researchers have studied the problems represented by evolution equations by using different kinds of fixed point theorems. In this method, the differential equation is recast as an integral equation in a function space. The most frequently used fixed point theorems are Banach fixed point theorem, Krasnoselskii's fixed point theorem, Schauder's fixed point theorem and Burton-Kirk's fixed point theorem.

Theorem 1.5.1. *(Banach Fixed point theorem) [72] Let $(\mathbb{X}, \|.\|)$ be a Banach space. Let $\mathscr{T} : \mathbb{X} \to \mathbb{X}$ be a mapping such that there exists a constant $k, 0 < k < 1$ and*

$$\|\mathscr{T}(u) - \mathscr{T}(v) \leq k\|u - v\|$$

for all $u, v \in \mathbb{X}$. Then $\mathscr{T}$ has a precisely one fixed point in $\mathbb{X}$.

Definition 1.5.2. *A bounded map $f : \mathbb{X} \to \mathbb{X}$ between Banach spaces $\mathbb{X}$ and $\mathbb{X}$ is said to be compact if the image of a bounded set S in $\mathbb{X}$ is precompact in $\mathbb{X}$.*

Theorem 1.5.2. *(Krasnoselskii's fixed point theorem) [119] Let* $\mathbb{X}$ *be Banach space,* E *be a bounded closed convex subset of* $\mathbb{X}$ *and* $F_1, F_2 : \mathbb{X} \to \mathbb{X}$ *be mappings of* E *into* $\mathbb{X}$ *such that* $F_1u + F_2v \in E$ *for every pair* $u, v \in E$*. If* F_1 *is a contraction and* F_2 *is completely continuous, then the equation* $w = F_1w + F_2w$ *has a solution in* E*.*

Theorem 1.5.3. *(Schauder's fixed point theorem) [120] Let* $E \subset \mathbb{X}$ *be a nonempty, closed, bounded, convex subset of a Banach space* $\mathbb{X}$*, and* $\mathscr{T} : E \to E$ *a compact operator. Then* $\mathscr{T}$ *has a fixed point.*

Theorem 1.5.4. *(Burton-Kirk's fixed point theorem) [35] Let* $\mathbb{X}$ *be a Banach space and* F_1, F_2 *be two operators satisfying*

(i) F_1 *is a contraction and*

(ii) F_2 *is completely continuous.*

Then, either the operator equation $u = F_1(u) + F_2(u)$ *possesses a solution, or the set* $\mathscr{E} = \{u \in \mathbb{X} : \delta F_1(\frac{u}{\delta}) + \delta F_2(u) = u, \text{ for some } 0 < \delta < 1\}$ *is unbounded.*

1.6 Mild solutions for abstract fractional Cauchy problem

Consider the following Cauchy problem of fractional evolution equation with Caputo derivative:

$$ {}^C D_t^\beta y(t) = Ay(t) + f(t, y(t)), \beta \in (0, 1], \text{ a.e. } t \in J = [0, T], \tag{1.8} $$

$$ y(0) = x_0, \tag{1.9} $$

where the state function y takes values in a Banach space $\mathbb{X}$, A is a closed linear operator which generates a C_0 semigroup $\{Q(t)\}_{t\geq 0}$ in $\mathbb{X}$ and f is a given function.

Two approaches are generally adopted to find the solution of the above problem: Integration method and Laplace transform method. In integration method, after taking the Riemann-Liouville integration on both sides, then only Laplace transform is employed on the resulting equations. On Laplace inversion, the solution is then obtained in terms of probability density function.

However, in the second method, the solution is obtained directly by applying Laplace transform. Here, the solution is then obtained in terms of Mittag-Leffler function.

The problem (1.8)-(1.9) is equivalent to the integral form

$$y(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1}[Ay(\tau) + f(\tau, y(\tau))]d\tau. \tag{1.10}$$

Let us take $f \in L^1_{loc}(J, \mathbb{X})$.

On taking Laplace transform and after simplification, equation (1.10) becomes

$$\int_0^\infty e^{-\lambda s} y(\tau)d\tau = \lambda^{\beta-1} \int_0^\infty e^{-\lambda^\beta s} Q(s)x_0 d\tau + \int_0^\infty e^{-\lambda^\beta s} Q(s)\omega(\lambda)d\tau, \tag{1.11}$$

where $\omega(\lambda) = \int_0^\infty e^{-\lambda s} f(\tau, y(\tau))d\tau$.

By applying inverse Laplace transform on (1.11), we get the following representation:

$$y(t) = \mathscr{T}(t)x_0 + \int_0^t (t-\tau)^{\beta-1} \mathscr{S}(t-\tau)f(\tau)d\tau. \tag{1.12}$$

The explicit formulation of $\mathscr{T}$ and $\mathscr{S}$ are given by [126]

$$\mathscr{T}(t)u = \int_0^\infty \phi_\beta(\theta)Q(t^\beta\theta)ud\theta, \tag{1.13}$$

$$\mathscr{S}(t)u = \beta \int_0^\infty \theta\phi_\beta(\theta)Q(t^\beta\theta)ud\theta, \tag{1.14}$$

where ϕ_β is the probability density function defined on $(0, \infty)$, that is, $\phi_\beta(\theta) \geq 0$ and $\int_0^\infty \phi_\beta(\theta)d\theta = 1$. Moreover, ϕ_β has the expression

$$\phi_\beta(\theta) = \frac{1}{\beta}\theta^{-1-\frac{1}{\beta}}\psi_\beta(\theta^{-\frac{1}{\beta}}),$$

where

$$\psi_\beta(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-\beta n-1}\frac{\Gamma(n\beta+1)}{n!}\sin(n\pi\beta).$$

For the second method, we take Laplace transform on both sides of equation (1.8), which results in

$$\lambda^\beta \mathscr{L}\{y(t)\} - \lambda^{\beta-1}y(0) = A\mathscr{L}\{y(t)\} + \mathscr{L}\{f(t, y(t))\}. \tag{1.15}$$

If $\lambda^{\beta} \in \rho(A)$, then (1.15) takes the form

$$\mathscr{L}\{y(t)\} = \frac{\lambda^{\beta-1}}{(\lambda^{\beta} I - A)} x_0 + \frac{1}{(\lambda^{\beta 1} I - A)} \mathscr{L}\{f(t, y(t))\}. \tag{1.16}$$

On Laplace inversion, we get

$$y(t) = E_{\beta,1}(At^{\beta}) x_0 + \int_0^t (t-\tau)^{\beta-1} E_{\beta,\beta}(A(t-\tau)^{\beta}) f(\tau, y(\tau)) d\tau. \tag{1.17}$$

Denote $E_{\beta,1}(At^{\beta}) = S_{\beta}(t)$ and $t^{\beta-1} E_{\beta,\beta}(At^{\beta}) = T_{\beta}(t)$. Then equation (1.17) can be expressed as

$$y(t) = S_{\beta}(t) x_0 + \int_0^t T_{\beta}(t-\tau) f(\tau, y(\tau)) d\tau. \tag{1.18}$$

Definition 1.6.1. *A function $y \in C(J, \mathbb{X})$ satisfying the integral equation (1.12) or (1.18) is called a mild solution of (1.8)-(1.9).*

1.7 Review of literature

The concept of fractional differential equation (FDE) has been growing as an essential place of research due to the fact that many complex systems with anomalous dynamics, such as the transport of chemical contaminants through water around rocks, the dynamics of viscoelastic materials as polymers, the effect of of speculation on the probability of stocks in financial markets and many more phenomena, can be better understood than the corresponding concept of classical differential equations. In real sense, the classical derivative models are not efficient to explain anomalous processes having macroscopic complex behaviour. Corresponding models of some of these processes have been addressed using tools from statistical physics.

FEE, i.e., evolution equation where the integer derivative with respect to time is replaced by a derivative of fractional order, is an emerging branch of research. There are several types of FDEs that arise in many fields of science and engineering and have great importance [71]. Zhang [122] proved that the solution of the FDE

$${}^{C}D_t^{\beta} y(t) = 0, \beta > 0 \tag{1.19}$$

is of the form $y(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1}, c_i \in \mathbb{R}, i = 0, 1, 2, ...n-1, n = [\beta] + 1$. Kilbas and Marzan [70] investigated the existence of solution of the nonlinear FDE of the form

$$ {}^{C}D_t^{\beta} y(t) = f(t), t \in [0, T], 0 < \beta < 1, \tag{1.20} $$

$$ y(0) = x_0. \tag{1.21} $$

They proved that a function y is a solution of the FDE if and only if y is a solution of the fractional integral equation

$$ y(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) d\tau. \tag{1.22} $$

In one of the pioneering works on FEE, El-Sayed [51] investigated the diffusion wave equation of the form

$$ D_t^{\beta} y(t) = Ay(t), t > 0, 0 < \beta \leq 2, \tag{1.23} $$

which became the standard diffusion equation when $\beta \to 1$ and wave equation as $\beta \to 2$. Under the assumption that $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ was a closed densely defined linear operator, he established the existence, uniqueness and the properties of the solution of the negative-direction fractional diffusion wave problem.

Bezhlekova [23] established a necessary and sufficient condition for the existence of solution of Cauchy problem of an abstract FEE of the form

$$ D_t^{\beta} y(t) = Ay(t), t > 0, 0 < \beta < 1, \tag{1.24} $$

with A as an unbounded closed linear operator.

El-Borai [50] investigated the existence of solution of the fractional Cauchy problem

$$ D_t^{\beta} y(t) = Ay(t) + f(t), 0 < \beta < 1, \tag{1.25} $$

$$ y(0) = x_0 \in D(A), \tag{1.26} $$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ was a closed linear operator defined on a Banach space $\mathbb{X}$ and f an abstract function defined on $[0, \infty)$ and with values in $\mathbb{X}$. He established that if f

satisfied a uniform Hölder condition, with exponent $\beta \in (0,1)$, then the unique solution of the Cauchy problem could be found as

$$y(t) = \int_0^\infty \xi_\beta(\theta) Q(t^\beta \theta) x_0 d\theta + \beta \int_0^t \int_0^\infty \theta (t-\eta)^{\beta-1} \xi_\beta(\theta) Q((t-\eta)^\beta \theta) f(\eta) d\theta d\eta.$$

Jaradat et al. [67] investigated the existence and uniqueness of mild solution for mixed Volterra-Fredholm integro-differential equation of the form

$$ {}^C D_t^\beta y(t) = Ay(t) + f(t, y(t), Gy(t), Sy(t)), t > t_0, 0 < \beta < 1, \tag{1.27}$$

$$ y(t_0) = x_0, \tag{1.28}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ was the infinitesimal generator of a C_0 semigroup $\{Q(t)\}_{t \geq 0}$ in a Banach space $\mathbb{X}$. They defined the mild solution of the problem as a continuous function $y(t)$ satisfying the integral equation

$$y(t) = Q(t - t_0)x_0 + \frac{1}{\Gamma(\beta)} \int_0^t Q(t-\tau) f(\tau, y(\tau), Gy(\tau), Sy(\tau)) d\tau.$$

The result was obtained with the help of Banach contraction principle and Grönwall's lemma.

Hernández et al. [61] proved by examples that the concept of mild solution given by Jaradat et al. [67] was not correct. Using the resolvent operator, they introduced a new concept of mild solution of the FDE

$$ {}^C D_t^\beta y(t) = Ay(t) + f(t, y(t)), t \in [0,T], 0 < \beta < 1, \tag{1.29}$$

$$ y(0) = x_0, \tag{1.30}$$

with A as the infinitesimal generator of a C_0 semigroup of bounded linear operators $\{Q(t)\}_{t\geq 0}$ in a Banach space $\mathbb{X}$, $x_0 \in \mathbb{X}$ and $f \in C([0,T] \times \mathbb{X}; \mathbb{X})$. They established that a function $x \in C([0, T_1]; \mathbb{X}), 0 < T_1 \leq T$, could be a mild solution of (1.29)-(1.30) if $\int_0^t \frac{y(\tau)}{(t-\tau)^{1-\beta}} d\tau \in D(A)$ for all $t \in [0, T_1]$ and

$$y(t) = x_0 + \frac{1}{\Gamma(\beta)} A \int_0^t \frac{y(\tau)}{(t-\tau)^{1-\beta}} d\tau + \frac{1}{\Gamma(\beta)} A \int_0^t \frac{f(\tau, y(\tau))}{(t-\tau)^{1-\beta}} d\tau, t \in [0,T].$$

BVPs with integral boundary conditions constitutes a fascinating and important class of problems. They include two, three, multi point and nonlocal BVP as special cases [66]. The nonlocal initial value problems generalizes the classical initial value problems. In fact the nonlocal conditions can be applied in better effects to a Cauchy problem in comparison to its corresponding initial condition [48]. The earliest work related to abstract evolution problem subject to nonlocal initial conditions was accomplished by Byszewski [36]. In this work, using the method of semigroup theory and the Banach fixed point theorem, the existence of mild and strong solutions for the following first order Cauchy problem was proved:

$$y'(t) = Ay(t) + f(t, y(t)), t \in [0, 1], \tag{1.31}$$

$$y(0) = g(y), \tag{1.32}$$

where A was an operator defined in a Banach space $\mathbb{X}$ which generated a C_0 semigroup $\{Q(t)\}_{t\geq 0}$, and the maps f and g two suitable $\mathbb{X}$-valued functions. Subsequently, Byszewski and Laksmikantham [37] investigated the same type of problem for a different class of evolution equations in a Banach space.

Deng [48] used

$$g(y) = \sum_{k=1}^{p} c_k y(t_k),$$

where $c_k, k = 1, 2, ..., p$ were given constants and $0 < t_1 < t_2 < ... < t_p \leq 1$, to represent the dissipation phenomenon of a small quantity of gas in a permeable tube [30, 87]. In this case, the Cauchy problem allowed additional measurements at $t_k, k = 1, 2, ..., p$. From the theoretical standpoint, the nonlocal condition above appeared more general than the classical initial value condition.

Balachandran and Park [18] studied the existence of solution of nonlocal Cauchy problem for abstract FEEs of the form

$${}^{C}D_t^{\beta} y(t) = A(t)y(t) + f(t, y(t)), t \in [0, T], 0 < \beta < 1, \tag{1.33}$$

$$y(0) + g(y) = x_0, \tag{1.34}$$

with $A(t)$ as a bounded linear operator and $f : J \times \mathbb{X} \to \mathbb{X}$ continuous. This problem is equivalent to integral equation of the form

$$y(t) = x_0 - g(y) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} A(\tau)y(\tau)d\tau + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau, y(\tau)d\tau. \tag{1.35}$$

By a local solution of an abstract Cauchy problem (1.33)-(1.34), they considered an abstract function y such that the following conditions were satisfied:

(i) $y \in C(J, \mathbb{X})$ and $y \in D(A(t))$ on J,

(ii) ${}^C D_t^\beta y(t)$ exists and is continuous on J, where $0 < \beta < 1$.

(iii) y satisfies equations (1.33)-(1.34) with initial condition $y(0) + g(y) = x_0$ or equivalently y satisfies the integral equation (1.35).

Existence of solution of the problem (1.33)-(1.34) was proved by Banach contraction theorem.

Balachandran and Trujillo [19] extended the nonlocal problem for quasi-linear integro-differential equation of the form

$$
\begin{aligned}
{}^C D_t^\beta y(t) &= A(t, y)y(t) + f(t, y(t), \int_0^t h(t, \tau, y(\tau))d\tau), t \in [0, T], &(1.36)\\
y(0) + g(y) &= x_0, &(1.37)
\end{aligned}
$$

with $A(t, x)$ as a bounded linear operator on $\mathbb{X}, 0 < \beta < 1$ and continuous functions $f : J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}, h : J \times J \times \mathbb{X} \to \mathbb{X}$. Debbouche [46] studied the nonlocal fractional Cauchy problem of the form

$$
\begin{aligned}
D^\beta y(t) + A(t)y(t) &= f(t, y(t)) + \int_{t_0}^t G(t-\tau)g(\tau, y(\tau))d\tau, 0 < \beta \le 1, 0 \le t_0 < t,\\
y(t_0) + h(y) &= x_0,
\end{aligned}
$$

where $(-A(t))$ generated an evolution operator in the Banach space $\mathbb{X}$, the function G was real-valued and locally integrable on $[t_0, \infty)$, the nonlinear map f and g were defined on $[t_0, \infty) \times \mathbb{X}$ into $\mathbb{X}$ and $h : C(J, \mathbb{X}) \to D(\bar{A})$ was a given function.

Bragdi and Hazi [34] considered the quasi-linear fractional integro-differential equation:

$$
\begin{aligned}
{}^C D_t^\beta y(t) + A(t, y(t))y(t) &= f(t, y(t), \int_0^t k(t, \tau, y(\tau))d\tau, \int_0^t h(t, \tau, y(\tau))d\tau), 0 < \beta \le 1,\\
y(0) + g(t_1, t_2, ..., t_p, y(.)) &= x_0, x_0 \in \mathbb{X},
\end{aligned}
$$

where $0 \le t_1 < t_2 < ... < t_p \le T, -A(t)$ generated an evolution operator in the Banach space $\mathbb{X}$. The nonlinear functions f, g, k, h were given.

Benchohra et al. [29] considered the following Caputo FDE with nonlinear integral boundary conditions in a Banach space $\mathbb{X}$:

$$^{C}D^{\beta}y(t) = f(t,y(t)), t \in J = [0,T], 1 < \beta \leq 2, \tag{1.38}$$

$$y(0) - y'(0) = \int_0^T g(\tau, y(\tau))d\tau, \tag{1.39}$$

$$y(t) - y'(T) = \int_0^T h(\tau, y(\tau))d\tau. \tag{1.40}$$

Using the concept of measure of noncompactness, they established the existence of mild solution of the equation under some assumptions on the functions f, g and h.

Benchohra and Slimani [31] studied the existence and uniqueness of solution of fractional impulsive differential equation of the form

$$^{C}D_t^{\beta}y(t) = f(t,y(t)), t \in J = [0,T], t \neq t_k, \tag{1.41}$$

$$\Delta y|t_k = I_k(y(t_k^-)), \tag{1.42}$$

$$y(0) = x_0, \tag{1.43}$$

where $k = 1, 2, 3..., p$, $0 < \beta \leq 1$. $f : J \times \mathbb{R} \to \mathbb{R}, I_k : R \to \mathbb{R}$ were given functions. $0 < t_1 < t_2 < ... < t_{p+1} \leq T$, $\Delta y|t_k = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ were right hand limit and left hand limit at $t = t_k$, respectively.

A function $y \in PC(J,\mathbb{R})$ is called a solution of (1.41)-(1.43) if y satisfies the equivalent integral equation

$$y(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau, y(\tau))d\tau, & t \in [0, t_1); \\ . & \\ . & \\ . & \\ x_0 + \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-1} f(\tau, y(\tau))d\tau + \dfrac{1}{\Gamma(\beta)} \int_{t_k}^{t} (t-\tau)^{\beta-1} f(\tau, y(\tau))d\tau & \\ + \sum_{0<t_k<t} I_k(y(t_k^-)), & t \in (t_k, t_{k+1}], k = 1, 2, \ldots N. \end{cases}$$

The existence of local solution has also been obtained in terms of piecewise continuous function. Wang and Li [113] investigated the existence and uniqueness of solutions for a mixed boundary value problem of nonlinear impulsive differential equation of fractional order $\beta \in (1, 2]$ given by

$$\begin{aligned} {}^C D_t^\beta y(t) &= f(t, y(t)), t \in J = [0, T], t \neq t_k, && (1.44)\\ \Delta y(t_k) &= I_k(y(t_k)), && (1.45)\\ \Delta y'(t_k) &= I_k^*(y(t_k)), k = 1, 2, ..., p, && (1.46)\\ Ty'(0) &= -cy(0) - dy(t), && (1.47)\\ Ty'(T) &= c_1 y(0) + d_1 y(t), && (1.48) \end{aligned}$$

where $c, d, c_1, d_1 \in \mathbb{R}, I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $\Delta y'(t_k) = y(t_k^+) - y(t_k^-)$. The result was established by using contraction mapping theorem. Using Krasnoselskii's fixed point theorem, Ahmad and Sivasundaram [7] investigated some existence results for a BVP of nonlinear impulsive differential equations of fractional order $\beta \in (1, 2]$ with integral boundary conditions of the form

$$\begin{aligned} {}^C D_t^\beta y(t) &= f(t, y(t)), t \in J = [0, 1], t \neq t_k, && (1.49)\\ \Delta y|t_k &= I_k(y(t_k)), \Delta y'|t_k = I_k^*(y(t_k)), k = 1, 2, ..., p, && (1.50)\\ cy(0) + dy'(0) &= \int_0^1 q_1(y(\tau))d\tau, && (1.51)\\ cy(1) + dy'(1) &= \int_0^1 q_2(y(\tau))d\tau, && (1.52) \end{aligned}$$

with $f : J \times \mathbb{R} \to \mathbb{R}$ continuous, $q_1, q_2 : \mathbb{R} \to \mathbb{R}$ and $c > 0, d \geq 0$.

Fečkan et al. [53] proved by examples that the proposed formulae of solution of impulsive FDEs were not correct.

For the following impulsive FDE

$$\begin{aligned} {}^C D_t^\beta y(t) &= f(t, y(t)), t \in J = [0, T], t \neq t_k, 0 < \beta < 1, && (1.53)\\ y(t_k^+) &= y(t_k^-) + y_k, && (1.54)\\ y(0) &= x_0, y_k \in \mathbb{R}, && (1.55) \end{aligned}$$

they proposed a new concept and defined a formula for the solution. According to them the correct formula of solutions for (1.53)-(1.55) should be

$$y(t) = \begin{cases} x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau, y(\tau)) d\tau, & t \in [0, t_1]; \\ x_0 + y_1 + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau, y(\tau)) d\tau, & t \in (t_1, t_2]; \\ . \\ . \\ . \\ x_0 + \sum_{i=1}^{m} y_i + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau, y(\tau)) d\tau, & t \in (t_N, T]. \end{cases}$$

Using this idea, Anguraj et al. [10] investigated the existence results for fractional integro-differential equations with impulsive and integral conditions of the form

$$\begin{aligned} {}^C D_t^\beta y(t) &= f(t, y(t), \int_0^t k(t, \tau, y(\tau)) d\tau), t \in J = [0, 1], t \neq t_k, & (1.56) \\ y(t_k^+) &= y(t_k^-) + y_k, y_k \in \mathbb{R}, k = 1, 2, ..., p, & (1.57) \\ y(0) &= \int_0^1 g(\tau) y(\tau) d\tau. & (1.58) \end{aligned}$$

Mophou [86] studied the mild solution of the abstract impulsive evolution equation of the form

$$\begin{aligned} {}^C D_t^\beta y(t) &= Ay(t) + f(t, y(t)), t \in J = [0, T], t \neq t_k, & (1.59) \\ \Delta y|t_k &= I_k(y(t_k^-)), k = 1, 2, ..., p, & (1.60) \\ y(0) &= x_0, & (1.61) \end{aligned}$$

where $0 < \beta < 1$, the operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ as the generator of a C_0 semigroup $\{Q(t)\}_{t\geq 0}$ in a Banach space $\mathbb{X}$, $f : J \times \mathbb{X} \to \mathbb{X}$ a given continuous function. He defined the mild solution for (1.59)-(1.61) as follows :
A function $y \in PC(J, \mathbb{X})$ satisfying the fractional integral equation

$$\begin{aligned} y(t) = Q(t)x_0 &+ \frac{1}{\Gamma(\beta)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\beta-1} Q(t-\tau) f(\tau, y(\tau)) d\tau \\ &+ \frac{1}{\Gamma(\beta)} \int_{t_k}^t (t-\tau)^{\beta-1} Q(t-\tau) f(\tau, y(\tau)) d\tau + \sum_{0<t_k<t} T(t-t_k) I_k(y(t_k^-)). \end{aligned}$$

The existence of mild solution has also been established by semigroup theory, the contraction mapping and Schaefer's fixed point theorems.

It has been observed that the memory effects and impulsive conditions are not properly utilized in the definitions of solution of the aforementioned problems. Shu et al. [104] observed that the classical solution of the considered problem did not satisfy the definition of a mild solution defined in [86, 108, 123]. The property $Q(t+s) = Q(t)Q(s)$ was not utilized correctly. For a sectorial type operator A on a Banach space $\mathbb{X}$, a piecewise continuous function $y : J \to \mathbb{X}$ is called a mild solution if it satisfies the following relation:

$$y(t) = \begin{cases} S_\beta(t)x_0 + \int_0^t T_\beta(t-\tau)f(\tau, y(\tau))d\tau, & t \in [0, t_1]; \\ S_\beta(t-t_1)(y(t_1^-)) + I_1(y(t_1^-)) + \int_{t_1}^t T_\beta(t-\tau)f(\tau, y(\tau))d\tau, & t \in (t_1, t_2]; \\ . \\ . \\ . \\ S_\beta(t-t_m)(y(t_m^-)) + I_m(y(t_m^-)) + \int_{t_m}^t T_\beta(t-\tau)f(\tau, y(\tau))d\tau, & t \in (t_m, T], \end{cases}$$

where $S_\beta(t), T_\beta(t)$ stand for the solutions operator and β-resolvent family, respectively. Afterward many mathematicians used this concept in their works [42, 98, 101].

In Banach space, Balachandran and Park [18] discussed the existence of nonlocal Cauchy problem for FEEs. Anguraj and Maheswari [11] studied the existence of solutions for fractional impulsive neutral functional infinite delay equations with nonlocal conditions.

Although the amalgamation of the concept of instantaneous impulses with differential equation can be used to explain the dynamics of many evolution processes, it has some drawbacks in modeling of phenomena involving continuous disturbance (however small time interval) experiencing by the trajectories in its evolution. For example, in the hemodynamical study, the introduction of drugs in the bloodstream and its absorption by the human body, which is a continuous and gradual process, can't be modeled by instantaneous impulses.

The differential equations with non-instantaneous impulsive condition was used for the first time by Hernández and O'Regan [60] for the following abstract problem:

$$y'(t) = Ay(t) + f(t, y(t)), t \in (s_i, t_{i+1}], i = 0, 1, ..., N, \quad (1.62)$$

$$y(t) = G_i(t, y(t)), t \in (t_i, s_i], i = 1, 2, ..., N, \tag{1.63}$$

$$y(0) = x_0, \tag{1.64}$$

with $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ as the generator of a C_0- semigroup of bounded operators$\{Q(t)\}_{t\geq 0}$ defined on a Banach space $\mathbb{X}$. The impulses in the problem started abruptly at the points t_i and their action continued on the interval $[t_i, s_i)$. To be precise, the function y took an abrupt impulse at s_i and followed different rules in the two sub intervals $(t_i, s_i]$ and $(s_i, t_{i+1}]$ of the interval $(t_i, t_{i+1}]$. At that point s_i, the function y was continuous. By using the variation of constant formula the definition of mild solution was given as follows:

Definition 1.7.1. *A function $y \in PC(J, \mathbb{X})$ is called a mild solution of the problem* (1.62)-(1.64) *if $y(0) = x_0, y(t) = G_i(t, y(t)), t \in (t_i, s_i]$ for each $i = 1, \dots N$, and*

$$y(t) = Q(t)x_0 + \int_0^t Q(t-\tau)f(\tau, y(\tau))d\tau, \ \forall t \in [0, t_1] \quad \textit{and}$$

$$y(t) = Q(t-s_i)G_i(s_i, y(s_i)) + \int_{s_i}^t Q(t-\tau)f(\tau, y(\tau))d\tau, \ \forall t \in [s_i, t_{i+1}], i = 1, \dots N.$$

Motivated by this work, several researchers investigated the problem of non-instantaneous impulses. Pierri et al. [95] studied the existence of solutions for a class of semi-linear abstract differential equation with non-instantaneous impulses. With the aid of analyticity of the semigroup generated by the linear part and the fractional power of it, the existence of mild solutions was established by Banach contraction principle and condensing operator. The existence of global solution of a class of FEE perturbed by non-instantaneous impulses was established with the help of Hausdorff measure of noncompactness [94].

Wang et al. [114] considered the problem with non-instantaneous impulses in fractional framework as

$$^C D_t^{\beta} y(t) = f(t, y(t)), t \in (s_i, t_{i+1}], i = 0, 1, ..., N, 0 < \beta < 1, \tag{1.65}$$

$$y(t) = G_i(t, y(t)), t \in (t_i, s_i], i = 1, 2, ..., N, \tag{1.66}$$

$$y(0) = x_0. \tag{1.67}$$

Wang et al. [113] extended the FDE with non-instantaneous impulses to periodic BVP. For $0 < \beta < 1$, Kumar et al. [73] studied the following fractional order problem with non-instantaneous impulses:

$$\begin{aligned} {}^{C}D_t^{\beta}y(t) + Ay(t) &= f(t, y(t), y(g(t))), t \in (s_i, t_{i+1}], i = 0, 1, ..., N, && (1.68)\\ y(t) &= G_i(t, y(t)), t \in (t_i, s_i], i = 1, 2, ..., N, && (1.69)\\ y(0) &= x_0, && (1.70)\end{aligned}$$

by using the Banach fixed point theorem and condensing map. They defined the mild solution in the following way:

Definition 1.7.2. *A function $y \in PC(J, \mathbb{X})$ is called a mild solution of the problem (1.68)-(1.70) if $y(0) = x_0, y(t) = G_i(t, y(t)), t \in (t_i, s_i]$ for each $i = 1, \ldots N$, and satisfies the following integral equation*

$$y(t) = \mathscr{T}x_0 + \int_0^t \mathscr{S}(t-\tau)(t-\tau)^{\beta-1} f(\tau, y(\tau), y(g(\tau)))d\tau, \ \forall t \in [0, t_1] \ \ \textit{and}$$

$$y(t) = \mathscr{T}(t-s_i)(G_i(s_i, y(s_i)) + \int_{s_i}^t \mathscr{S}(t-\tau)(t-\tau)^{\beta-1} f(\tau, y(\tau), y(g(\tau)))d\tau,$$

$$\forall t \in [s_i, t_{i+1}], i = 1, \ldots N.$$

Dabas and Gautam [43] considered the following impulsive differential equation with state dependent delay and nonlocal conditions for $0 < \beta < 1$:

$$\begin{aligned} {}^{C}D_t^{\beta}y(t) &= Ay(t) + t^n f(t, y_{k(t,y_t)}) + \int_0^t q(t-\tau)h(\tau, y_{k(\tau,y_\tau)}), t \in (s_i, t_{i+1}], && (1.71)\\ y(t) &= G_i(t, y(t)), t \in (t_i, s_i], i = 1, 2, ..., N, && (1.72)\\ y(t) + g(y) &= \phi(t), t \in (-\infty, 0]. && (1.73)\end{aligned}$$

For secotrial operator A, the concept of mild solution was given in the same as 1.7.2.

1.8 Motivation

A strong motivation to study FDE, in particular impulsive FDE, comes from the fact that many real world problems can be modeled more vigorously through fractional calculus. Here we present some physical problems of fractional calculus, which motivate us to study the existence of solution of FDE.

Abel's integral equation: The pioneering work on fractional calculus was reported by Abel (1802-1829). The Abel's integral equation

$$g(t) = \int_0^t f(\tau)(t-\tau)^{1-\beta} d\tau, 0 < \beta < 1$$

can be expressed in terms of fractional derivative

$$D_t^\beta g(t) = f(t),$$

which is a FDE. This is a fundamental integral equation that frequently appears in many physical and engineering problems. Although solutions of many applied problems lead to integral equations not readily in Abel's form, yet they can be transformed to that of Abel's integral equation for obtaining the solution immediately. For extensive applications of Abel integral equations, reader is referred to [55, 105].

Anomalous diffusion process: Consider the following diffusion equation with memory:

$$\frac{\partial \mathscr{U}}{\partial t}(\mathscr{X}, t) = \int_0^t f(t-\tau)\Delta \mathscr{U}(\mathscr{X}, \tau) d\tau, t > 0, \mathscr{X} \in \mathbb{R}^n.$$

For the material density (ρ) concentrated at a single point $t = \xi$, $f(t) = \rho^2 \delta(t-\xi)$, with $\delta(t-\xi)$ being Dirac delta function, the equation represents the classical diffusion equation

$$\frac{\partial \mathscr{U}}{\partial t}(\mathscr{X}, t) = \rho^2 \Delta \mathscr{U}(\mathscr{X}, t). \tag{1.74}$$

Equation (1.74) takes the form of time fractional diffusion equation

$$D^\beta \mathscr{U}(\mathscr{X}, t) = \rho^2 \Delta \mathscr{U}(\mathscr{X}, t) \tag{1.75}$$

on replacement of the classical temporal derivative by fractional deriavtive of order $\beta, 0 < \beta < 1$. For $\beta \in (1, 2)$, equation (1.75) describes the super-diffusion process.

Viscoelasticity: The fractional derivatives and integrals are widely used in the mathematical modeling of viscoelastic materials. For an ideal solid, the relation between the stress vector $\sigma(t)$ and strain vector $\varepsilon(t)$ is given by

$$\sigma(t) = c\varepsilon(t), \; c \; \text{ a generic constant,} \tag{1.76}$$

whereas for an ideal fluid, it takes the form

$$\sigma(t) = cD\varepsilon(t). \tag{1.77}$$

Real solid and real fluid both are virtual concepts. Real materials show behaviour which combines characteristic features of those two limiting cases (e.g., firmness). For this intermediate material, stress may be proportional to the stress derivative of non-integer order $\beta \in (0, 1)$. In other words

$$\sigma(t) = c{D_t}^{\beta}\varepsilon(t),$$

where E and β are material dependent constants. For details, see [96].

Advection dispersion equation (ADE): The transport process in porous media is mainly governed by the process of advection and dispersion. The classical advection dispersion equation contains first order time derivative and second order space derivative. From the laboratory experiment, Benson et al. [32] observed that solutes moving through a highly porous aquifer violates the basic assumptions of local-second order theories because of large deviations from the stochastic process of Brownian motion. Let $\mathscr{D} \subset \mathbb{R}^n$ be a bounded connected set and $\mathscr{U}(t, \mathscr{X})$ be the concentration of a solute at a spatial coordinate $\mathscr{X} \in \mathscr{D}$ and temporal coordinate t. The fractional ADE can be written as

$$\frac{\partial \mathscr{U}}{\partial t} = -\Delta(\nu\mathscr{U}) - \Delta(\Delta^{-\beta}(-K\Delta\mathscr{U})) + f, \text{ in } \mathscr{D}, \tag{1.78}$$

where ν, K are constants, $0 \leq \beta < 1$, $\nu\mathscr{U}$ and $-K\Delta\mathscr{U}$ denote the mass flux from advection and dispersion, respectively. For additional information, one can refer to [119].

Chapter 2

Non-instantaneous impulsive Volterra-Fredholm integro FDE

2.1 Introduction

Ordinary or partial differential equations of integer order fail to completely address the phenomena involving after-effects in physics. The concept of integro-differential equation resolves this problem. They are considered as an alternative model of nonlinear differential equation. There are some recent works which treat integro-differential equation of fractional order [8, 34, 39, 115].

Mixed integro-differential equations appear in mathematical modeling of various phenomena in diverse disciplines like electromagnetic theory, biological sciences and elasticity. Although different analytic approaches, viz., differential transform method [44], homotopy analysis method [103] etc. are used to solve Volterra-Fredholm integro-differential equation, the solution of FDE of this kind is extremely difficult due to the absence of product rule differentiation and separation of variables method in fractional calculus. Recently many researchers have paid attention to the qualitative properties of solution of FDE with Volterra-Fredholm argument in the source terms. The existence of mild solution for Volterra-Fredholm FDE in an abstract space was investigated by Matar [81]. He proposed a mild solution by using the classical variational constant formula. Nirmalkumar and Murugesu [89] discussed an approximate controllability problem on Volterra-Fredholm FEE with infinite delay in Banach spaces.

After the introduction of non-instantaneous impulses by Hernández and O'Regan [60], who treated a dynamical system that was influenced by an impulsive action starting from a random fixed point and staying dynamic for a certain interval, many mathematicians have investigated different classes of differential equations in both classical as well as fractional setup [4, 40, 54, 79, 95, 107, 113, 114].

Existence of mild solution for the class of Volterra-Fredholm integro FDE was investigated by Ravichandran and Arjunan [97]:

$$\begin{aligned} {}^{C}D_t^{\beta} y(t) &= Ay(t) + f\Big(t, y(t), \int_0^t \mathcal{A}(t,\tau,y(\tau))d\tau, \int_0^{T_0} \mathcal{B}(t,\tau,y(\tau))d\tau\Big), && (2.1)\\ \Delta y|_{t_k} &= I_k(y(t_k^-)), \quad k = 1, 2, \dots, N, && (2.2)\\ y(0) &= x_0. && (2.3) \end{aligned}$$

The same problem as above, without the Fredholm arguments, was investigated by Anguraj and Kanjanadevi [8] and they studied the existence results using the fixed point theorem for condensing map and resolvent operator. Motivated by the above literature, here we consider the following impulsive functional differential equation with non-instantaneous impulses:

$$\begin{aligned} {}^{C}D_t^{\beta} y(t) &= -Ay(t) + f(t, y(t), \mathscr{I}_y(t), \mathscr{J}_y(t)),\ a.e.\ t \in (s_i, t_{i+1}],\ i = 0, 1, ..., N, && (2.4)\\ y(t) &= G_i(t, y(t)), \quad t \in (t_i, s_i], \quad i = 1, 2, \dots, N, && (2.5)\\ y(0) &= x_0, && (2.6) \end{aligned}$$

where $\mathscr{I}_y(t) = \int_0^t \mathcal{A}(t,\tau,y(\tau))d\tau$, $\mathscr{J}_y(t) = \int_0^{T_0} \mathcal{B}(t,\tau,y(\tau)))d\tau$, $(-A)$ is the infinitesimal generator of a compact semigroup of bounded linear operators $\{Q(t)\}_{t\geq 0}$ on a Banach space $\mathbb{X}$, $0 < \beta < 1$, $J = [0, T_0]$, $\mathcal{B} : D \times \mathbb{X} \longrightarrow \mathbb{X}$, $\mathcal{A} : D \times \mathbb{X} \longrightarrow \mathbb{X}$, $f : J \times \mathbb{X} \times \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}$ are given functions, $D = \{(t,s) \in J \times J, s \leq t\}$, $0 = t_0 = s_0 < t_1 \leq s_1 \cdots < t_N \leq s_N < t_{N+1} = T_0$, $G_i \in C((t_i, s_i] \times \mathbb{X}, \mathbb{X})$, $i = 1, 2, \dots, N$.

2.2 Preliminaries

We assume that $\{Q(t)\}_{t\geq 0}$ is uniformly bounded by $M > 1$, that is, $\|Q(t)\| \leq M$, $t > 0$. We consider the Banach space $PC(J, \mathbb{X})$, formed by all functions $y : J \longrightarrow \mathbb{X}$ such that $y(.)$ is continuous at $t \neq t_i$, $y(t_i^-) = y(t_i)$ and $y(t_i^+)$ exist for all $i = 1, 2, \ldots, N$ endowed with the uniform norm defined by

$$\|y\|_{PC} := \sup_{t\in J} \|y(t)\|.$$

For a function $y \in PC(J, \mathbb{X})$ and $i = 0, 1, \ldots, N$, we introduce the function $\tilde{y}_i \in C([t_i, t_{i+1}], \mathbb{X})$ given by

$$\tilde{y}_i(t) = \begin{cases} y(t), & t \in (t_i, t_{i+1}], \\ y(t_i^+), & t = t_i. \end{cases}$$

Let $B \subset PC(J, \mathbb{X})$ and we define $\tilde{B}_i = \{\tilde{y}_i : y \in B\}$.

Lemma 2.2.1. *[60] A set $B \subseteq PC(J, \mathbb{X}))$ is relatively compact in $PC(J, \mathbb{X})$ if and only if the set $\tilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}], \mathbb{X})$.*

2.3 Concept of mild solution

Definition 2.3.1. *By the mild solution of the impulsive Cauchy problem, we mean that the function $y \in PC(J, \mathbb{X})$ satisfies*

$$\begin{aligned}
y(0) &= x_0, && (2.7)\\
y(t) &= G_i(t, y(t)), \quad t \in (t_i, s_i],\ i = 1, 2, \ldots, N, && (2.8)\\
y(t) &= \mathscr{T}(t)x_0 + \int_0^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))d\tau,\ t \in [0, t_1], && (2.9)\\
y(t) &= \mathscr{T}(t-s_i)G_i(s_i, y(s_i)) + \int_{s_i}^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))d\tau,\\
&\quad t \in [s_i, t_{i+1}]. && (2.10)
\end{aligned}$$

Lemma 2.3.1. *[119] The operators $\mathscr{T}$ and $\mathscr{S}$ have the following properties:*

(i) For any fixed $t \geq 0$, $\mathscr{T}(t)$ and $\mathscr{S}(t)$ are linear bounded operators, that is, for any $u \in \mathbb{X}$,

$$\|\mathscr{T}(t)u\| \leq M\|u\|, \quad \|\mathscr{S}(t)u\| \leq \frac{M\beta}{\Gamma(\beta+1)}\|u\|.$$

(ii) $\{\mathscr{T}(t)\}_{t>0}$ *and* $\{\mathscr{S}(t)\}_{t>0}$ *are strongly continuous.*

(iii) For every $t > 0$, $\mathscr{T}(t)$ *and* $\mathscr{S}(t)$ *are compact operators if* $\{Q(t)\}_{t>0}$ *is compact.*

2.4 Existence of mild solutions

To establish the result of existence of mild solution, we use the following hypotheses:

(H1) Let $\Omega \subset Dom(f)$ be an open subset of $J \times \mathbb{X} \times \mathbb{X} \times \mathbb{X}$. For each $(t, u_1, u_2, u_3) \in \Omega$, there is a neighbourhood $V_1 \subset \Omega$ of (t, u_1, u_2, u_3) such that the nonlinear map

$$f : J \times \mathbb{X} \times \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{X}$$

satisfies the following condition:

$$\|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)\| \leq L_f[\|u_1 - v_1\| + \|u_2 - v_2\| + \|u_3 - v_3\|] \tag{2.11}$$

$\forall$ $(t, u_1, u_2, u_3), (t, v_1, v_2, v_3) \in V_1$ and $L_f = L_f(., u_1, u_2, u_3)>0$, a constant.

(H2) The functions G_i are continuous and there are positive constants $L_{G_i} \in (0, \frac{1}{1+M})$ such that

$$\|G_i(t, u) - G_i(t, v)\| \leq L_{G_i}\|u - v\| \tag{2.12}$$

for all $u, v \in \mathbb{X}$, $t \in (t_i, s_i], i = 0, 1, \ldots, N$.

(H3) There exist positive numbers $M_{\mathcal{A}}, M_{\mathcal{B}}$ such that

$$\|\mathscr{I}_y(t) - \mathscr{I}_z(t)\| \leq M_{\mathcal{A}}\|y - z\|_{PC}, \tag{2.13}$$

$$\|\mathscr{J}_y(t) - \mathscr{J}_z(t)\| \leq M_{\mathcal{B}}\|y - z\|_{PC} \tag{2.14}$$

for all $y, z \in PC(J, \mathbb{X})$.

(H4) For all $u, v, w \in \mathbb{X}$, the function $f(., u, v, w)$ is strongly measurable on $[0, T_0]$, $f(t, ., ., .) \in C(\mathbb{X} \times \mathbb{X} \times \mathbb{X}; \mathbb{X})$ for a.e. $t \in J$ and there exists a function $m(t) \in L^1(J, \mathbb{R}^+)$ such that

$${}_{s_i}D_t^{-\beta}m \in C(J_i, \mathbb{R}^+), \quad \lim_{t\to s_i^+} {}_{s_i}D_t^{-\beta}m(t) = 0, \quad J_i = [s_i, t_{i+1}] \tag{2.15}$$

and for any $\eta > 0$

$$\|f(t, u, v, w)\| \leq m(t), \ (u, v, w) \in \mathbb{X} \times \mathbb{X} \times \mathbb{X} \text{ satisfying } \|u\|, \|v\|, \|w\| \leq \eta \text{ and a.e. } t \in J. \tag{2.16}$$

2.5 Main Results:

Theorem 2.5.1. *Assuming that the hypotheses* $(H1)$-$(H3)$ *hold, then the system of equations* (2.4)-(2.6) *has a unique mild solution* $x \in \mathbb{X}$, *provided*

$$L = \max_{1\leq i\leq N}\left\{ML_{G_i} + \frac{ML_f(t_{i+1}^{\beta} - s_i^{\beta})}{\Gamma(\beta+1)}[1 + M_{\mathcal{A}} + M_{\mathcal{B}}], \frac{ML_f {t_1}^{\beta}}{\Gamma(\beta+1)}[1 + M_{\mathcal{A}} + M_{\mathcal{B}}]\right\} < 1. \tag{2.17}$$

Proof. We define the operator

$$\mathcal{T} : PC(J, \mathbb{X}) \longrightarrow PC(J, \mathbb{X}) \quad \text{as}$$

$$\mathcal{T}y(0) = x_0, \tag{2.18}$$

$$\mathcal{T}y(t) = G_i(t, y(t)), t \in (t_i, s_i], i = 1, \ldots, N, \tag{2.19}$$

$$\mathcal{T}y(t) = \mathscr{T}(t)x_0 + \int_0^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))d\tau, t \in [0, t_1], \tag{2.20}$$

$$\mathcal{T}y(t) = \mathscr{T}(t-s_i)G_i(s_i, y(s_i)) + \int_{s_i}^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))d\tau,$$
$$t \in [s_i, t_{i+1}], i = 1, \ldots, N. \tag{2.21}$$

Clearly, $\mathcal{T}$ is well-defined. For $y, z \in PC(J, \mathbb{X})$ and $t \in [0, t_1]$, we get

$$\begin{aligned}
&\|\mathcal{T}y(t) - \mathcal{T}z(t)\| \\
&= \|\textstyle\int_0^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))d\tau \\
&- \textstyle\int_0^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, z(\tau), \mathscr{I}_z(\tau), \mathscr{J}_y(\tau))d\tau\| \\
&\leq \frac{M}{\Gamma(\beta)}L_f \int_0^t (t-\tau)^{\beta-1}[\|y(\tau) - z(\tau)\| + \|\mathscr{I}_y(\tau) - \mathscr{I}_z(\tau)\| + \|\mathscr{J}_y(\tau) - \mathscr{J}_z(\tau)\|]d\tau \\
&\leq \frac{ML_f t_1^{\beta}}{\Gamma(\beta+1)}[1 + M_{\mathcal{A}} + M_{\mathcal{B}}]\|y - z\|_{PC}.
\end{aligned}$$

For $t \in [s_i, t_{i+1}]$,

$$\begin{aligned}
&\|\mathcal{T}y(t) - \mathcal{T}z(t)\| \\
&= \|\mathscr{T}(t-s_i)G_i(s_i, y(s_i)) + \textstyle\int_{s_i}^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))d\tau \\
&\quad - \mathscr{T}(t-s_i)G_i(s_i, z(s_i)) - \textstyle\int_{s_i}^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, z(\tau), \mathscr{I}_z(\tau), \mathscr{J}_z(\tau))d\tau\|
\end{aligned}$$

$$\le ML_{G_i}\|y-z\|_{PC} + \frac{M}{\Gamma(\beta)} L_f \int_{s_i}^t (t-\tau)^{\beta-1} d\tau [1 + M_{\mathcal{A}} + M_{\mathcal{B}}] \|y-z\|_{PC}$$
$$= \left[ML_{G_i} + \frac{ML_f (t_{i+1}^\beta - s_i^\beta)}{\Gamma(\beta+1)} (1 + M_{\mathcal{A}} + M_{\mathcal{B}}) \right] \|y-z\|_{PC}.$$

For $t \in (t_i, s_i]$,

$$\|\mathcal{T}y(t) - \mathcal{T}z(t)\| \le M\, L_{G_i}\|y-z\|_{PC}. \tag{2.22}$$

Thus, $\|\mathcal{T}y - \mathcal{T}z\| \le L\|y-z\|_{PC}$.

Therefore, $\mathcal{T}$ is a contraction on $PC(J,\mathbb{X})$ and there exists a unique mild solution of (2.4)-(2.6). □

In the next result, we establish the existence of mild solution via Krasnoselskii's fixed point theorem. Here we assume that the semigroup $\{Q(t)\}_{t>0}$ is compact.

For $\eta > 0$, let B_η be a closed ball in $PC(J,\mathbb{X})$ with radius η and center at 0, that is,

$$B_\eta = \{y \in PC(J,\mathbb{X}) : \|y\|_{PC} \le \eta\}.$$

Then B_η is a closed, convex and bounded set in $PC(J,\mathbb{X})$.

Theorem 2.5.2. *Suppose that the assumptions* $(H2)$, $(H3)$ *and* $(H4)$ *hold, the functions* $G_i(.,0)$ *are bounded, (2.17) holds and for each* $x_0 \in \mathbb{X}$, *let* $\eta > 1$ *and* $0 < \delta < 1$ *be such that*

$$M\|x_0\| + (1+M)\|G_i(.,0)\| \le (1-\delta)\eta,$$
$$L_{G_i}(1+M) + \frac{1}{\nu} \sup_{t\in[s_i.t_{i+1}]} \frac{M\beta}{\Gamma(\beta+1)} \int_{s_i}^t (t-\tau)^{(\beta-1)} m(\tau) d\tau \le \delta\eta, \nu \ge \eta, i = 1,2,\dots N,$$
$$\frac{1}{\nu} \sup_{t\in[0,t_1]} \frac{M\beta}{\Gamma(\beta+1)} \int_0^t (t-\tau)^{(\beta-1)} m(\tau) d\tau \le \delta\eta, \nu \ge \eta.$$

Then there exists a mild solution $y \in PC(J,\mathbb{X})$ *of the impulsive problem* (2.4)-(2.6).

Proof. Consider the map introduced in Theorem 2.5.1 and its decomposition

$$\mathcal{T} = \sum_{i=0}^N \mathcal{T}_i^1 + \sum_{i=0}^N \mathcal{T}_i^2,$$

where $\mathcal{T}_i^j : B_\eta \longrightarrow B_\eta$, $i = 0, \dots, N; j = 1, 2$ are given by

$$\mathcal{T}_i^1 y(t) = \begin{cases} \mathscr{T}(t)x_0, & t \in [0, t_1], \\ G_i(t, y(t)), & t \in (t_i, s_i], \quad i \geq 1, \\ \mathscr{T}(t - s_i)G_i(s_i, y(s_i)), & t \in (s_i, t_{i+1}], i \geq 1, \\ 0, & t \notin (t_i, t_{i+1}], \end{cases}$$

$$\mathcal{T}_i^2 y(t) = \begin{cases} \int_{s_i}^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))d\tau, & t \in (s_i, t_{i+1}], \\ 0, & t \notin (s_i, t_{i+1}]. \end{cases}$$

Our aim is to prove that $\mathcal{T}_1 = \sum_{i=0}^{N} \mathcal{T}_i^1$ is a contraction and $\mathcal{T}_2 = \sum_{i=0}^{N} \mathcal{T}_i^2$ is a completely continuous map. For this purpose, we divide the proof into several steps.

Step 1: To show that $\mathcal{T}$ maps B_η to B_η.

For $i \geq 1$, let $t \in (t_i, t_{i+1}]$ and $y \in B_\eta$,

$$\begin{aligned} \|\mathcal{T}y(t)\| &= \|G_i(t, y(t)) + \mathscr{T}(t - s_i)G_i(s_i, y(s_i)) \\ &\quad + \int_{s_i}^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))d\tau\| \\ &\leq L_{G_i}\|y(t)\| + \|G_i(t, 0)\| + M[L_{G_i}\|y(t)\|_{PC} + \|G_i(t, 0)\|] \\ &\quad + \frac{M\beta}{\Gamma(\beta+1)} \int_{s_i}^t (t-\tau)^{\beta-1}m(\tau)d\tau \\ &\leq L_{G_i}(1 + M)\|y\|_{PC} + (1 + M)\|G_i(t, 0)\| \\ &\quad + \frac{M\beta}{\Gamma(\beta+1)} \sup_{t\in[s_i,t_{i+1}]} \int_{s_i}^t (t-\tau)^{\beta-1}m(\tau)d\tau. \end{aligned}$$

For $t \in [0, t_1]$,

$$\begin{aligned} \|\mathcal{T}y(t)\| &= \|\mathscr{T}(t)x_0 + \int_0^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))d\tau\| \\ &\leq M\|x_0\| + \frac{M\beta}{\Gamma(\beta+1)} \int_0^t (t-\tau)^{\beta-1}m(\tau)d\tau \\ &\leq M\|x_0\| + \frac{M\beta}{\Gamma(\beta+1)} \sup_{t\in[0,t_1]} \int_0^t (t-\tau)^{\beta-1}m(\tau)d\tau, \end{aligned}$$

from which we get

$$\|\mathcal{T}y\|_{PC} \leq \eta \quad \text{for any} \quad y \in B_\eta.$$

Step 2: To show that the map $\mathcal{T}_1 = \sum_{i=0}^{N} \mathcal{T}_i^1$ is a contraction on B_η.

For $t \in (t_i, t_{i+1}]$ and $y, z \in B_\eta)$, $i = 1, 2. \ldots, N$,

$$\|\mathcal{T}_i^1 y(t) - \mathcal{T}_i^1 z(t)\| \leq (1 + M) L_{G_i} \|y - z\|_{C((t_i, t_{i+1}], \mathbb{X})},$$

which implies that

$$\|\sum_{i=0}^{N} \mathcal{T}_i^1 y - \sum_{i=0}^{N} \mathcal{T}_i^1 z\|_{PC} \leq \|y - z\|_{PC}.$$

Therefore $\mathcal{T}_1$ is a contraction.

Step 3: To prove $\bigcup \mathcal{T}_i^2 B_\eta$ is relatively compact in $PC(J, \mathbb{X})$.

Let $s_i < t \leq t_{i+1}$ be fixed. For each $\epsilon \in (0, t - s_i)$ and $\forall \psi > 0$, define the operator $\mathcal{T}_\epsilon^\psi$ on B_η as

$$\begin{aligned}
(\mathcal{T}_\epsilon^\psi y)(t) &= \beta \int_{s_i}^{t-\epsilon} \int_{\psi}^{\infty} \theta (t - \tau)^{\beta - 1} \xi_\beta(\theta) Q((t - \tau)^\beta \theta) f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau)) d\tau d\theta \\
&= \beta Q(\epsilon^\beta \psi) \int_{s_i}^{t-\epsilon} \int_{\psi}^{\infty} \theta (t - \tau)^{\beta - 1} \xi_\beta(\theta) Q((t - \tau)^\beta \theta - \epsilon^\beta \psi) \\
&\times \ f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau)) d\tau d\theta,
\end{aligned}$$

where $y \in B_\eta$.

For $\epsilon^\beta \psi > 0, Q(\epsilon^\beta \psi)$ is compact, gives the closure of the set

$$V_\epsilon^\psi(t) = \{(\mathcal{T}_\epsilon^\psi y)(t) | y \in B_\eta\}$$

is compact in $PC(J, \mathbb{X})$.

Moreover, for any $y \in B_\eta$,

$$\begin{aligned}
\|(\mathcal{T}_i^2 y)(t) - (\mathcal{T}_\epsilon^\psi y)(t)\|_{PC} &\leq \beta \| \int_{s_i}^{t} \int_{0}^{\psi} \theta (t - \tau)^{\beta - 1} \xi_\beta(\theta) Q((t - \tau)^\beta (\theta)) \\
&\times f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau)) d\tau \| \\
&+ \beta \| \int_{t-\epsilon}^{t} \int_{\psi}^{\infty} \theta (t - \tau)^{\beta - 1} \xi_\beta(\theta) Q((t - \tau)^\beta (\theta)) \\
&\times f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau)) d\tau \| \\
&\leq \beta M \int_{s_i}^{t} (t - \tau)^{\beta - 1} m(\tau) d\tau \int_{0}^{\psi} \theta \xi_\beta(\theta) d\theta
\end{aligned}$$

$$+\frac{M}{\Gamma(\beta)}\int_{t-\epsilon}^{t}(t-\tau)^{\beta-1}m(\tau)d\tau$$
$$\to 0 \quad \text{as} \quad \epsilon\to 0, \psi\to 0.$$

Thus the set $\bigcup \mathcal{T}_i^2 B_\eta$ is arbitrarily close to a relatively compact set. Therefore the closure of the set $\bigcup \mathcal{T}_i^2 B_\eta$ is compact in $PC(J,\mathbb{X})$.

Step 4: We prove that the set of functions $[\widetilde{\mathcal{T}_i^2 B_\eta}]_i$, $i=0,\ldots,N$ is an equicontinuous subset of $C([t_i,t_{i+1}],\mathbb{X})$.

Let, $\xi_1,\xi_2\in[s_i,t_{i+1}]$, $s_i<\xi_1<\xi_2$ and $y\in B_\eta$.

We have

$$\begin{aligned}
\|(\mathcal{T}_i^2)y(\xi_2)-(\mathcal{T}_i^2)y(\xi_1)\| &= \|\int_{s_i}^{\xi_2}(\xi_2-\tau)^{\beta-1}\mathscr{S}(\xi_2-\tau)f(\tau,y(\tau),\mathscr{I}_y(\tau),\mathscr{J}_y(\tau))d\tau\\
&-\int_{s_i}^{\xi_1}(\xi_1-\tau)^{\beta-1}\mathscr{S}(\xi_1-\tau)f(\tau,y(\tau),\mathscr{I}_y(\tau),\mathscr{J}_y(\tau))d\tau\|\\
&=\|\int_{\xi_1}^{\xi_2}(\xi_2-\tau)^{\beta-1}\mathscr{S}(\xi_2-\tau)f(\tau,y(\tau),\mathscr{I}_y(\tau),\mathscr{J}_y(\tau))d\tau\|\\
&+\|\int_{s_i}^{\xi_1}(\xi_2-\tau)^{\beta-1}\mathscr{S}(\xi_2-\tau)f(\tau,y(\tau),\mathscr{I}_y(\tau),\mathscr{J}_y(\tau))d\tau\\
&-\int_{s_i}^{\xi_1}(\xi_1-\tau)^{\beta-1}\mathscr{S}(\xi_2-\tau)f(\tau,y(\tau),\mathscr{I}_y(\tau),\mathscr{J}_y(\tau))d\tau\|\\
&+\|\int_{s_i}^{\xi_1}(\xi_1-\tau)^{\beta-1}[(\mathscr{S}(\xi_2-\tau)-\mathscr{S}(\xi_1-\tau))]\\
&\times f(\tau,y(\tau),\mathscr{I}_y(\tau),\mathscr{J}_y(\tau))d\tau\|\\
&\le\frac{M\beta}{\Gamma(\beta+1)}\int_{\xi_1}^{\xi_2}(\xi_2-\tau)^{\beta-1}m(\tau)d\tau\\
&+\frac{M\beta}{\Gamma(\beta+1)}\int_{s_i}^{\xi_1}[(\xi_1-\tau)^{\beta-1}-(\xi_2-\tau)^{\beta-1}]m(\tau)d\tau\\
&+\int_{s_i}^{\xi_1}(\xi_1-\tau)^{\beta-1}\|\mathscr{S}(\xi_2-\tau)-\mathscr{S}(\xi_1-\tau)\|m(\tau)d\tau\\
&\le\frac{M\beta}{\Gamma(\beta+1)}\|\int_{s_i}^{\xi_2}(\xi_2-\tau)^{\beta-1}m(\tau)d\tau-\int_{s_i}^{\xi_1}(\xi_1-\tau)^{\beta-1}m(\tau)d\tau\|\\
&+\frac{2M\beta}{\Gamma(\beta+1)}\int_{s_i}^{\xi_1}[(\xi_1-\tau)^{\beta-1}-(\xi_2-\tau)^{\beta-1}]m(\tau)d\tau\\
&+\int_{s_i}^{\xi_1}(\xi_1-\tau)^{\beta-1}\|\mathscr{S}(\xi_2-\tau)-\mathscr{S}(\xi_1-\tau)\|m(\tau)d\tau
\end{aligned}$$

$$= I_1 + I_2 + I_3.$$

Since ${}_{s_i}D_t^{-\beta} m \in C(J_i, \mathbb{R}^+)$, therefore

$$I_1 \to 0 \quad \text{as} \quad \xi_2 \to \xi_1.$$

For $\xi_1 < \xi_2$,

$$I_2 \leq \frac{2M\beta}{\Gamma(\beta+1)} \int_{s_i}^{\xi_1} (\xi_1 - \tau)^{\beta-1} m(\tau) d\tau.$$

Then by Lemma 1.1.3, we have $I_2 \to 0 \quad \text{as} \quad \xi_2 \to \xi_1$.

For $\epsilon > 0$ small enough,

$$\begin{aligned}
I_3 \quad &= \int_{s_i}^{\xi_1-\epsilon} (\xi_1 - \tau)^{\beta-1} \|\mathscr{S}(\xi_2 - \tau) - \mathscr{S}(\xi_1 - \tau)\| m(\tau) d\tau \\
&+ \int_{\xi_1-\epsilon}^{\xi_1} (\xi_1 - \tau)^{\beta-1} \|\mathscr{S}(\xi_2 - \tau) - \mathscr{S}(\xi_1 - \tau)\| m(\tau) d\tau \\
&\leq \int_{s_i}^{\xi_1-\epsilon} (\xi_1 - \tau)^{\beta-1} m(\tau) d\tau \sup_{\tau \in [s_i, \xi_1-\epsilon]} \|\mathscr{S}(\xi_2 - \tau) - \mathscr{S}(\xi_1 - \tau)\| \\
&+ \int_{s_i}^{\xi_1} (\xi_1 - \tau)^{\beta-1} \|\mathscr{S}(\xi_2 - \tau) - \mathscr{S}(\xi_1 - \tau)\| m(\tau) d\tau \\
&- \int_{s_i}^{\xi_1-\epsilon} (\xi_1 - \tau)^{\beta-1} \|\mathscr{S}(\xi_2 - \tau) - \mathscr{S}(\xi_1 - \tau)\| m(\tau) d\tau \\
&\leq \int_{s_i}^{\xi_1-\epsilon} (\xi_1 - \tau)^{\beta-1} m(\tau) d\tau \sup_{\tau \in [s_i, \xi_1-\epsilon]} \|\mathscr{S}(\xi_2 - \tau) - \mathscr{S}(\xi_1 - \tau)\| \\
&+ \frac{2M\beta}{\Gamma(\beta+1)} \left[\int_{s_i}^{\xi_1} (\xi_1 - \tau)^{\beta-1} m(\tau) d\tau - \int_{s_i}^{\xi_1-\epsilon} (\xi_1 - \tau)^{\beta-1} m(\tau) d\tau\right] \\
&\leq \int_{s_i}^{\xi_1-\epsilon} (\xi_1 - \tau)^{\beta-1} m(\tau) d\tau \sup_{\tau \in [s_i, \xi_1-\epsilon]} \|\mathscr{S}(\xi_2 - \tau) - \mathscr{S}(\xi_1 - \tau)\| \\
&+ \frac{2M\beta}{\Gamma(\beta+1)} \left[\int_{s_i}^{\xi_1} (\xi_1 - \tau)^{\beta-1} m(\tau) d\tau - \int_{s_i}^{\xi_1-\epsilon} (\xi_1 - \epsilon - \tau)^{\beta-1} m(\tau) d\tau\right] \\
&+ \frac{2M\beta}{\Gamma(\beta+1)} \int_{s_i}^{\xi_1-\epsilon} [(\xi_1 - \epsilon - \tau)^{\beta-1} - (\xi_1 - \tau)^{\beta-1}] m(\tau) d\tau \\
&= I_{31} + I_{32} + I_{33}.
\end{aligned}$$

Since compact operator forms an equicontinuous family, therefore $I_{31} \to 0$ as $\xi_2 \to \xi_1$.

As seen in the proof of I_2 and I_3, similarly $I_{32} \to 0$, $I_{33} \to 0$ as $\epsilon \to 0$, and therefore

$||(\mathcal{T}_i^2 y)(\xi_2) - (\mathcal{T}_i^2 y)(\xi_1)|| \to 0$ independent of $y \in B_\eta$ as $\xi_2 \to \xi_1$.
Similarly the case $\xi_1 = s_i$ can also be verified.

Thus, $\widetilde{[\mathcal{T}_i^2 B_\eta]}_i$ is equicontinuous.

Step 5: To show that $\mathcal{T}^2$ is continuous in B_η.

Let $y^n (n = 1, 2, 3, ...), y \in B_\eta$ such that $y^n \to y$.
Since $f, \mathscr{I}_y, \mathscr{J}_y$ are continuous, it implies that

$$f(t, y^n(t), \mathscr{I}_{y^n}(t), \mathscr{J}_{y^n}(t)) \to f(t, y(t), \mathscr{I}_y(t), \mathscr{J}_y(t)) \text{ as } n \to \infty.$$

For each $t \in J_i = [s_i, t_{i+1}]$, we obtain

$$\begin{aligned}&(t-\tau)^{\beta-1}\|f(\tau, y^n(\tau), \mathscr{I}_{y^n}(\tau), \mathscr{J}_{y^n}(\tau)) - f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))\| \\ &\leq 2(t-\tau)^{\beta-1} m(\tau) \text{ a.e. } \tau \in [s_i, t).\end{aligned}$$

By hypothesis $(H4)$, the function $\tau \to (t-\tau)^{\beta-1} 2m(\tau)$ is integrable for $\tau \in [s_i, t]$ and $t \in [s_i, t_{i+1}]$. In view of Lemma 1.1.3, we get

$$\int_{s_i}^t (t-\tau)^{\beta-1} |f(\tau, y^n(\tau), \mathscr{I}_{y^n}(\tau), \mathscr{J}_{y^n}(\tau)) - f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))| d\tau \to 0 \text{ as } n \to \infty$$

This gives for each $t_i < t \leq t_{i+1}$,

$$\begin{aligned}&\|(\mathcal{T}^2 y^n)(t) - (\mathcal{T}^2 y)(t)\| \\ &\leq \frac{M}{\Gamma(\beta)} \int_{s_i}^t (t-\tau)^{\beta-1} \|f(\tau, y^n(\tau), \mathscr{I}_{y^n}(\tau), \mathscr{J}_{y^n}(\tau)) - f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))\| d\tau \\ &\leq \frac{M}{\Gamma(\beta)} \int_{s_i}^t (t-\tau)^{\beta-1} \|f(\tau, y^n(\tau), \mathscr{I}_{y^n}(\tau), \mathscr{J}_{y^n}(\tau)) - f(\tau, y(\tau), \mathscr{I}_y(\tau), \mathscr{J}_y(\tau))\| d\tau \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Therefore, $\mathcal{T}^2 y^n \to \mathcal{T}^2 y$ is pointwise on $(t_i, t_{i+1}]$ as $n \to \infty$. Hence from the equicontinuity it follows that $\mathcal{T}^2 y^n \to \mathcal{T}^2 y$ uniformly on $(t_i, t_{i+1}]$ as $n \to \infty$ and so $\mathcal{T}^2$ is continuous. From the above steps, $\mathcal{T}_2$ is completely continuous. Consequently, Krasnoselskii's fixed point theorem ensures that $\mathcal{T}$ has a fixed point which gives rise to a mild solution. □

2.6 Examples

In this section we present two examples to validate the results established in Theorem 2.5.1 and Theorem 2.5.2.

Consider the fractional partial integro-differential equation of the form

$$\begin{aligned}
{}^{C}D_t^{\beta} y(t,w) &= \frac{\partial^2 y}{\partial w^2} + P\Big(t, y(t,w), \int_0^t \mathcal{A}(t,\tau,y(\tau,w))d\tau, \int_0^{T_0} \mathcal{B}(t,\tau,y(\tau,w))d\tau\Big), \\
& \quad a.e. \quad (t,w) \in \cup_{i=1}^{N}[s_i, t_{i+1}] \times [0,\pi], && (2.23)\\
y(t,0) &= y(t,\pi) = 0, t \in [0,T_0], && (2.24)\\
y(0,w) &= x_0(w), w \in [0,\pi], && (2.25)\\
y(t,w) &= \mathscr{G}_i(t, y(t,w)), \quad t \in (t_i, s_i], w \in [0,\pi], \quad i = 1,2,\dots,N, && (2.26)
\end{aligned}$$

where $0 = t_0 = s_0 < t_1 \leq s_1 ... < t_N \leq s_N < t_{N+1} = T_0, P \in C([0,T_0] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\mathscr{G}_i \in C((t_i, s_i] \times \mathbb{R}, \mathbb{R})$ for all $i = 1, 2, ..., N$.

Let $\mathbb{X} = L^2[0,\pi]$ and define the operator $Ax = -\dfrac{\partial^2 y}{\partial w^2}$ with $D(A) = \{y \in \mathbb{X} : \dfrac{\partial y}{\partial w}, \dfrac{\partial^2 y}{\partial w^2} \in \mathbb{X}; y(0) = y(\pi) = 0\}$.

Then it is known that A is the infinitesimal generator of compact semigroup $\{Q(t)\}_{t\geq 0}$ on $\mathbb{X}$.

Set

$$\begin{aligned}
& y(t)w = y(t,w), \mathscr{G}_i(t, y(t))w = \mathscr{G}_i(t, y(t,w)), \\
& f(t, y(t), I_y(t), J_y(t))w = P\Big(t, y(t,w), \int_0^t \mathcal{A}(t,\tau,y(\tau,w))d\tau, \int_0^{T_0} \mathcal{B}(t,\tau,y(\tau,w))d\tau\Big).
\end{aligned}$$

Thus with this set-up, equations (2.23)-(2.26) can be written in the abstract form for (2.4)-(2.6).

We define

$$\begin{aligned}
f(t, y(t), \mathscr{I}_y(t), \mathscr{J}_y(t))w &= \frac{e^{-t} y(t,w)}{(9+e^t)(1+|y(t,w)|)} + \frac{1}{10}\int_0^t e^{-\frac{1}{3}y(\tau,w)}d\tau \\
&+ \frac{1}{10}\int_0^{T_0} e^{-\frac{1}{5}y(\tau,w)}d\tau, t \in [0,T_0], w \in [0,\pi].
\end{aligned}$$

Let $\mathscr{I}_y(t) = \int_0^t e^{-\frac{1}{3}y(\tau)}d\tau$ and $\mathscr{J}_y(t) = \int_0^{T_0} e^{-\frac{1}{5}y(\tau)}d\tau$. Then $\mathscr{I}_y$ and $\mathscr{J}_y$ satisfy (2.13) and (2.14), respectively, with $M_{\mathcal{A}} = \frac{1}{3}$ and $M_{\mathcal{B}} = \frac{1}{5}$.

Also, $f : J \times \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is a continuous function satisfying

$$\|f(t, y, \mathscr{I}_y, \mathscr{J}_y) - f(t, z, \mathscr{I}_z, \mathscr{J}_z)\| \leq \frac{1}{10}(\|y - z\| + \|\mathscr{I}_y - \mathscr{I}_z\| + \|\mathscr{J}_y - \mathscr{J}_z\|).$$

The impulse function is taken as

$$\mathscr{G}_i(t, y(t))w = \frac{\cos t|y(t, w)|}{2(1 + |y(t, w)|)}, t \in (t_i, s_i], i = 1, 2, ..., N, \ y \in \mathbb{X}, w \in [0, \pi].$$

For each i, $\mathscr{G}_i : (t_i, s_i] \times \mathbb{X} \to \mathbb{X}$ is continuous and satisfies (2.12) with $L_{\mathscr{G}_i} = \frac{1}{2}$. Thus the assumptions in Theorem 2.5.1 are satisfied.

Now in order to validate the result obtained in Theorem 2.5.2, let us take $\beta = \frac{1}{3}$ and define

$$\begin{aligned} f(t, y(t), \mathscr{I}_y(t), \mathscr{J}_y(t))w &= (t - s_i)^{-1/5} \sin(\int_0^t e^{-\frac{1}{3}y(\tau, w)} d\tau + \int_0^{T_0} e^{-\frac{1}{5}y(\tau, w)} d\tau), \\ & \quad t \in (s_i, t_{i+1}], y \in \mathbb{X}, w \in [0, \pi] \ \text{ for } i = 1, 2, \ldots, N. \end{aligned}$$

Let us choose

$$m(t) = \begin{cases} (t - s_i)^{-1/5}, & t \in (s_i, t_{i+1}], \\ t^{-1/5}, & t \in (t_i, s_i]. \end{cases}$$

Hypothesis $(H4)$ is satisfie by the function $m(t) \in L(J, \mathbb{R}^+)$. Along with this f and the same impulse function from the previous result, th problem satisfies the conditions of Theorem 2.5.2.

2.7 Conclusion

In this chapter we study the existence of mild solution of a class of FEE with non-instantaneous impulses. The proof of the main results are presented through two theorems based on the Banach contraction theorem and Krasnoselskii's fixed point theorem. Both results are validated through appropriate examples.

Chapter 3

Non-instantaneous impulsive FEE with non-dense domain

In this chapter, we establish sufficient conditions for existence and uniqueness of integral solution for some non-densely defined non-instantaneous impulsive evolution equation on a Banach space involving Caputo fractional derivative. The results are obtained by means of characteristic functions based on probability density. Finally, the main results are illustrated through examples.

3.1 Introduction

Da Prato and Sinestrari [41] initiated the study of evolution equations with a non-densely defined linear operator. It was shown that the density condition was not necessary in dealing with partial functional differential equations. The main method used in their work was based on integrated semigroup theory. Some results on existence of integral solution of non-densely defined evolution equation without impulse have been proved under suitable hypotheses for any $\mathbb{X}$-valued continuous function f and any $x_0 \in \overline{D(A)}$. For more details and examples on non-densely defined operators and the concept of integrated semigroup, we refer the reader to [2,27,52]. Thieme [110] showed that integral solution reduced to mild solution when $f \in \overline{D(A)}$. Zhang and Liu [124] corrected the error in the formulation of integral solution for non-densely defined fractional differential equation with impulsive effects that was observed in some past works, e.g., [24,84]. The solution was obtained by integrated semigroup theory and some probability densities.

Motivated by the above discussion, we consider the semi-linear impulsive Cauchy problem with not instantaneous impulses of the following form:

$$ {}^{C}D_t^{\beta} y(t) = Ay(t) + f(t, y(t)), \; a.e. \; t \in (s_i, t_{i+1}], i = 0, 1, ..., N, \tag{3.1} $$

$$ y(t) = G_i(t, y(t)), t \in (t_i, s_i], i = 1, 2, ..., N, \tag{3.2} $$

$$ y(0) = x_0, \tag{3.3} $$

with $0 < \beta < 1; A : D(A) \subset \mathbb{X} \to \mathbb{X}$ not necessarily a densely defined closed linear operator on the Banach space $(\mathbb{X}, \|.\|)$; $f : J \times \mathbb{X} \to \mathbb{X}$ a given function; $G_i \in (C((t_i, s_i] \times \mathbb{X}, \overline{D(A)}), i = 1, 2, ..., N$.

We follow $y(t), PC(J, \mathbb{X}), B, \tilde{y}_i, \tilde{B}_i$ from preliminaries of Chapter 2.

We use contraction mapping principle and Krasnoselskii's fixed point theorem to prove the existence of the integral solution of problem (3.1)-(3.3).

3.2 Preliminaries

Let $\mathbb{X}_0 = \overline{D(A)}$ and A_0 be the part of A in $\overline{D(A)}$ defined by

$$ D(A_0) = \{u \in D(A) : Au \in \overline{D(A)}\}, \; A_0(u) = A(u). $$

Throughout our analysis, the following hypotheses will be considered:

(H1) $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ satisfies the Hille-Yosida condition, that is, there exist two constants $\omega \in \mathbb{R}$ and $M_0 \geq 0$ such that $(\omega, \infty) \subset \rho(A)$ and

$$ \|(\lambda I - A)^{-n}\|_{B(\mathbb{X})} \leq \frac{M_0}{(\lambda - \omega)^n}, \text{ for all } \lambda > \omega, n \geq 1. $$

(H2) The part A_0 of A generates a compact C_0 semigroup $\{(Q(t)\}_{t \geq 0}$ in $\mathbb{X}_0$ which is uniformly bounded, that is, there exists $M \geq 1$ such that $\sup_{t \in [0,\infty)} \|Q(t)\| < M$.

Let $B_\lambda = \lambda R(\lambda, A) := \lambda(\lambda I - A)^{-1}$. Then for all $u \in \mathbb{X}_0, B_\lambda u \to u$ as $\lambda \to \infty$. Also from Hille-Yosida condition, it is clear that $\lim\limits_{\lambda \to \infty} \|B_\lambda\| \leq M_0$.

Lemma 3.2.1. *[56] By the integral solution $y(t)$ of the non-homogenous fractional order evolution system (with continuous source f)*

$$\begin{aligned} {}_{0+}^{C}D_t^{\beta}y(t) &= Ay(t) + f(t), t \in (0,T], &(3.4)\\ y(0) &= x_0 \in \mathbb{X}_0, &(3.5) \end{aligned}$$

we mean a continuous function $y : J \to \mathbb{X}$ which satisfies the following conditions:

(i) ${}_{0+}I_t^{\beta}y(t) \in \mathbb{X}_0$ *for* $t \in J = [0,T]$ *and*

(ii) $y(t) = x_0 + A_{0+}I_t^{\beta}y(t) +_{0+} I_t^{\beta}f(t), \quad t \in J = [0,T].$

Lemma 3.2.2. *[56] If y is an integral solution of (3.4)-(3.5), then for all $t \in J, y(t) \in \mathbb{X}_0$. In particular, $y(0) = x_0 \in \mathbb{X}_0$.*

Lemma 3.2.3. *[56] The integral solution $y(t) = x_0 + A({}_{0+}I_t^{\beta}y(t)) + {}_{0+}I_t^{\beta}f(t),\ t \in J, x_0 \in \mathbb{X}_0$ of the auxiliary problem*

$$\begin{aligned} {}_{0+}{}^{C}D_t^{\beta}y(t) &= A_0y(t) + f(t), t \in (0,T], &(3.6)\\ y(0) &= x_0, &(3.7) \end{aligned}$$

can be expressed as

$$z(t) = S_\beta(t)x_0 + \lim_{\lambda\to\infty}\int_0^\infty K_\beta(t-\tau)B_\lambda f(\tau)d\tau,$$

where

$$S_\beta(t) =_{0+} I_t^{1-\beta}K_\beta(t), K_\beta(t) = t^{\beta-1}P_\beta(t), P_\beta(t) = \int_0^\infty \beta\theta M_\beta(\theta)Q(t^\beta\theta)d\theta.$$

Lemma 3.2.4. *[119] $P_\beta(t)$ is continuous in the uniform operator topology for $t > 0$.*

Lemma 3.2.5. *[127] For any fixed $t > 0$, $K_\beta(t)$ and $S_\beta(t)$ are linear operators, and for any $u \in \mathbb{X}_0$,*

$$\|K_\beta(t)u\| \le \frac{Mt^{\beta-1}}{\Gamma(\beta)}\|u\| \text{ and } \|S_\beta(t)u\| \le M\|u\|.$$

Lemma 3.2.6. *[127] $\{K_\beta(t)\}_{t>0}$ and $\{S_\beta(t)\}_{t>0}$ are strongly continuous.*

Theorem 3.2.1. *[56] $y(t)$ is an integral solution of (3.4)-(3.5) if and only if*

$$y(t) = S_\beta(t)x_0 + \lim_{\lambda\to\infty} \int_0^\infty K_\beta(t-\tau)B_\lambda f(\tau)d\tau, \text{ for } t \in J \text{ and } x_0 \in \mathbb{X}_0.$$

Definition 3.2.1. *[56] The operator defined by*

$$\phi_\beta(t)u = \lim_{\lambda\to\infty} \int_0^\infty K_\beta(t-\tau)B_\lambda u d\tau = \lim_{\lambda\to\infty} \int_0^\infty K_\beta(s)B_\lambda u d\tau \text{ for } u \in \mathbb{X} \text{ and } t \geq 0$$

exists as a bounded linear operator for $x \in \mathbb{X}$ and $t \geq 0$.

Remark [56] We say that A generates the operator $\{\phi_\beta(t)\}_{t\geq 0}$. When $\beta = 1$, $\{\phi_\beta(t)\}_{t\geq 0}$ degenerates into $\{S(t)\}_{t\geq 0}$, which is the integrated semigroup generated by A in [69].

3.3 Integral solution to a nonlinear Cauchy problem

Here we take $a = \dfrac{\beta - 1}{1-\beta_1} \in (-1, 0)$.
Motivated by Theorem 3.13 of [56], we adopt the following concept of integral solution of our problem:

Definition 3.3.1. *A function $y \in PC(J, \mathbb{X}_0)$ is said to be an integral solution of the Cauchy problem (3.1)-(3.3) if it satisfies $y(0) = x_0 \in \mathbb{X}_0$, $\quad y(t) = G_i(t, y(t))$ for all $t \in (t_i, s_i]$, $i = 1, 2, \ldots, N$:*

$$y(t) = S_\beta(t)x_0 + \lim_{\lambda\to\infty} \int_0^t K_\beta(t-\tau)B_\lambda f(\tau, y(\tau))d\tau, \qquad t \in [0, t_1] \text{ and}$$

$$y(t) = S_\beta(t-s_i)G_i(s_i, y(s_i)) + \lim_{\lambda\to\infty} \int_{s_i}^t K_\beta(t-\tau)B_\lambda f(\tau, y(\tau))d\tau, \qquad t \in [s_i, t_{i+1}],$$

$$i = 1, 2, \ldots, N.$$

To study the existence and uniqueness of the integral solution of impulsive fractional evolution equation, we require the following assumptions:
(H3) $f : J \times \mathbb{X} \to \mathbb{X}$ is continuous and there exist a constant $\beta_1 \in (0, \beta)$ and a function $\mu \in L^{\frac{1}{\beta_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t, u) - f(t, v)\| \leq \mu(t)\|u - v\| \text{ for all } u, v \in \mathbb{X} \text{ and almost all } t \in J.$$

(H4) The functions $G_i \in C((t_i, s_i] \times \mathbb{X}, \mathbb{X}_0)$ and there are positive constants L_{G_i} such that $\|G_i(t,u) - G_i(t,v)\| \leq L_{G_i}\|u - v\|$ for all $u, v \in \mathbb{X}$, $t \in (t_i, s_i]$, $i = 1, 2, ..., N$.

Theorem 3.3.1. *Assume that the hypotheses (H1)-(H4) hold and*

$$
\begin{aligned}
k &= \max\left\{ \max_{1\leq i\leq N}\left\{ ML_{G_i} + \frac{MM_0}{\Gamma(\beta)} \frac{(t_{i+1}-s_i)^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}} \|\mu\|_{L^{\frac{1}{\beta}1}([s_i,t_{i+1}])} \right\}, \right. \\
&\quad \left. \frac{MM_0}{\Gamma(\beta)} \frac{{t_1}^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}} \|\mu\|_{L^{\frac{1}{\beta}1}([0,t_1])} \right\} < 1.
\end{aligned}
$$

Then there exists a unique integral solution in $PC(J, \mathbb{X})$ of the problem (3.1)-(3.3) provided $x_0 \in \mathbb{X}_0$.

Proof. Define the operator $F : PC(J, \mathbb{X}) \to PC(J, \mathbb{X})$ by $Fy(0) = x_0$, $Fy(t) = G_i(t, y(t))$ for $t \in (t_i, s_i]$ and

$$
Fy(t) = S_\beta(t-s_i)G_i(s_i, y(s_i)) + \lim_{\lambda\to\infty}\int_{s_i}^{t} K_\beta(t-\tau)B_\lambda f(\tau, y(\tau))d\tau, t \in [s_i, t_{i+1}],\ i \geq 0.
$$

By the hypothesis $(H3)$ and Hölder's inequality, the operator F is well-defined. Let $y, z \in PC(J, \mathbb{X})$. For $t \in [s_i, t_{i+1}], i = 1, 2, ..., N$, we get

$$
\begin{aligned}
&\|Fy(t) - Fz(t)\| \\
&\leq \|S_\beta(t-s_i)G_i(s_i, y(s_i)) - S_\beta(t-s_i)G_i(s_i, z(s_i))\| \\
&+ \lim_{\lambda\to\infty}\int_{s_i}^{t} \|K_\beta(t-\tau)B_\lambda f(\tau, y(\tau)) - K_\beta(t-\tau)B_\lambda f(\tau, z(\tau))\|d\tau \\
&\leq ML_{G_i}\|y - z\|_{PC} + \frac{MM_0}{\Gamma(\beta)}\int_{s_i}^{t}(t-\tau)^{\beta-1}\mu(s)d\tau\|y - z\|_{PC} \\
&\leq ML_{G_i}\|y - z\|_{PC} + \frac{MM_0}{\Gamma(\beta)}\left[\int_{s_i}^{t}((t-\tau)^{\beta-1})^{\frac{1}{1-\beta_1}}d\tau\right]^{1-\beta_1}\|\mu\|_{L^{\frac{1}{\beta}1}([s_i,t_{i+1}])}\|y - z\|_{PC} \\
&\leq \left[ML_{G_i} + \frac{MM_0}{\Gamma(\beta)}\frac{(t_{i+1}-s_i)^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}}\|\mu\|_{L^{\frac{1}{\beta}1}([s_i,t_{i+1}])}\right]\|y - z\|_{PC}.
\end{aligned}
$$

For $t \in [0, t_1]$,

$$
\|Fy(t) - Fz(t)\| \leq \left[\frac{MM_0}{\Gamma(\beta)}\frac{{t_1}^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}}\|\mu\|_{L^{\frac{1}{\beta}1}([0,t_1],\mathbb{R}^+)}\right]\|y - z\|_{PC}.
$$

For $t \in (t_i, s_i], i = 1, 2, ..., N$, we have

$$
\|Fy(t) - Fz(t)\| \leq L_{G_i}\|y - z\|_{PC} \leq ML_{G_i}\|y - z\|_{PC}.
$$

From above, we observe that

$$\|F(y) - F(z)\|_{PC} \leq k\|y - z\|_{PC},$$

which implies that $F(.)$ is a contraction and there exists a unique integral solution of (3.1)-(3.3).

To prove the next theorem, we add the following assumptions:

For $\eta > 0$, let B_η be a closed ball in $PC(J, \mathbb{X})$ with radius η and center at 0, that is,

$$B_\eta = \{y \in PC(J, \mathbb{X}_0) : \|y\|_{PC} \leq \eta\}.$$

Then B_η is a closed, convex and bounded set in $PC(J, \mathbb{X})$.

(H5) For each $t \in J$, the function $f(t, .) : \mathbb{X} \to \mathbb{X}$ is continuous and for each $u \in \mathbb{X}$, the function $f(., u) : J \to \mathbb{X}$ is strongly measurable.

(H6) There exist a constant $\beta_1 \in (0, \beta)$ and a function $m \in L^{\frac{1}{\beta_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t, y)\| \leq m(t) \text{ for all } y \in B_\eta \text{ and almost all } t \in J.$$

□

Theorem 3.3.2. *Assume that the conditions (H1)-(H6) are satisfied with the exception of condition $(H3)$, the functions $G_i(., 0)$ are bounded and*

$$\kappa = (M + 1)L_{G_i} < 1, \qquad \forall\, i = 1, 2, \ldots, N.$$

Then there exists at least one integral solution in B_η of the problem (3.1)-(3.3).

Proof. Let $\eta > 1$ and $0 < \zeta < 1$ be such that

$$M\|x_0\| + (1 + M) \max_{i=1,2,\ldots,N} \|G_i(., 0)\|_{C((t_i, s_i], \mathbb{X})} < (1 - \zeta)\eta,$$

$$\max_{i=1,2,\ldots,N} \left\{(1 + M)L_{G_i} + \frac{1}{s} \frac{MM_0(t_{i+1} - s_i)^{(1+a)(1-\beta_1)}}{\Gamma(\beta)(1 + a)^{1-\beta_1}} \|m\|_{L^{\frac{1}{\beta_1}}([s_i, t_{i+1}])}\right\} < \zeta, \; s \geq \eta,$$

$$\frac{1}{s} \frac{MM_0 t_1^{(1+a)(1-\beta_1)}}{\Gamma(\beta)(1 + a)^{1-\beta_1}} \|m\|_{L^{\frac{1}{\beta_1}}([0, t_1])} < \zeta, \; s \geq \eta.$$

For any $y \in B_\eta$, we define the operator F as follows

$$F=\sum_{i=0}^{m}F_i^1+\sum_{i=0}^{m}F_i^2=F^1+F^2,$$

where

$$(F_i^1y)(t) = \begin{cases} S_\beta(t)x_0, & t\in[0,t_1],\\ G_i(t,y(t)), & t\in(t_i,s_i],\ i\geq 1,\\ S_\beta(t-s_i)G_i(s_i,y(s_i)), & t\in(s_i,t_{i+1}],\ i\geq 1,\\ 0, & t\notin(t_i,t_{i+1}],\ i\geq 0,\end{cases}$$

$$(F_i^2y)(t) = \begin{cases} \lim_{\lambda\to\infty}\int_{s_i}^{t}K_\beta(t-\tau)B_\lambda f(\tau,y(\tau))d\tau, & t\in(s_i,t_{i+1}],\ i\geq 0,\\ 0, & \text{otherwise}.\end{cases}$$

We claim that the map F is a χ-contraction map from B_η into B_η. This consists of the following steps.

Step 1: To show that $F(B_\eta)\subset B_\eta$.

Let $y\in B_\eta$. For $i\geq 1$, and $t\in(t_i,t_{i+1}]$,

$$\begin{aligned}
&\|(Fy)(t)\|\\
&\leq L_{G_i}\|y(t)\|+\|G_i(t,0)\|+M(L_{G_i}\|y(t)\|+\|G_i(t,0)\|)+\|\lim_{\lambda\to\infty}\int_{s_i}^{t}K_\beta(t-\tau)B_\lambda f(\tau,y(\tau))d\tau\|\\
&\leq (M+1)L_{G_i}r+(1+M)\|G_i(.,0)\|_{C((t_i.s_i],\mathbb{X})}+\frac{MM_0}{\Gamma(\beta)}\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)d\tau\\
&\leq (M+1)L_{G_i}r+(1-\eta)r+\frac{MM_0}{\Gamma(\beta)}\left[\int_{s_i}^{t}(t-\tau)^{\frac{\beta-1}{1-\beta_1}}d\tau\right]^{1-\beta_1}\|m\|_{L^{\frac{1}{\beta_1}}([s_i,t_{i+1}])}\\
&\leq (M+1)L_{G_i}r+(1-\eta)r+\frac{MM_0(t_{i+1}-s_i)^{(1+a)(1-\beta_1)}}{\Gamma(\beta)(1+a)^{1-\beta_1}}\|m\|_{L^{\frac{1}{\beta_1}}([s_i,t_{i+1}]}\\
&\leq (1-\zeta)\eta+\eta\zeta\\
&=\eta.
\end{aligned}$$

Therefore,

$$\|Fy\|_{C((t_i,t_{i+1}],\mathbb{X})}\leq\eta \quad \text{for all } i\geq 1.$$

For $t\in[0,t_1]$,

$$\|(Fy)(t)\|\leq\|S_\beta(t)x_0\|+\|\lim_{\lambda\to\infty}\int_{0}^{t}K_\beta(t-\tau)B_\lambda f(\tau,y(\tau))d\tau\|$$

$$\leq M\|x_0\|+\frac{MM_0t_1^{(1+a)(1-\beta_1)}}{\Gamma(\beta)(1+a)^{1-\beta_1}}\|m\|_{L^{\frac{1}{\beta_1}}([0,t_1])}$$

$$\leq \eta.$$

Therefore,

$$\|Fy\|_{C([0,t_1],\mathbb{X})} \leq \eta.$$

Thus it follows that $\|Fy\|_{PC}\leq \eta$ and F maps B_η into B_η.

By the same line of argument, one can show that for any $y, z \in B_\eta$, $F^1y + F^2z \in B_\eta$.

Step 2: To show that the map $F^1 = \sum_{i=0}^{N} F_i^1$ is a contraction on B_η.

Let $y, z \in B_\eta$, $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, N$. Then

$$\|(F_i^1 y)(t) - (F_i^1 z)(t)\|\leq (M+1)L_{G_i}\|y - z\|_{C((t_i,t_{i+1}],\mathbb{X})}.$$

Therefore,

$$\|\sum_{i=0}^{N} F_i^1 y - \sum_{i=0}^{N} F_i^1 z\|_{PC(J,\mathbb{X})}\leq \kappa\|y - z\|_{PC(J,\mathbb{X})}$$

and F^1 is a contraction on B_η.

Step 3: To show that for $i = 0, 1, \ldots, N$ and for any $s_i < s < t < t_{i+1}$, the set $V(l) = \cup_{l\in[s,t]}\{(F_i^2 y)(l) : y \in B_\eta\}$ is relatively compact in $\mathbb{X}$.

For $\forall\epsilon \in (s_i, s)$ and $\delta > 0$, we define an operator $(F_i^2)_{\epsilon,\delta}$ on B_η by the formula

$$\begin{aligned}((F_i^2)_{\epsilon,\delta}y)(l) &= \lim_{\lambda\to\infty} \beta \int_{s_i}^{l-\epsilon} \int_{\delta}^{\infty} \theta(l-\tau)^{\beta-1}M_\beta(\theta)Q((l-\tau)^\beta\theta)B_\lambda f(\tau, y(\tau))d\tau d\theta\\ &= \beta\epsilon^\beta\delta \lim_{\lambda\to\infty} \int_{s_i}^{l-\epsilon} \int_{\delta}^{\infty} \theta(l-s)^{\beta-1}M_\beta(\theta)Q((l-\tau)^\beta\theta - \epsilon^\beta\delta)B_\lambda f(\tau, y(\tau))d\tau d\theta.\end{aligned}$$

From the compactness of $Q(\epsilon^\beta\delta)$, $(\epsilon^\beta\delta > 0)$, we obtain that the set $V_{\epsilon,\delta}(l) = \{((F_i^2)_{\epsilon,\delta}y)(l) : y \in B_\eta\}$ is relatively compact in $\mathbb{X}$ for $\forall\epsilon \in (s_i, s)$ and $\delta > 0$.

Moreover for any $x \in B_\eta$, we have

$$\begin{aligned}&\|(F_i^2 y)(l) - ((F_i^2)_{\epsilon,\delta}y)(l)\|\\ &\leq \|\beta \lim_{\lambda\to\infty} \int_{s_i}^{l} \int_{0}^{\delta} \theta(l-\tau)^{\beta-1}M_\beta(\theta)Q((l-\tau)^\beta\theta)B_\lambda f(\tau, y(\tau))d\tau d\theta\|\\ &+ \beta \lim_{\lambda\to\infty} \|\int_{l-\epsilon}^{l} \int_{\delta}^{\infty} \theta(l-\tau)^{\beta-1}M_\beta(\theta)Q((l-\tau)^\beta\theta)B_\lambda f(\tau, y(\tau))d\tau d\theta\|\end{aligned}$$

$$\leq \beta M M_0 \int_{s_i}^{l} (l-\tau)^{\beta-1} m(\tau) d\tau \int_0^{\delta} \theta M_\beta(\theta) d\theta$$

$$+ \beta M M_0 \int_{l-\epsilon}^{l} (l-\tau)^{\beta-1} m(\tau) d\tau \int_0^{\infty} \theta M_\beta(\theta) d\theta$$

$$\leq \beta M M_0 \frac{(l-s_i)^{(a+1)(1-\beta_1)}}{(a+1)^{1-\beta_1}} \|m\|_{L^{\frac{1}{\beta_1}}[s_i,t_{i+1}]} \int_0^{\delta} \theta M_\beta(\theta) d\theta$$

$$+ \frac{\beta M M_0}{\Gamma(1+\beta)} \frac{\epsilon^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}} \|m\|_{L^{\frac{1}{\beta_1}}[s_i,t_{i+1}]}$$

$$\longrightarrow 0 \quad \text{as } \epsilon, \delta \to 0.$$

Hence the closure of the set $V(l)$ is compact in $\mathbb{X}$.

Step 4: We prove that the set of functions $[\widetilde{F_i^2 B_\eta}]_i$, $i = 0, \ldots, N$, is an equicontinuous subset of $C([t_i, t_{i+1}], \mathbb{X})$.

Let, $l_1, l_2 \in [s_i, t_{i+1}]$, $s_i < l_1 < l_2$ and $x \in B_\eta$.

We have

$$\begin{aligned}
\|(F_i^2)x(l_2) - (F_i^2)x(l_1)\| &= \| \lim_{\lambda\to\infty} \int_{s_i}^{l_2} (l_2-\tau)^{\beta-1} P_\beta(l_2-\tau) B_\lambda f(\tau, y(\tau)) d\tau \\
&\quad - \lim_{\lambda\to\infty} \int_{s_i}^{l_1} (l_1-\tau)^{\beta-1} P_\beta(l_1-\tau) B_\lambda f(\tau, y(\tau)) d\tau \| \\
&= \| \lim_{\lambda\to\infty} \int_{l_1}^{l_2} (l_2-\tau)^{\beta-1} P_\beta(l_2-\tau) B_\lambda f(\tau, y(\tau)) d\tau \\
&\quad + \|\beta \lim_{\lambda\to\infty} \int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} P_\beta(l_2-\tau) B_\lambda f(\tau, y(\tau)) d\tau \\
&\quad - \lim_{\lambda\to\infty} \int_{s_i}^{l_1} (l_1-\tau)^{\beta-1} P_\beta(l_2-\tau) B_\lambda f(\tau, y(\tau)) d\tau \| \\
&\quad + \|\beta \lim_{\lambda\to\infty} \int_{s_i}^{l_1} (l_1-\tau)^{\beta-1} P_\beta(l_2-\tau) B_\lambda f(\tau, y(\tau)) d\tau \\
&\quad - \beta \lim_{\lambda\to\infty} \int_{s_i}^{l_1} (l_1-\tau)^{\beta-1} P_\beta((l_1-\tau) B_\lambda f(\tau, y(\tau)) d\tau \| \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \frac{M M_0}{\Gamma(\beta)} \left| \int_{l_1}^{l_2} (l_2-\tau)^{\beta-1} m(\tau) d\tau \right|, \\
I_2 &= \frac{M M_0}{\Gamma(\beta)} \int_{s_i}^{a} [(l_1-\tau)^{\beta-1} - (l_2-\tau)^{\beta-1}] m(\tau) d\tau,
\end{aligned}$$

$$I_3 = M_0\int_{s_i}^{l_1-\epsilon}(l_1-\tau)^{\beta-1}\|P_\beta(l_2-\tau)-P_\beta(l_1-\tau)\|m(\tau)d\tau.$$

Now,

$$\begin{aligned}
I_1 &\le \frac{MM_0}{\Gamma(\beta)}\left[\int_{l_1}^{l_2}(l_2-\tau)^{\frac{\beta-1}{1-\beta_1}}d\tau\right]^{1-\beta_1}\Big(\int_{l_1}^{l_2}|m(\tau)|^{\frac{1}{\beta_1}}d\tau\Big)^{\beta_1}\\
&\le \frac{MM_0}{\Gamma(\beta)}\frac{(l_2-l_1)^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}}\|m\|_{L^{\frac{1}{\beta_1}}([l_1,l_2],\mathbb{R}^+)}\\
&\to 0 \quad \text{as} \quad l_2\to l_1.
\end{aligned}$$

Also,

$$\begin{aligned}
I_2 &\le \frac{MM_0}{\Gamma(\beta)}\left(\int_{s_i}^{l_1}[(l_1-\tau)^{\beta-1}-(l_2-\tau)^{\beta-1}]^{\frac{1}{1-\beta_1}}d\tau\right)^{1-\beta_1}\\
&\le \frac{MM_0}{\Gamma(\beta)}\left(\int_{s_i}^{l_1}[(l_1-\tau)^{a}-(l_2-\tau)^{a}]d\tau\right)^{1-\beta_1}\|m\|_{L^{\beta_1}([s_i,l_1])}\\
&\le \frac{MM_0}{\Gamma(\beta)(1+a)^{1-\beta_1}}((l_2-l_1)^{a+1}-((l_2-s_i)^{a+1}-(l_1-s_i)]^{a+1}))^{1-\beta_1}\|m\|_{L^{\beta_1}([s_i,l_1],\mathbb{R}^+)}\\
&\le \frac{MM_0}{\Gamma(\beta)(1+a)^{1-\beta_1}}((l_2-l_1)^{(a+1)(1-\beta_1)}\|m\|_{L^{\frac{1}{\beta_1}}([s_i,l_1],\mathbb{R}^+)}\\
&\to 0 \quad \text{as} \quad l_2\to l_1.
\end{aligned}$$

For $\epsilon>0$ small enough,

$$\begin{aligned}
I_3 &= M_0\int_{s_i}^{l_1-\epsilon}(l_1-\tau)^{\beta-1}\|P_\beta(l_2-\tau)-P_\beta(l_1-\tau)\|m(\tau)d\tau\\
&+ M_0\int_{l_1-\epsilon}^{l_1}(l_1-\tau)^{\beta-1}\|P_\beta(l_2-\tau)-P_\beta(l_1-\tau)\|m(\tau)d\tau\\
&\le \int_{s_i}^{l_1-\epsilon}(l_1-\tau)^{\beta-1}m(\tau)d\tau \sup_{s\in[s_i,l_1-\epsilon]}\|P_\beta(l_2-\tau)-P_\beta(l_1-\tau)\|\\
&+ \int_{s_i}^{l_1}(l_1-\tau)^{\beta-1}\|P_\beta(l_2-\tau)-P_\beta(l_1-\tau)\|m(\tau)d\tau\\
&- \int_{s_i}^{l_1-\epsilon}(l_1-\tau)^{\beta-1}\|P_\beta(l_2-\tau)-P_\beta(l_1-\tau)\|m(\tau)d\tau\\
&\le M_0\int_{s_i}^{l_1-\epsilon}(l_1-\tau)^{\beta-1}m(\tau)d\tau \sup_{s\in[s_i,l_1-\epsilon]}\|P_\beta(l_2-\tau)-P_\beta(l_1-\tau)\|\\
&+ \frac{2M_0\beta}{\Gamma(\beta+1)}[\int_{s_i}^{l_1}(l_1-\tau)^{\beta-1}m(\tau)d\tau-\int_{s_i}^{l_1-\epsilon}(l_1-\tau)^{\beta-1}m(\tau)d\tau]
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{s_i}^{l_1-\epsilon}(l_1-\tau)^{\beta-1}m(\tau)d\tau \sup_{s\in[s_i,l_1-\epsilon]}\|P_\beta(l_2-\tau)-P_\beta(l_1-\tau)\| \\
&+ \frac{2M_0M_1}{\Gamma(\beta)}[\int_{s_i}^{l_1}(l_1-\tau)^{\beta-1}m(\tau)d\tau - \int_{s_i}^{l_1-\epsilon}(l_1-\epsilon-\tau)^{\beta-1}m(\tau)d\tau] \\
&+ \frac{2M_0M_1}{\Gamma(\beta)}\int_{s_i}^{l_1-\epsilon}[(l_1-\epsilon-\tau)^{\beta-1}-(l_1-\tau)^{\beta-1}]m(\tau)d\tau \\
&= I_{31}+I_{32}+I_{33}.
\end{aligned}$$

Since $P_\beta(t), t>0$ is continuous in the uniform operator topology, so $I_{31}\to 0$ as $l_2\to l_1$.

As seen in the proof of I_2 and I_3, similarly $I_{32}\to 0$, $I_{33}\to 0$ as $\epsilon\to 0$, and therefore $||(F_i^2y)(l_2)-(F_i^2y)(l_1)||\to 0$ independent of $y\in B_\eta$ as $l_2\to l_1$.
Similarly the case $l_1=s_i$ can also be verified.
Thus, $\widetilde{[F_i^2B_\eta]}_i$ is an equicontinuous subset of $C([t_i,t_{i+1}],\mathbb{X})$.
Step 5: To establish that F^2 is continuous in B_η.

Let $\{y^n\}$ be a sequence of functions in B_η such that $y^n\to y\in B_\eta$. By $(H3)$, we have

$$f(t,y^n(t))\to f(t,y(t)) \text{ as } n\to\infty.$$

For each $t\in J$, we obtain

$$(t-\tau)^{\beta-1}|f(\tau,y^n(\tau))-f(\tau,y(\tau))|\leq 2(t-\tau)^{\beta-1}m(\tau) \text{ a.e. } \in[s_i,t).$$

By Hölder's inequality, the RHS of the above inequality is integrable, for $s\in[s_i,t)$, $s_i\leq t\leq t_{i+1}$, and hence by Lemma 1.1.3, we obtain

$$\int_{s_i}^t (t-\tau)^{\beta-1}\|f(\tau,y^n(\tau))-f(\tau,y(\tau))\|d\tau\to 0 \text{ as } n\to\infty.$$

Thus, for $t_i<t\leq t_{i+1}$, we obtain

$$\begin{aligned}
&\|(F^2y^n)(t)-(F^2y)(t)\| \\
&\leq \|\lim_{\lambda\to\infty}\int_{s_i}^t K_\beta(t-\tau)B_\lambda f(\tau,y^n(\tau))-f(\tau,y(\tau))d\tau\| \\
&\leq \frac{MM_0}{\Gamma(\beta)}\int_{s_i}^t(t-\tau)^{\beta-1}\|f(\tau,y^n(\tau))-f(\tau,y(\tau))\|d\tau\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Therefore, $F^2y^n \to F^2y$ is point wise on $(t_i, t_{i+1}]$ as $n \to \infty$. Hence it follows from Step IV that $Fy^n \to Fy$ uniformly on $(t_i, t_{i+1}]$ as $n \to \infty$ and so F is continuous.

From the above discussion, F^2 is a completely continuous operator. Then Krasnoselskii's fixed point theorem ensures that T has a fixed point which gives rise to a mild solution. □

3.4 Application

As an application of our results, we consider the following fractional time partial differential equation:

$$ {}^C D^{\beta} y(t,w) = \frac{\partial^2}{\partial z^2} y(t,w) + \mathcal{F}(t, y(t,w)), \ a.e. \ (t,w) \in \bigcup_{i=1}^{N} [s_i, t_{i+1}] \times [0,\pi], 0 < \beta < 1, \tag{3.8} $$

$$ y(t,w) = \mathcal{G}_i(t, y(t,w)), w \in [0,\pi], t \in (t_i, s_i], i = 1, 2, ..., N, \tag{3.9} $$

$$ y(t,0) = y(t,\pi) = 0, t \in [0,T], \tag{3.10} $$

$$ y(0,w) = y_0(w), w \in [0,\pi], \tag{3.11} $$

where $0 = t_0 = s_0 < t_1 \leq s_1 < ... < t_N \leq s_N < t_{N+1} = T$ are fixed real numbers, $\mathcal{F} \in C([0,T] \times \mathbb{R}, \mathbb{R})$ and $\mathcal{G}_i \in C((t_i, s_i] \times \mathbb{R}, \mathbb{R})$ for all $i = 1, ..., N$. Let

$$ \begin{aligned} y(t)w &= y(t,w), t \in [0,T], w \in [0,\pi], \\ f(t,y)(w) &= \mathcal{F}(t, y(t,w)), t \in [0,T], w \in [0,\pi], \\ G_i(t,y)(w) &= \mathcal{G}_i(t, y(t,w)), w \in [0,\pi], t \in (t_i, s_i], \ i = 1, 2, ..., N. \end{aligned} $$

We choose $\mathbb{X} = C([0,\pi], \mathbb{R})$ endowed with the uniform topology and consider the operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ defined by

$$ D(A) = \{ y \in C^2([0,\pi], \mathbb{R}) : y(0) = y(\pi) = 0 \}, Ay = y''. $$

This shows that the problem (3.1)-(3.3) is an abstract formulation of the problem (3.8)-(3.11).

From [41], $\rho(A) = (0, \infty)$ and for $\lambda > 0, \|R(\lambda; A)\| \leq \frac{1}{\lambda}$ and

$$\overline{D(A)} = \{y \in \mathbb{X} : y(0) = y(\pi) = 0\} \neq \mathbb{X}.$$

This implies that A satisfies $(H1)$ with $M_0 = 1$. Since it is well known that A generates a compact C_0 semigroup $\{(Q(t))\}_{t>0}$ on $\mathbb{X}_0$ such that $\|Q(t)\| \leq 1$, therefore $(H2)$ is satisfied with $M = 1$.

For the validation of Theorem 3.3.1, let us take,

$$\begin{aligned} f(t, y)(w) &= \frac{e^{-t}}{e^t + e^{-t}} \left(\frac{|y(t, w)|}{1 + |y(t, w)|} \right) + e^{-t}, \\ G_i(t, y)(w) &= \frac{\cos t|y(t, w)|}{5(1 + |y(t, w)|)}, t \in (t_i, s_i], i = 1, ..., N, y \in \mathbb{X}, w \in [0, \pi]. \end{aligned}$$

Then clearly $f : [0, T] \times \mathbb{X} \to \mathbb{X}$ is a continuous function and

$$\|f(t, y) - f(t, z)\| \leq \frac{e^{-t}}{e^t + e^{-t}} \|y - z\|, \text{for all } y, z \in \mathbb{X}.$$

And if we let $\mu(t) = \dfrac{e^{-t}}{e^t + e^{-t}}$, it follows that $\mu \in L^{\frac{1}{\beta_1}}([0, T], \mathbb{R}^+)$. Also $G_i : (t_i, s_i] \times \mathbb{X} \to \mathbb{X}$ are continuous functions such that

$$\|G_i(t, y) - G_i(t, z)\| \leq L_{G_i} \|y - z\|,$$

with $L_{G_i} = \frac{1}{5}$. Thus the functions f and G_i satisfy the hypotheses $(H3)$ and $(H4)$ respectively. We deduce that that the system (3.8)-(3.11) has a unique integral solution.

On the other hand, for the validation of Theorem 3.3.2, let us take $\beta = \frac{1}{3}$ and

$$\begin{aligned} f(t, y(t)) &= t^{-1/4} \sin(y(t)), \\ G_i(t, y) &= \frac{\cos(t|y(t, w)|)}{5(1 + |y(t, w)|)}, t \in (t_i, s_i], i = 1, ..., N, y \in \mathbb{X}, w \in [0, \pi]. \end{aligned}$$

Choose $m(t) = t^{-1/4}$. Then the function G_i satisfies $(H4)$ and f satisfies the assumptions $(H5)$ and $(H6)$. Thus the problem (3.8)-(3.11) has a solution.

3.5 Conclusion

The main purpose of this work is to extend the existence results of the non-instantaneous impulsive evolution equations to the case when the operator A is not dense and satisfies a Hille-Yosida condition. Under a set of sufficient conditions, the existence of integral solutions is obtained. The result is supported by a suitable example.

Chapter 4

Non-instantaneous impulsive FEE with finite delay

In this chapter we consider a class of non-instantaneous impulsive fractional evolution equation with finite delay. With the aid of Burton-Kirk's fixed point theorem, fractional calculus and semigroup theory, we establish a set of sufficient conditions for existence of the mild solution. The theory is illustrated by an example.

4.1 Introduction

Functional differential equations may be used to represent a model in which derivative of the state variable does not depend only on present state but also on the knowledge of past time. Abada et al. [1] established sufficient conditions for the existence of mild and extremal solutions for some densely defined impulsive functional differential equations in separable Banach spaces of the form

$$\begin{aligned} y'(t) - Ay(t) &= f(t, y_t), \text{ a.e. } t \in J = [0,T], t \neq t_k, k = 1,2,\ldots N, \\ \Delta y|_{t=t_K} &= I_k(y(t_k^-)), k = 1,2,\ldots N, \\ y(t) &= \phi(t), \phi \in \mathscr{D}, \end{aligned}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ generates a C_0 semigroup, $f, I_k, k = 1,2,\ldots N$ are given functions and $\mathscr{D} = \{\psi : [-r,0] \to \mathbb{X}, \psi$ is continuous everywhere except a finite number of points s at which $\psi(s^-), \psi(s^+)$ exist and $\psi(s^-) = \psi(s^+)\}$.

Agarwal at el. [6] studied the existence of solution of a class of fractional neutral functional differential equation with bounded delay of the form

$$\begin{aligned} {}^C D_t^\beta (y(t) - g(t, y_t)) &= f(t, y_t), t \in (t_0, \infty), t_0 \geq 0, \\ y_{t_0} &= \phi \in C([-r, 0], \mathbb{R}^n). \end{aligned}$$

The existence of solution was achieved by using Burton and Kirk's fixed point theorem. Liang [68] used analytic semigroup theory of linear operator and fixed point theory to prove the existence of mild solutions of a class of semilinear fractional differential with finite delay. Using Banach fixed point theorem and Schauder's fixed point theorem, existence of solutions of impulsive fractional functional differential equations was studied in Guo and Wang [58]. Bellmekki et al. [25] established sufficient conditions for the existence and uniqueness of solution of semi-linear functional differential equations with finite delay.

In this work, we establish sufficient conditions for the existence of mild solution for a class of impulsive fractional functional differential equation with finite delay of the form

$$\begin{aligned} {}^C D_t^\beta y(t) &= Ay(t) + f(t, y_t), t \in (s_i, t_{i+1}], i = 0, 1, ..., N, &(4.1)\\ y(t) &= G_i(t, y_t), t \in (t_i, s_i], i = 1, 2, ..., N, &(4.2)\\ y(t) &= \phi(t), \phi \in \mathscr{D}, &(4.3) \end{aligned}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is the generator of a C_0 semigroup of bounded linear operators $\{Q(t)\}_{t\geq 0}$ on $\mathbb{X}$, $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < ... < t_N \leq s_N \leq t_{N+1} = T$ is a partition on the interval $[0, T]$, f and G_i are suitable functions, $\mathscr{D}$ as defined in [1] is a Banach space with respect to the norm $\|\phi\|_{\mathscr{D}} = \sup_{-r\leq s\leq 0} \|\phi(s)\|$, y_t represents the history of the state from $t - r$ upto the present time t. We also assume that the semigroup $\{Q(t)\}_{t>0}$ is uniformly bounded by $M > 1$.

In section 4.2 we recall some definitions and preliminaries which are required to develop the work. In section 4.3, we give sufficient conditions for the existence of mild-solution of the system (4.1)-(4.3). At the end, an example is presented to support the obtained results.

4.2 Preliminaries

We consider the Banach space $\mathscr{D}_T = \{y : [-r, T] \to \mathbb{X}$ such that $y|_{J_k} \in C(J_k, \mathbb{X})$, for $k = 0, 1, 2, ..., N$, $y(t_k{}^+), y(t_k{}^-)$ exist, $y(t_k{}^-) = y(t_k), k = 0, 1, 2, ..., N, x_0 = \phi \in \mathscr{D}$ and $\sup_{t\in[-r,T]} \|y(t)\| < \infty\}$, endowed with the norm

$$\|y\|_{\mathscr{D}_T} = \sup_{t\in[-r,T]} \|y(t)\|.$$

If $y \in \mathscr{D}_T$, then for each $t > 0$, y_t is an element of $\mathscr{D}$ and $y_t(\theta) = x(t+\theta), \theta \in [-r, 0]$. If $y \in \mathscr{P}_T$, then for any $i = 0, 1, 2, \dots N$, the function $\tilde{y}_i \in C([t_i, t_{i+1}], \mathbb{X})$ is constructed as follows:

$$\tilde{y}_i(t) = \begin{cases} y(t), \ \text{for} \ \ t \in (t_i, t_{i+1}], \\ y(t_i^+), \ \text{for} \ \ t = t_i. \end{cases}$$

For $\mathscr{B} \subset PC(J, \mathbb{X})$, we denote $\tilde{\mathscr{B}}_i = \{\tilde{y}_i : y \in \mathscr{B}\}$.

4.3 Existence of PC-mild solution

In this section we first formulate the definition of PC-mild solution of our problem and then prove the existence of solutions with finite delay.

In view of definitions of mild solution in [119] and PC-mild solution in [60], we define the mild solution as follows:

Definition 4.3.1. *A function $y \in \mathscr{D}_T$ is called PC-mild solution of the problem* (4.1)-(4.3) *if it satisfies the following integral equation:*

$$y(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \mathscr{T}(t)x_0 + \int_0^t \mathscr{S}(t-\tau)f(\tau, y_\tau)d\tau, & t \in [0, t_1], \\ G_i(t, y_t), & t \in (t_i, s_i], \\ \mathscr{T}(t-s_i)G_i(s_i, y_{s_i}) + \int_{s_i}^t \mathscr{S}(t-\tau)f(\tau, y_\tau)d\tau, & t \in [s_i, t_{i+1}], \end{cases}$$

We introduce the following hypotheses:

(H1) the functions G_i are continuous and there are constants $L_{G_i} > 0$ such that $\|G_i(t, \psi_1) - G_i(t, \psi_2)\| \leq L_{G_i}\|\psi_1 - \psi_2\|_{\mathscr{D}}$, for all $\psi_1, \psi_2 \in \mathscr{D}, t \in (t_i, s_i]$ and each

$i = 1, 2, ..., N$;

(H2) For each $\phi \in \mathscr{D}$, the function $f(., \phi) : J \to \mathbb{X}$ is strongly measurable and for each $t \in J$, the function $f(t, .) : \mathscr{D} \to \mathbb{X}$ is continuous.

(H3) there exist a constant $\beta_1 \in (0, \beta)$ and a function $m \in L^{\frac{1}{\beta_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t, \psi)\| \leq m(t)W(\|\psi\|_{\mathscr{D}}), \text{a.e. } t \in J, \psi \in \mathscr{D},$$

where $W : [0, \infty) \to \mathbb{R}^+$ is a continuous nondecreasing function with

$$K_1 \int_{s_i}^{t_{i+1}} (t - \tau)^{\beta - 1} m(\tau) d\tau < \int_{K_0}^{\infty} \frac{d\tau}{W(\tau)}.$$

Here

$$K_0 = M\|\phi(0)\|, \quad K_1 = \frac{M}{\Gamma(\beta)}, \text{ for } t \in [0, t_1], \text{ and}$$

$$K_0 = \frac{M\|G_i(t, 0)\|}{1 - ML_{G_i}}, K_1 = \frac{M}{(1 - ML_{G_i})\Gamma(\beta)} \text{ for } t \in [s_i, t_{i+1}], i = 1, 2 \dots N.$$

(H4) the operator A is the infinitesimal generator of a compact semigroup of uniformly bounded linear operators $\{Q(t)\}_{t \geq 0}$ such that there exists $M > 1$ such that

$$\|Q(t)\| \leq M.$$

It is to be noted that $a = \dfrac{\beta - 1}{1 - \beta_1} \in (-1, 0)$.

Theorem 4.3.1. *Assume that the above hypotheses hold and* $\|G_i(., 0)\|$ *are bounded for each* $i = 1, 2, \dots N$. *Then, for every initial* $\phi \in \mathscr{D}$, *the system of equations* (4.1)-(4.3) *has a unique PC-mild solution* $y \in \mathscr{D}_T$, *provided* $(1 + M)L_{G_i} < 1$.

Proof. Let $\mathscr{F} : \mathscr{D}_T \to \mathscr{D}_T$ be defined by

$$\mathscr{F}y(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ G_i(t, y_t), & t \in (t_i, s_i], \\ \mathscr{T}(t)\phi(0) + \int_0^t \mathscr{S}(t - \tau) f(\tau, y_\tau) d\tau, & t \in [0, t_1], \\ \mathscr{T}(t - s_i)G_i(s_i, y_{s_i}) + \int_{s_i}^t \mathscr{S}(t - \tau) f(\tau, y_\tau) d\tau, & t \in [s_i, t_{i+1}]. \end{cases}$$

By hypothesis $(H3)$ and the work in [40], it is easy to verify that the operator is well-defined. To apply Burton-Kirk's fixed point theorem, we use the following decomposition of $\mathscr{F}$:

$$\mathscr{F} = \mathscr{F}^1 + \mathscr{F}^2 = \sum_{i=0}^{N} \mathscr{F}_i^1 + \sum_{i=0}^{N} \mathscr{F}_i^2,$$

where $\mathscr{F}_i^j : \mathscr{D}_T \to \mathscr{D}_T, i = 1, \ldots, N, j = 1, 2$ are defined as

$$\mathscr{F}_i^1 y(t) = \begin{cases} G_i(t, y_t), & t \in (t_i, s_i], \quad i \geq 1, \\ \mathscr{T}(t - s_i)G_i(s_i, y_{s_i}), & t \in (s_i, t_{i+1}], i \geq 1, \\ 0, & t \notin (t_i, t_{i+1}], \\ \mathscr{T}(t)x_0, & t \in [0, t_1], \end{cases}$$

and

$$\mathscr{F}_i^2 y(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \int_{s_i}^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, y_\tau)d\tau, & t \in (s_i, t_{i+1}], i \geq 0, \\ 0, \text{ otherwise.} \end{cases}$$

Our proof consists of six steps.

Step 1: The function $\mathscr{F}^2$ is continuous.

Let $\{y^n\}_{n=1}^\infty$ be a sequence of functions in $\mathscr{D}_T$ such that y^n converges to $y \in \mathscr{D}_T$.

Then $\lim_{m\to\infty} y^n(\tau) = y(\tau)$, for $\tau \in [-r, T]$.

Since $\|y_\tau\| \leq \|y\|_\infty$, for $\tau \in J$, by the condition $(H2)$,

$$\lim_{m\to\infty} f(\tau, y_\tau^n) = f(\tau, y_\tau) \quad \text{for each } \tau \in J_i.$$

Now, for each $\tau \in J_i$,

$$\|(\mathscr{F}_i^2 y^n)(\tau) - (\mathscr{F}_i^2 y)(\tau)\| \leq \frac{M}{\Gamma(\beta)} \frac{t_{i+1}^\beta}{\beta} \sup_{\tau\in[s_i,t_{i+1}]} \|f(\tau, y_\tau^n) - f(\tau, y_\tau)\|.$$

Hence by Lemma 1.1.3 we have

$$\|(\mathscr{F}_i^2 y^n) - (\mathscr{F}_i^2 y)\| \to 0 \text{ as } n \to \infty.$$

Hence $\mathscr{F}^2$ is continuous in $\mathscr{D}_T$.

Step 2: $\mathscr{F}^2$ sends a bounded set to a bounded set in $\mathscr{D}_T$.

For $\eta > 0$, consider the ball $B_\eta = \{y \in \mathscr{D}_T : \|y\|_{\mathscr{D}_T} \leq \eta\}$.

Now, for any $y \in B_\eta$ and $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned} \|(\mathscr{F}_i^2 y)(t)\| &= \|\int_{s_i}^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, y_\tau)d\tau\| \\ &\leq \frac{M}{\Gamma(\beta)} \int_{s_i}^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)m(\tau)W(\|y_\tau\|)d\tau \end{aligned}$$

$$\leq \frac{M}{\Gamma(\beta)} W(\eta) \int_{s_i}^{t} (t-\tau)^{\beta-1} m(\tau) d\tau$$
$$\leq \frac{M}{\Gamma(\beta)} W(\eta) \frac{(t_{i+1}-s_i)^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}} \|m\|_{L^{\frac{1}{\beta_1}}([s_i,t_{i+1}])}$$
$$=: l_i, \text{ a finite quanitity.}$$

Hence for each $y \in B_\eta$ and $i = 0, 1, \dots N$,

$$\|(\mathscr{F}_i^2 y)(t)\| \leq l_i.$$

Also the boundedness of $(\mathscr{F}_i^2 y)(t)$ is trivial for any $t \notin J_i$.

Step 3: The set of functions $[\widetilde{\mathscr{F}^2 y : y} \in B_\eta]_i, i = 0, 1, \dots, N$, is an equicontinuous set in $C([t_i, t_{i+1}]; \mathbb{X})$.

Let $y \in B_\eta$ and $s_i < \tau_1 < \tau_2 \leq t_{i+1}$.

Now

$$\begin{aligned}
\|(\mathscr{F}_i^2 y)(\tau_2) - (\mathscr{F}_i^2 y)(\tau_1)\| &= \|\int_{s_i}^{\tau_2} (\tau_2-\tau)^{\beta-1} \mathscr{S}(\tau_2-\tau) f(\tau, y_\tau) d\tau \\
&- \int_{s_i}^{\tau_1} (\tau_1-\tau)^{\beta-1} \mathscr{S}(\tau_1-\tau) f(\tau, y_\tau) d\tau\| \\
&\leq \|\int_{\tau_1}^{\tau_2} (\tau_2-\tau)^{\beta-1} \mathscr{S}(\tau_2-\tau) f(\tau, y_\tau) d\tau\| \\
&+ \|\int_{s_i}^{\tau_1} \mathscr{S}(\tau_2-\tau)[(\tau_1-\tau)^{\beta-1} - (\tau_2-\tau)^{\beta-1}] f(\tau, y_\tau) d\tau\| \\
&+ \|\int_{s_i}^{\tau_1} (\tau_1-\tau)^{\beta-1} [\mathscr{S}(\tau_2-\tau) f(\tau, y_\tau) - \mathscr{S}(\tau_1-\tau) f(\tau, y_\tau)] d\tau\| \\
&\leq \frac{M}{\Gamma(\beta)} \int_{\tau_1}^{\tau_2} (\tau_2-\tau)^{\beta-1} m(\tau) W(\|y_\tau\|_D) d\tau \\
&+ \frac{M}{\Gamma(\beta)} \int_{s_i}^{\tau_1} [(\tau_1-\tau)^{\beta-1} - (\tau_2-\tau)^{\beta-1}] m(\tau) W(\|y_\tau\|_D) d\tau \\
&+ \int_{s_i}^{\tau_1} (\tau_1-\tau)^{\beta-1} \|\mathscr{S}(\tau_2-\tau) - \mathscr{S}(\tau_1-\tau)\| m(\tau) W(\|y_\tau\|_D) dt \\
&\leq \frac{M}{\Gamma(\beta)} W(\eta) \int_{\tau_1}^{\tau_2} (\tau_2-\tau)^{\beta-1} m(\tau) d\tau \\
&+ \frac{M}{\Gamma(\beta)} W(\eta) \int_{s_i}^{\tau_1} [(\tau_1-\tau)^{\beta-1} - (\tau_2-\tau)^{\beta-1}] m(\tau) d\tau \\
&+ W(\eta) \int_{s_i}^{\tau_1} (\tau_1-\tau)^{\beta-1} \|\mathscr{S}(\tau_2-\tau) - \mathscr{S}(\tau_1-\tau)\| m(\tau) d\tau
\end{aligned}$$

$$=: I_1 + I_2 + I_3.$$

We have

$$\begin{aligned} I_1 &= \frac{M}{\Gamma(\beta)} W(\eta) \int_{\tau_1}^{\tau_2} (\tau_2 - \tau)^{\beta - 1} m(\tau) d\tau \\ &\leq \frac{M}{\Gamma(\beta)} W(\eta) \frac{(\tau_2 - \tau_1)^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}} \|m\|_{L^{\frac{1}{\beta_1}}[s_i, t_{i+1}]} \\ &\to 0 \text{ as } \tau_2 \to \tau_1. \end{aligned}$$

For $\tau_1 < \tau_2$,

$$\begin{aligned} I_2 &\leq \frac{M}{\Gamma(\beta)} W(\eta) [\int_{s_i}^{\tau_1} [(\tau_1 - \tau)^{\beta-1} - (\tau_2 - \tau)^{\beta - 1}]^{\frac{1}{1-\beta_1}} d\tau]^{1-\beta_1} \|m\|_{L^{\frac{1}{\beta_1}}[s_i, t_{i+1}]} \\ &\leq \frac{M}{\Gamma(\beta)} W(\eta) [\int_{s_i}^{\tau_1} [(\tau_1 - \tau)^a - (\tau_2 - \tau)^a] d\tau]^{1-\beta_1} \|m\|_{L^{\frac{1}{\beta_1}}[s_i, t_{i+1}]} \\ &\leq \frac{M}{\Gamma(\beta)(1+a)^{(1-\beta_1)}} W(\eta) [(\tau_2 - \tau_1)^{1+a} - ((\tau_2 - s_i)^{1+a} - (\tau_1 - s_i)^{1+a})]^{1-\beta_1} \|m\|_{L^{\frac{1}{\beta_1}}[s_i, t_{i+1}]} \\ &\leq \frac{M}{\Gamma(\beta)(1+a)^{(1-\beta_1)}} W(\eta) (\tau_2 - \tau_1)^{(1+a)^{(1-\beta_1)}} \\ &\to 0 \text{ as } \tau_2 \to \tau_1. \end{aligned}$$

For $\epsilon > 0$ small enough, we have

$$\begin{aligned} I_3 &\leq W(\eta) \int_{s_i}^{\tau_1 - \epsilon} (\tau_1 - \tau)^{\beta-1} \|\mathscr{S}(\tau_2 - \tau) - \mathscr{S}(\tau_1 - \tau)\| m(\tau) d\tau \\ &+ W(\eta) \int_{\tau_1 - \epsilon}^{\tau_1} (\tau_1 - \tau)^{\beta-1} \|\mathscr{S}(\tau_2 - \tau) - \mathscr{S}(\tau_1 - \tau)\| m(\tau) d\tau \\ &\leq W(\eta) \int_{s_i}^{\tau_1} (\tau_1 - \tau)^{\beta - 1} m(\tau) d\tau \sup_{s \in [s_i, t_1 - \epsilon]} \|\mathscr{S}(\tau_2 - \tau) - \mathscr{S}(\tau_1 - \tau)\| \\ &+ \frac{2M}{\Gamma(\beta)} \int_{t_1 - \epsilon}^{t_1} (t_1 - s)^{(\beta - 1)} m(\tau) d\tau. \end{aligned}$$

The first term on the right-hand side tends to zero as $\tau_2 \to \tau_1$, since $\mathscr{S}(t)$ is compact for $t > 0$ and hence continuous in the uniform operator topology. The second term tends to zero as $\epsilon \to 0$ by I_2.

Step 4: For $i = 0, 1, \ldots, N$ and $s_i < s < t \leq t_{i+1}$, the set $V(l) = \bigcup_{l \in [s,t]} \{(\mathscr{F}_i^2 y)(l : y \in$

$B_\eta\}$ is a precompact set in $\mathbb{X}$. For $0 < \epsilon < t - s_i$ and any $\delta > 0$, define an operator $(\mathscr{F}_i^2)_\epsilon^\delta$ on B_η by the formula

$$
\begin{aligned}
(\mathscr{F}_i^2)_\epsilon^\delta y(l) &= \beta \int_{s_i}^{l-\epsilon} \int_{\delta}^{\infty} \theta(l-s)^{\beta-1} M_\beta(\theta) Q((l-\tau)^\beta\theta) f(\tau, y_\tau) d\theta d\tau \\
&= \beta Q(\epsilon^\beta \delta) \int_{s_i}^{l-\epsilon} \int_{\delta}^{\infty} \theta(l-\tau)^{\beta-1} M_\beta(\theta) Q((l-\tau)^\beta\theta - \epsilon^\beta\delta) f(\tau, y_\tau) d\theta d\tau.
\end{aligned}
$$

From the compactness of the operator $Q(\epsilon^\beta\delta)$, we say that the set $V_\epsilon^\delta(l) = \{(\mathscr{F}_i^2)_\epsilon^\delta y(l) : y \in B_\eta\}$ is relatively compact in $\mathbb{X}$. Moreover for any $y \in B_\eta$ we have

$$
\begin{aligned}
\|(\mathscr{F}_i^2)y(l) - (\mathscr{F}_i^2)_\epsilon^\delta y(l)\| &\leq \beta \| \int_{s_i}^{l} \int_{0}^{\delta} \theta(l-\tau)^{\beta-1} M_\beta(\theta) Q((l-\tau)^\beta\theta - \epsilon^\beta\delta) f(\tau, y_\tau) d\theta d\tau \| \\
&+ \beta \| \int_{l-\epsilon}^{l} \int_{\delta}^{\infty} \theta(l-\tau)^{\beta-1} M_\beta(\theta) Q((l-\tau)^\beta\theta - \epsilon^\beta\delta) f(\tau, y_\tau) d\theta d\tau \| \\
&\leq W(\eta)\beta M \int_{s_i}^{l} (l-\tau)^{\beta-1} m(\tau) d\tau \int_0^\delta M_\beta(\theta) d\theta \\
&+ W(\eta) \frac{M}{\Gamma(\beta)} \int_{l-\epsilon}^{l} (l-\tau)^{\beta-1} m(\tau) d\tau \\
&\to 0 \text{ as } \epsilon \to 0, \delta \to 0.
\end{aligned}
$$

Therefore, the set $V(l)$ is precompact in $\mathbb{X}$. Hence the operator $\mathscr{T}^2 : \mathscr{D}_T \to \mathscr{D}_T$ is completely continuous.

Step 5: $\mathscr{F}^1$ is a contraction on B_η.

Let $y, z \in B_\eta$ and $t \in (t_i, t_{i+1}], i = 1, 2, \ldots N$. Then

$$
\begin{aligned}
\|\mathscr{F}_i^1 y(t) - \mathscr{F}_i^1 z(t)\| &\leq (1+M) L_{G_i} \|y_t - z_t\|_{\mathscr{D}} \\
&\leq (1+M) L_{G_i} \|y - z\|_\infty.
\end{aligned}
$$

This implies

$$
\|\mathscr{F}^1 y - \mathscr{F}^1 z\|_{\mathscr{D}_T} \leq \Theta \|y - z\|_{\mathscr{D}_T},
$$

which is a contraction.

Step 6: To show a priori bounds.

Consider the set

$$
\mathscr{E} = \{y \in \mathscr{D}_T : y = \delta \mathscr{F}^2(y) + \delta \mathscr{F}^1\left(\frac{y}{\delta}\right) \text{ for some } 0 < \delta < 1\}.
$$

For each $t \in [0, t_1]$, we have

$$y(t) = \delta \mathscr{F}(t)\phi(0) + \delta \int_0^t (t-\tau)^{\beta-1} \mathscr{S}(t-\tau) f(\tau, y_\tau) d\tau.$$

Hence for each $t \in [0, t_1]$, we have

$$\|y(t)\| \leq M\|\phi(0)\| + \frac{M}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} m(\tau) W(\|y_\tau\|_{\mathscr{D}}) d\tau. \tag{4.4}$$

But $\|y_t\|_{\mathscr{D}} \leq \{\sup \|y(\tau)\| : -r \leq \tau \leq t\}, 0 \leq t \leq t_1$.

If we define $\mu(t) = \{\sup \|y(\tau)\| : -r \leq \tau \leq t\}, 0 \leq t \leq t_1$, then (4.4) becomes

$$\|y(t)\| \leq M\|\phi(0)\| + \frac{M}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} m(\tau) W(\mu(\tau)) d\tau. \tag{4.5}$$

Hence from the definition of μ, we have

$$\mu(t) \leq M\|\phi(0)\| + \frac{M}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} m(\tau) W(\mu(\tau)) d\tau.$$

Thus we have

$$\mu(t) \leq K_0 + K_1 \int_0^t (t-\tau)^{\beta-1} m(\tau) W(\mu(\tau)) d\tau,$$

where $K_0 = M\|\phi(0)\|, K_1 = \frac{M}{\Gamma(\beta)}$.

If we take the right hand side inequality as $\mathcal{V}(t)$, then

$$\mu(t) \leq \mathcal{V}(t) \ \forall \ t \in [0, t_1], \ \mathcal{V}(0) = K_0,$$

and

$$\mathcal{V}'(t) = (s-t)^{\beta-1} m(t) W(\mu(t)).$$

This gives

$$\mathcal{V}'(t) \leq (s-t)^{\beta-1} m(t) W(\mathcal{V}(t)).$$

Therefore,

$$\int_{\mathcal{V}(0)}^{\mathcal{V}(t)} \frac{du}{W(u)} \leq K_1 \int_0^t (t-\tau)^{\beta-1} m(\tau) d\tau < \int_{K_0}^{\infty} \frac{du}{W(u)}.$$

Hence there exists a constant C such that

$$\mu(t) \leq \mathcal{V}(t) \leq C, \ \forall \ t \in [0, t_1].$$

Now from the definition of μ, it follows that

$$\|y\|_{\mathscr{D}_T} \leq \mu(t_1) \leq C, \ \forall\ x \in \mathscr{E}.$$

For each $t \in (t_i, s_i], i = 1, 2, \ldots N$,

$$y(t) = \delta G_i\left(t, \frac{y_t}{\delta}\right).$$

This implies that for each $t \in (t_i, s_i]$,

$$\|y(t)\| \leq L_{G_i}\|y_t\| + \delta\|G_i(t, 0)\|, \text{ and}$$

$$\|y(t)\| \leq L_{G_i}\|y_t\|_{\mathscr{D}_T} + \|G_i(t, 0)\|. \tag{4.6}$$

If $\|y_t\|_{\mathscr{D}_T} \leq \mu(t)$, then (4.6) becomes

$$\|y(t)\| \leq L_{G_i}\mu(t) + \|G_i(t, 0)\|. \tag{4.7}$$

Using the definition of μ in (4.7), we have

$$\mu(t) \leq L_{G_i}\mu(t) + \|G_i(t, 0)\|. \tag{4.8}$$

Thus,

$$\mu(t) \leq \frac{\|G_i(t, 0)\|}{1 - L_{G_i}} = M_{t_i}.$$

This gives $\mu(t) \leq M_{t_i}, t \in (t_i, s_i]$.

Hence from (4.7), we have

$$\|y(t)\| \leq L_{G_i}M_{t_i} + \|G_i(t, 0)\| =: L_i.$$

Thus

$$\|y\|_{\mathscr{D}_T} \leq L_i.$$

Finally for $t \in [s_i, t_{i+1}], i = 1, 2, \ldots N$,

$$y(t) = \delta\mathscr{T}(t - s_i)G_i(s_i, \frac{y_{s_i}}{\delta}) + \delta\int_{s_i}^{t}(t - \tau)^{\beta-1}\mathscr{S}(t - \tau)f(\tau, y_\tau)d\tau.$$

Hence for each $t \in [s_i, t_{i+1}]$,

$$\|y(t)\| \leq ML_{G_i}\|y_{s_i}\| + M\delta\|G_i(t, 0)\| + \delta\frac{M}{\Gamma(\beta)}\int_{s_i}^{t}(t - \tau)^{\beta-1}m(\tau)W(\|y_\tau\|_{\mathscr{D}})d\tau.$$

Therefore,

$$\|y(t)\| \leq ML_{G_i}\|y_{s_i}\| + M\|G_i(t,0)\| + \frac{M}{\Gamma(\beta)}\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)W(\|y_\tau\|_{\mathscr{D}})d\tau. \quad (4.9)$$

If $\|y_t\|_{\mathscr{D}_T} \leq \mu(t)$, then (4.9) becomes

$$\|y(t)\| \leq ML_{G_i}\mu(t) + M\|G_i(t,0)\| + \frac{M}{\Gamma(\beta)}\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)W(\mu(\tau))d\tau. \quad (4.10)$$

Using the definition of μ in (4.10), we have

$$\mu(t) \leq \tilde{K}_0 + \tilde{K}_1\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)W(\mu(\tau))d\tau, \quad (4.11)$$

where

$$\tilde{K}_0 = \frac{M\|G_i(t,0)\|}{1-ML_{G_i}}, \tilde{K}_1 = \frac{M}{(1-ML_{G_i})\Gamma(\beta)}.$$

If we take the right hand side inequality as $\mathcal{V}(t)$, then

$$\mu(t) \leq \mathcal{V}(t), \ \forall\, t \in [s_i, t_{i+1}], \ v(s_i) = \tilde{K}_0,$$

and

$$\mathcal{V}'(t) = \tilde{K}_1(s-t)^{\beta-1}m(t)W(\mu(t)), \ t \in [s_i, t_{i+1}].$$

By the increasing property of W, we obtain

$$\mathcal{V}'(t) \leq \tilde{K}_1(s-t)^{\beta-1}m(t)W(\mathcal{V}(t)), \ t \in [s_i, t_{i+1}].$$

Hence on integration, we get

$$\int_{\mathcal{V}(s_i)}^{\mathcal{V}(t)}\frac{d\tau}{W(\tau)} \leq \tilde{K}_1\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)d\tau < \int_{\tilde{K}_0}^{\infty}\frac{d\tau}{W(\tau)}.$$

Therefore, there exists a constant $\tilde{C}_i$ such that $\mu(t) \leq \mathcal{V}(t) \leq \tilde{C}_i, \ \forall t \in [s_i, t_{i+1}]$.
Hence

$$\|y\|_{\mathscr{D}_T} \leq ML_{G_i}\tilde{C}_i + M\|G_i(t,0)\| + \frac{M}{\Gamma(\beta)}\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)W(\tilde{C}_i)d\tau.$$

This implies that the set $\mathscr{E}$ is bounded.

Thus by Burton-Kirk's fixed point theorem, the operator $\mathscr{F}$ has a fixed point in $\mathscr{D}_T$ which is a mild solution of the system (4.1)-(4.3). □

4.4 Example

We consider the fractional reaction-diffusion equation with delay described by

$$\begin{aligned} {}^C D_t^\beta y(t,w) &= \frac{\partial^2}{\partial z^2} y(t,w) + F(t, y(t-r,w)), t \in J_{i+1}, i = 0,1,...,N, w \in [0,\pi], \quad (4.12)\\ y(t,w) &= G_i(t, y(t-r,w)), w \in [0,\pi], t \in (t_i, s_i], i = 1,2,...,N, \quad (4.13)\\ y(t,0) &= y(t,\pi) = 0, t \in [0,T], \quad (4.14)\\ y(t,w) &= \phi(t,w), t \in [-r,0], w \in [0,\pi], \quad (4.15)\end{aligned}$$

where $J_i = (s_i, t_{i+1}], r > 0, \phi \in \mathscr{D} = \{\psi : [-r,0]\times[0,\pi] \to \mathbb{R}, \psi$ is continuous everywhere except at a finite number of points s at which $\psi(s^-), \psi(s^+)$ exist and $\psi(s^-) = \psi(s^+)\}$, the impulse time t_i satisfies $t_0 = s_0 < t_1 \leq s_1 < t_2 < ... < t_N \leq s_N < t_{N+1} = T$ and F, G_i are given functions.

Let us take $\mathbb{X} = L^2([0,\pi])$ and define $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ by $Ay = y''$ with domain

$$D(A) = \{y \in \mathbb{X} : y, y' \text{ are absolutely continuous }, y'' \in \mathbb{X}, y(0) = y(\pi) = 0\}.$$

Then

$$Ax = \sum_{n=1}^{\infty} e^{-n^2 t}(w, w_n)w_n, w \in \mathbb{X},$$

where $(.,.)$ is an inner product in L^2 and $w_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, n = 1,2,\dots$ is the orthogonal set of eigenvectors in A which generates a C_0 semigroup $\{Q(t)\}_{t\geq 0}$ on $\mathbb{X}$. Since A is compact and analytic, there exit $M \geq 1$ such that

$$\|Q(t)\|_{B(\mathbb{X})} \leq M.$$

Let $y(t)w = y(t,w), t \in J, w \in [0,\pi]$.

For the case $(t,\phi) \in [-r,b] \times \mathscr{D}$: Assume that (i) For all $i = 0,1,\dots N$, the function $f : [s_i, t_{i+1}] \times \mathscr{D} \to \mathbb{X}$ defined by $f(t, y_t)w = F(t, y(t-r,w)), t \in J_i, w \in [0,\pi]$, is continuous and satisfies the hypotheses $(H2)$ and $(H3)$.

(ii) For all $i = 1,\dots N$, the functions $G_i : (t_i, s_i] \times \mathscr{D} \to \mathbb{X}$ defined by $G_i(t, y_t)w = G_i(t, y(t-r,w)), t \in (t_i, s_i], z \in [0,\pi]$, are continuous and satisfy the hypothesis $(H1)$. With the above setting, the system of equations (4.12)-(4.14) gets transformed to the abstract form (4.1)-(4.3). Since all the conditions of Theorem 4.3.1 are satisfied, therefore the problem (4.12)-(4.14) has a mild solution y on $[-r,T] \times [0,\pi]$.

4.5 Conclusion

In this chapter we prove the existence of mild solution of a class non-instantaneous impulsive fractional functional differential equation with finite delay. Our result is based on Burton-Kirk's fixed point theorem.

Chapter 5

Non-instantaneous impulsive FEE with infinite delay

In this chapter, we extend the result obtained in the previous chapter to a class of impulsive fractional functional evolution equations with infinite delay. Under the assumption that the linear part of the equation generates a compact analytic semigroup and the delay portion lies in an abstract phase space, a set of sufficient conditions is established to ensure the existence of a mild solution.

5.1 Introduction

Integro evolution equations in abstract spaces generalize many partial differential equations and integro-differential equations appearing in scientific and engineering problems, for instance, presence of hereditary influence in population dynamics [74], equation of plate with memory [45] etc. As presence of delay in dynamical systems causes unpredictable behavior, the classical solution of such type of equations is rarely possible. It is meaningful to discuss the nature of solution of these equations in qualitative ways.

Chang et al. [39] discussed the controllability of a first order impulsive functional differential system with infinite delay in Banach spaces. Hu et al. [65] established the existence of mild solution of a class of Riemann-Liouville fractional evolution equation with nonlocal conditions and infinite delay in a Banach space, in which the linear part was the infinitesimal generator of a compact analytic semigroup. Mahmudov and Zorlu [80] investigated the approximate controllability of fractional evolution equations with a compact analytic semigroup. The controllability of an impulsive neutral functional integro-differential equation in a Banach space can be found in [108]. Some interesting results on existence of solution of impulsive fractional evolution equations with infinite delay are obtained in [17, 21, 57, 117, 118].

Motivated by the above consideration, here we investigate the existence of mild solution of a class of fractional evolution equation with infinite delay of the following form:

$$ {}^{C}D_t^{\beta} y(t) = -Ay(t) + f(t, y_t, \int_0^t G(t,\tau,y_\tau)d\tau), t \in (s_i, t_{i+1}], i = 0, 1, \cdots, N, \quad (5.1) $$

$$ y(t) = G_i(t, y_t), t \in (t_i, s_i], i = 1, 2, ..., N, \quad (5.2) $$

$$ y(t) = \phi(t), t \in (-\infty, 0], \quad (5.3) $$

where $x(.)$ takes values in a Banach space $(\mathbb{X}, \|.\|), 0 < \beta < 1, -A : D(A) \subset \mathbb{X} \longrightarrow \mathbb{X}$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators on $\mathbb{X}$, $0 = s_0 < t_1 \leq s_1 \leq t_2 \ldots \leq t_N \leq s_N < t_{N+1} = T$ is a partition of the interval $J = [0, T]$, the functions $G_i \in C((t_i, s_i] \times \mathscr{P}_0, \mathbb{X}_\alpha)$ for each $i = 1, 2, ...N$ and $f : [0, T] \times \mathscr{P}_0 \times \mathbb{X}_\alpha \to \mathbb{X}_\alpha$ is a suitable function. Here $\mathscr{P}_0$ is a phase space and $\mathbb{X}_\alpha$ is a fractional power space defined in the next section. For a function y defined on $(-\infty, T]$ and any $t \in J$, we denote $y_t(.)$ to represent the portion of the function from $-\infty$ to present time t, that is,

$$ y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0]. $$

5.2 Preliminaries

The definitions and the theorems related to analytic semigroup theory are taken from [87, 92].

Let $0 \in \rho(A)$. Then for any $0 < \alpha < 1$, we can define $A^{-\alpha}$ as a closed linear operator on its domain $D(A^{-\alpha})$:

$$ A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} Q(t) dt. \quad (5.4) $$

The operator defined by (5.4) is a bounded linear operator and each $A^{-\alpha}$ is an injective continuous endomorphism of $\mathbb{X}$. Therefore it is possible to define A^{α} for $0 < \alpha < 1$, as a closed linear operator on its domain $D(A^{\alpha})$. The subspace $D(A^{\alpha})$ is dense in $\mathbb{X}$, and the expression

$$\|u\|_{\alpha} = \|A^{\alpha}u\|$$

defines a norm on $D(A^{\alpha})$ which makes it a Banach space. We write $\mathbb{X}_{\alpha} = D(A^{\alpha})$. For $0 < \gamma < \alpha \leq 1, \mathbb{X}_{\alpha} \hookrightarrow \mathbb{X}_{\gamma}$, the embedding is compact whenever the resolvent operator of A is compact.

Lemma 5.2.1. *[87] A^{α} and $Q(t)$ have the following properties:*

(i) There exists a constant $M > 1$ such that $\|Q(t)\| \leq M$.

(ii) $Q(t) : \mathbb{X} \to \mathbb{X}_{\alpha}$ for each $t > 0$ and $\alpha \geq 0$.

(iii) $A^{\alpha}Q(t)x = Q(t)A^{\alpha}x$ for each $x \in \mathbb{X}_{\alpha}$ and $t \geq 0$.

(iv) For every $t > 0$, $A^{\alpha}Q(t)$ is bounded in $\mathbb{X}$ and there exists $M_{\alpha} > 0$ such that

$$\|A^{\alpha}Q(t)\| \leq M_{\alpha}t^{-\alpha}.$$

(v) $A^{-\alpha}$ is a bounded linear operator in $\mathbb{X}$ with $D(A^{\alpha}) = Im(A^{-\alpha})$.

Lemma 5.2.2. *[87, 112] The restriction $Q_{\alpha}(t)$ of $Q(t)$ to $\mathbb{X}_{\alpha}$ is exactly the part of $Q(t)$ to $\mathbb{X}_{\alpha}$. Also $\{Q(t)\}_{t\geq 0}$ is a family of strongly continuous semigroup on $\mathbb{X}_{\alpha}$ and $\|Q_{\alpha}(t)\| \leq \|Q(t)\| \leq M$ for all $t \geq 0$.*

To find the solution of a differential equation with infinite delay at any time not only requires the knowledge of the state at current time but also of the state in the past. Thus the choice of the phase space is one of the most important characteristics in solution of such equations. In view of Hino et al. [64], it is a usual practice to take the phase space as a semi-normed space satisfying some axioms. Here we define the phase space in the following ways.

Definition 5.2.1. *[35] $\mathscr{P}_0$ is a linear space of functions from $(-\infty, 0]$ to $\mathbb{X}_\alpha$ endowed with a seminorm $\|.\|_{\mathscr{P}_0}$, which satisfies the following axioms:*

(A1): *If $y : (-\infty, T] \to \mathbb{X}, T > 0$ is such that $x_0 \in \mathscr{P}_0$ and for every $t \in [0, T)$, the following conditions hold:*

(i) $y_t \in \mathscr{P}_0$,

(ii) $\|y(t)\|_\alpha \leq C\|y_t\|_{\mathscr{P}_0}$,

(iii) $\|y_t\|_{\mathscr{P}_0} \leq C_1(t) \sup_{t\in[0,T]} \|y(t)\|_\alpha + C_2\|x_0\|_{\mathscr{P}_0}$, *where $C > 0$ is a constant, $C_1, C_2 : [0, \infty) \to [0, \infty), C_1$ is continuous, C_2 is locally bounded and C_1, C_2 are independent of $y(.)$.*

(A2): *For a function $y(.)$ in (**A1**), y_t is a $\mathscr{P}_0$-valued function for $t \in [0, T)$.*

(A3): *The space $\mathscr{P}_0$ is complete.*

Lemma 5.2.3. *[33] Let the functions $f_1, f_2 : J \to \mathbb{R}^+$ be continuous. Assume that there exist $c > 0$ and $h : \mathbb{R} \to (0, \infty)$ a continuous non-decreasing function such that*

$$f_2(t) \leq c + \int_a^t f_1(\tau)h(f_2(\tau))d\tau, \forall t \in J = [a, b].$$

Then

$$f_2(t) \leq H^{-1}\Big(\int_a^t f_1(\tau)d\tau\Big), \forall t \in J$$

provided

$$\int_c^\infty \frac{du}{h(u)} > \int_a^b f_1(\tau)d\tau.$$

Here $H(u) = \int_c^u \frac{d\tau}{h(\tau)}$ for $\tau \geq c$.

$PC(J, \mathbb{X}_\alpha)$, the space of piecewise continuous functions from J into $\mathbb{X}_\alpha$, is a Banach space with respect to the norm

$$\|y\|_{PC(\mathbb{X}^\alpha)} = \sup_{t\in J} \|A^\alpha y(t)\|.$$

To deal with impulsive as well as delay conditions, we consider the space $\mathscr{P}_T = \{y : (-\infty, T] \to \mathbb{X}_\alpha$ to be such that $y_k \in C(J_k, \mathbb{X}_\alpha)$, for $k = 0, 1, 2, ..., N$, $y(t_k{}^+), y(t_k{}^-)$

exist, $y(t_k^-) = y(t_k), k = 0, 1, 2, ..., N, x_0 = \phi \in \mathscr{P}_0$ and $\sup_{t\in[0,T]} \|y(t)\|_\alpha < \infty\}$, endowed with the norm

$$\|y\|_{\mathscr{P}_T} = \|y\|_{PC(\mathbb{X}^\alpha)} + \|\phi\|_{\mathscr{P}_0}.$$

If $y \in \mathscr{P}_T$, then for any $i = 0, 1, 2, \ldots N$, the function $\tilde{y}_i \in C([t_i, t_{i+1}], \mathbb{X}_\alpha)$ is constructed as follows:

$$\tilde{y}_i(t) = \begin{cases} y(t), & \text{for } t \in (t_i, t_{i+1}], \\ y(t_i^+), & \text{for } t = t_i. \end{cases}$$

For $\mathscr{B} \subset PC(J, \mathbb{X}_\alpha)$, we denote $\tilde{\mathscr{B}}_i = \{\tilde{y}_i : y \in \mathscr{B}\}$.

Lemma 5.2.4. *[60] A set $\mathscr{B} \subset PC(J, \mathbb{X}_\alpha)$ is relatively compact in $PC(J, \mathbb{X}_\alpha)$ if and only if each set $\tilde{\mathscr{B}}_i$ is relatively compact in $C([t_i, t_{i+1}], \mathbb{X}_\alpha)$.*

Set

$$\widetilde{C_1} = \sup_{t\in J} C_1(t) \text{ and } \widetilde{C_2} = \sup_{t\in J} C_2(t).$$

Definition 5.2.2. *[126] Consider the fractional evolution equation*

$$^C D_t^\beta y(t) = -Ay(t) + f(t, y_t), t \in J, 0 < \beta < 1, \quad (5.5)$$

$$y(t) = \phi \in \mathscr{P}_0. \quad (5.6)$$

For $f : J \times \mathscr{P}_0 \to \mathbb{X}$ and A generating an analytic semigroup $\{Q(t)\}_{t\geq 0}$, a continuous function $x : J \to \mathbb{X}_\alpha$ satisfying the integral equation

$$y(t) = \mathscr{T}(t)\phi(0) + \int_0^t (t-\tau)^{\beta-1} \mathscr{S}(t-\tau) f(\tau, y_\tau) d\tau$$

is called a mild-solution of (5.5)-(5.6).

Lemma 5.2.5. *[112] The operators $\mathscr{T}$ and $\mathscr{S}$ have the following properties:*

(i) For fixed $t \geq 0$ and any $u \in \mathbb{X}_\alpha$, we have

$$\|\mathscr{T}(t)u\|_\alpha \leq M\|u\|_\alpha, \|\mathscr{S}(t)u\|_\alpha \leq \frac{M\beta}{\Gamma(1+\beta)}\|u\|_\alpha.$$

(ii) $\mathscr{T}_\alpha(t)$ and $\mathscr{S}_\alpha(t), t > 0$ are uniformly continuous, where

$$\mathscr{T}_\alpha(t) = \int_0^\infty \xi_\beta(\theta) Q_\alpha(t^\beta\theta) d\theta \text{ and } \mathscr{S}_\alpha(t) = \beta \int_0^\infty \theta \xi_\beta(\theta) Q_\alpha(t^\beta\theta) d\theta.$$

5.3 Definition of PC-mild solution and assumptions

Motivated by the definition 5.2.2 and the work in [60], we define the mild solution as follows:

Definition 5.3.1. *A function $y \in \mathscr{P}_T$ is said to be PC-mild solution of the problem (5.1)-(5.3) if*

$$y(t) = \begin{cases} \phi(t), \quad t \in (-\infty, 0], \\ \mathscr{T}(t)x_0 + \int_0^t \mathscr{S}(t-\tau)(t-\tau)^{\beta-1} f(\tau, y_\tau, \int_0^\tau G(\tau, s, y_s)ds)d\tau, t \in J_1 = [0, t_1], \\ G_i(t, y_t), t \in (t_i, s_i], i = 1 \dots N, \\ \mathscr{T}(t-s_i)G_i(s_i, y_{s_i}) + \int_{s_i}^t \mathscr{S}(t-\tau)(t-\tau)^{\beta-1} f(\tau, y_\tau, \int_0^\tau G(\tau, s, y_s)ds)d\tau, \\ \qquad\qquad t \in J_i = [s_i, t_{i+1}], i = 1 \dots N. \end{cases}$$

We introduce the following hypotheses:

(H1) $f : J \times \mathscr{P}_0 \times \mathbb{X}_\alpha \to \mathbb{X}_\alpha$ is continuous and there exist $\beta_1 \in (0, \beta)$ and $L_f \in L^{\frac{1}{\beta_1}}(J, \mathbb{R}^+)$ such that

$$\|f(t, \psi_1, u_1) - f(t, \psi_2, v_1)\|_\alpha \le \Omega(t)[\|\psi_1 - \psi_2\|_{\mathscr{P}_0} + \|u_1 - v_1\|_\alpha],$$

for all $\psi_1, \psi_2 \in \mathscr{P}_0, u, v \in \mathbb{X}_\alpha, t \in J_i$ and $i = 0, 1, ..., N$.

(H2) Let $D = \{(t, s) \in J \times J : s \le t\}$, $G : D \times \mathscr{P}_0 \to \mathbb{X}_\alpha$ be continuous and there exists a constant M_G such that for all $(t, s) \in D, \xi_1, \xi_2 \in \mathscr{P}_0$

$$\| \int_0^t [G(t, \tau, \psi_1) - G(t, \tau, \psi_2)]d\tau \|_\alpha \le M_G \|\psi_1 - \psi_2\|_0.$$

(H3) There exist constants $L_{G_i} > 0$, for all $\psi_1, \psi_2 \in \mathscr{P}_0, t \in J_i$ and $i = 1, ..., N$ such that

$$\|G_i(t, \psi_1) - G_i(t, \psi_2)\|_\alpha \le L_{G_i} \|\psi_1 - \psi_2\|_{\mathscr{P}_0}$$

and $G_i \in C((t_i, s_i] \times \mathscr{P}_0, \mathbb{X}_\alpha)$, for all $i = 1, 2, ..., N$.

5.4 Existence of PC-mild solution

Theorem 5.4.1. *Assume that the hypotheses* (H1)-(H3) *hold and $\phi \in \mathbb{X}_\alpha$. Then the system of equations* (5.1)-(5.3) *has a unique PC-mild solution $y \in \mathscr{P}_T$, provided*

$$\Theta = \max\Bigg\{\frac{M}{\Gamma(\beta)}\tilde{C}_1\frac{{t_1}^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}}\|L_f\|_{L^{\frac{1}{\beta_1}}J_1},$$
$$M[\tilde{C}_1 L_{G_i} + \frac{\tilde{C}_1(1+M_G)}{\Gamma(\beta)}\frac{(t_{i+1}-s_i)^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}}\|\Omega\|_{L^{\frac{1}{\beta_1}}J_i}]\Bigg\} < 1, i = 1, 2, ..., N.$$

Proof. We define the operator

$$\mathscr{F} : \mathscr{P}_T \longrightarrow \mathscr{P}_T \quad as$$

$$\mathscr{F}(x)t = \begin{cases} \phi(t), \ \ t \in (-\infty, 0], \\ \mathscr{T}(t)x_0 + \int_0^t \mathscr{S}(t-\tau)(t-\tau)^{\beta-1} f(\tau, y_\tau, \int_0^\tau G(\tau, s, y_s)ds)d\tau, \ \ t \in [0, t_1], \\ G_i(t, y_t), \qquad t \in (t_i, s_i], \\ \mathscr{T}(t-s_i)G_i(s_i, y_{s_i}) + \int_{s_i}^t \mathscr{S}(t-\tau)(t-\tau)^{\beta-1} f(\tau, y_\tau, \int_0^\tau G(\tau, s, y_s)ds)d\tau, \ t \in J_i, \\ \qquad\qquad\qquad i = 1, \cdots, N. \end{cases}$$

Consider the extension Φ of $\phi \in \mathscr{P}_0$ as

$$\Phi(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathscr{T}(t)\phi(0), & t \in [0, t_1], \\ 0, & t \in (t_1, T]. \end{cases}$$

Then $\Phi \in \mathscr{P}_T$.

Let $y(t) = z(t) + \Phi(t), -\infty < t \leq T$. If y satisfies the integral equation in definition (5.2.1), then $z_0 = 0, y_t = z_t + \Phi_t$, for every $t \in J$ and the function $z(t)$ satisfies

$$z(t) = \begin{cases} \int_0^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, z_\tau + \Phi_\tau, \int_0^\tau G(\tau, s, z_s + \Phi_s)ds)d\tau, t \in J_1, \\ G_i(t, z_t + \Phi_t), t \in (t_i, s_i], i = 1, \cdots, N, \\ \int_{s_i}^t (t-\tau)^{\beta-1}\mathscr{S}(t-\tau)f(\tau, z_\tau + \Phi_\tau, \int_0^\tau G(\tau, s, z_s + \Phi_s)ds)d\tau, \\ \qquad\qquad t \in J_i, i = 1, \cdots, N. \end{cases}$$

Let

$$\tilde{\mathscr{P}}_T = \{z \in \mathscr{P}_T : z_0 = 0\}.$$

For any $z \in \tilde{\mathscr{P}}_T$, we have

$$\|z\|_{\tilde{\mathscr{P}}_T} = \sup_{t\in J}\|z(t)\|_\alpha.$$

Thus $\tilde{\mathscr{P}}_T, \|.\|_{\tilde{\mathscr{P}}_T})$ is a Banach space. We define the operator $\tilde{\mathscr{F}} : \tilde{\mathscr{P}}_T \to \tilde{\mathscr{P}}_T$ by

$$\tilde{\mathscr{F}}(z)t = \begin{cases} 0, & t \in (-\infty, 0], \\ \int_0^t (t-\tau)^{\beta-1} \mathscr{S}(t-\tau) f(\tau, z_\tau + \Phi_\tau, \int_0^\tau G(\tau, s, z_s + \Phi_s) ds) d\tau, & t \in J_1, \\ G_i(t, y_t), & t \in (t_i, s_i], \\ \mathscr{T}(t-s_i) G_i(s_i, z_{s_i} + \Phi_{s_i}) + \\ \int_{s_i}^t (t-\tau)^{\beta-1} \mathscr{S}(t-\tau) f(\tau, z_\tau + \Phi_\tau, \int_0^\tau G(\tau, s, z_s + \Phi_s) ds) d\tau, & t \in J_i, i, \cdots, N. \end{cases}$$

It is clear that the operator $\tilde{\mathscr{F}}$ determines the fixed point of the operator $\mathscr{F}$. We show that $\tilde{\mathscr{F}}$ is a contraction mapping.

For $\mathcal{U}, \mathcal{V} \in \acute{\mathscr{P}}_T$ and $t \in J_1$, we get

$$\begin{aligned}
&\|\tilde{\mathscr{F}}\mathcal{U}(t) - \tilde{\mathscr{F}}\mathcal{V}(t)\|_\alpha \\
&= \|\int_0^t (t-\tau)^{\beta-1} \mathscr{S}(t-\tau)[f(\tau, \mathcal{U}_\tau + \Phi_\tau, \int_0^\tau G(\tau, s, \mathcal{U}_s + \Phi_s) ds) \\
&- f(s, \mathcal{V}_s + \Phi_s, \int_0^\tau G(\tau, s, \mathcal{V}_s + \Phi_s) ds] d\tau\|_\alpha \\
&\leq \frac{M}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \Omega(\tau)[\|\mathcal{U}_\tau - \mathcal{V}_\tau\|_{\mathscr{P}_0} d\tau \\
&+ \|\int_0^\tau G(\tau, s, \mathcal{U}_s + \Phi_s) ds - \int_0^\tau G(\tau, s, \mathcal{V}_s + \Phi_s) ds\|_\alpha] d\tau \\
&\leq \frac{M}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \Omega(\tau)(1 + M_G)\|\mathcal{U}_\tau - \mathcal{V}_\tau\|_{\mathscr{P}_0} d\tau \\
&\leq \frac{M}{\Gamma(\beta)} (1 + M_G) \tilde{C}_1 \Big(\int_0^t (t-\tau)^{\frac{\beta-1}{1-\beta_1}} \Big)^{1-\beta_1} d\tau \|L_f\|_{L^{\frac{1}{\beta_1}} J_1} \sup_{t \in J_1} \|\mathcal{U}(t) - \mathcal{V}(t)\|_\alpha \\
&\leq \frac{M}{\Gamma(\beta)} \tilde{C_1} \frac{t_1^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}} \|\Omega\|_{L^{\frac{1}{\alpha_1}} J_1} \|\mathcal{U} - \mathcal{V}\|_{\tilde{\mathscr{P}}_T}.
\end{aligned}$$

For $t \in J_i, i = 1 \dots N$,

$$\begin{aligned}
&\|\tilde{\mathscr{F}}\mathcal{U}(t) - \tilde{\mathscr{F}}\mathcal{V}(t)\|_\alpha \\
&\leq \|\mathscr{T}(t-s_i) G_i(s_i, \mathcal{U}_{s_i} + \Phi(s_i)) - \mathscr{T}(t-s_i) G_i(s_i, \mathcal{V}_{s_i} + \Phi_{s_i})\|_\alpha \\
&+ \|\int_{s_i}^t (t-\tau)^{\beta-1} \mathscr{S}(t-\tau)(f(\tau, \mathcal{U}_\tau + \Phi_\tau, \int_0^\tau G(\tau, s, \mathcal{U}_s + \Phi_s) ds) d\tau \\
&- f(\tau, \mathcal{V}_\tau + \Phi_\tau, \int_0^\tau G(\tau, s, \mathcal{U}_s + \Phi_s) ds) d\tau) ds\|_\alpha \\
&\leq L_{G_i} M \|\mathcal{U}_{s_i} - \mathcal{V}_{s_i}\|_{\mathscr{P}_0} + \frac{M}{\Gamma(\beta)} \int_{s_i}^t (t-\tau)^{\beta-1} \Omega(\tau)(1 + M_G)\|\mathcal{U}_\tau - \mathcal{V}_\tau\|_{\mathscr{P}_0} d\tau
\end{aligned}$$

$$\leq L_{G_i} M \sup_{t\in J_i} \|\mathcal{U}(t)-\mathcal{V}(t)\|_\alpha + \tilde{C}_1 \frac{M}{\Gamma(\beta)}(1+M_G)\frac{(t_{i+1}-s_i)^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}}\|\Omega\|_{L^{\frac{1}{\beta_1}}J_i}$$
$$\times \sup_{t\in J_i} \|\mathcal{U}(t)-\mathcal{V}(t)\|_\alpha$$
$$\leq M[\tilde{C}_1 L_{G_i} + \frac{\tilde{C}_1(1+M_G)}{\Gamma(\beta)}\frac{(t_{i+1}-s_i)^{(1+a)(1-\beta_1)}}{(1+a)^{1-\beta_1}}\|\Omega\|_{L^{\frac{1}{\beta_1}}J_i}]\|\mathcal{U}-\mathcal{V}\|_{\acute{\mathscr{P}}_T}.$$

For $t \in (t_i, s_i]$,

$$\|\tilde{\mathscr{F}}\mathcal{U}(t) - \tilde{\mathscr{F}}\mathcal{V}(t)\| \leq \tilde{C}_1 L_{G_i}\|\mathcal{U}-\mathcal{V}\|_{\tilde{\mathscr{P}}_T}.$$

Thus,

$$\|\tilde{\mathscr{F}}\mathcal{U} - \tilde{\mathscr{F}}\mathcal{V}\|_{\tilde{\mathscr{P}}_T} \leq \Theta\|\mathcal{U}-\mathcal{V}\|_{\tilde{\mathscr{P}}_T}.$$

Therefore, $\tilde{\mathscr{F}}$ is a contraction on $\tilde{\mathscr{P}}_T$ and there exists a unique PC-mild solution of (5.1)-(5.3).

□

In the previous theorem, we established the existence and uniqueness of PC-mild solution by using Lipschitz conditions on both source as well as impulse functions. In our next result, we relax the Lipschitz condition on the source function and use Burton-Kirk's fixed point theorem to obtain the existence of PC-mild solution.

For $\eta > 0$, consider the ball $B_\eta = \{z \in \tilde{\mathscr{P}}_T : \|z\|_{\tilde{\mathscr{P}}_T} \leq \eta\}$. Clearly B_η is a closed bounded convex set in $\acute{\mathscr{P}}_T$.

In the sequel, we make the following assumptions:

(H4) For each $t \in J$, the function $f(t,.,.) : \mathscr{P}_0 \times \mathbb{X}_\alpha \to \mathbb{X}_\alpha$ is continuous and for each $\psi \in \mathscr{P}_0, u \in \mathbb{X}_\alpha$, the function $f(.,\psi,u) : J \to \mathbb{X}_\alpha$ is strongly measurable.

(H5) There exist a function $m \in L(J,\mathbb{R}^+)$ and a non decreasing function $W_f \in C([0,\infty),\mathbb{R}^+)$ such that

$$_{s_i}D_t^{-\beta} m \in C(J_i,\mathbb{R}^+), \quad \lim_{t\to s_i^+} {}_{s_i}D^{-\beta}m(t) = 0, \quad i = 0,1,2,\dots N,$$

and

$$\|f(t,y_t,\int_0^t G(t,s,y_s)ds\|_\alpha \leq m(t)W_f(\|y_t\|_{\mathscr{P}_0}), \quad y \in \mathscr{P} \text{ and almost all } t \in J.$$

Theorem 5.4.2. *Assume that the hypotheses (H3)-(H5) hold. If* $\|G_i(t,0)\|_\alpha, i = 1, 2, \cdots, N$ *are bounded,* $L_1 = \max_{1\le i\le N} \tilde{C}_1 M L_{G_i} < 1$ *and the following condition holds:*

$$K_1 \int_{s_i}^{t_{i+1}} (t-\tau)^{\beta-1} m(\tau) d\tau < \int_{K_0}^{\infty} \frac{d\tau}{W_f(\tau)}, i = 0, 1, \dots N,$$

where

$$K_0 = \frac{\tilde{C}_1 M \|G_i(t,0)\|_\alpha + \tilde{C}_1 \|\phi(0)\|_\alpha + \tilde{C}_2 \|\phi\|_{\mathscr{P}_0}}{1 - \tilde{C}_1 M L_{G_i}}, \; K_1 = \frac{\tilde{C}_1 M}{\Gamma(\beta)(1 - \tilde{C}_1 M L_{G_i})}, t \in [s_i, t_{i+1}];$$

$$K_0 = M\tilde{C}_1 \|\phi(0)\|_\alpha + \tilde{C}_2 \|\phi\|_{\mathscr{P}_0}, K_1 = \frac{M}{\Gamma(\beta)} \tilde{C}_1, \; t \in [0, t_1].$$

Then the problem possesses at least one mild solution on $(-\infty, T]$.

Proof. To apply Burton-Kirk's fixed point theorem, let us split our operator $\tilde{\mathscr{F}} : \acute{\mathscr{P}}_T \to \acute{\mathscr{P}}_T$, introduced in the previous theorem, in two parts, given by

$$\tilde{\mathscr{F}} = \sum_{i=0}^{N} \tilde{\mathscr{F}}_i^1 + \sum_{i=0}^{N} \tilde{\mathscr{F}}_i^2,$$

where $\tilde{\mathscr{F}}_i^j : \tilde{\mathscr{P}}_T \longrightarrow \tilde{\mathscr{P}}_T$, $i = 0, \dots, N; j = 1, 2$ are given by

$$\tilde{\mathscr{F}}_i^1 y(t) = \begin{cases} 0, & t \in (-\infty, t_1], \\ G_i(t, z_t + \Phi_t), & t \in (t_i, s_i], i \ge 1, \\ \mathscr{T}(t - s_i) G_i(s_i, z_{s_i} + \Phi_{s_i}), & t \in J_i, i \ge 1, \end{cases}$$

$$\tilde{\mathscr{F}}_i^2 y(t) = \begin{cases} \int_{s_i}^t \mathscr{S}(t-\tau) f(\tau, z_\tau + \Phi_\tau, \int_0^\tau G(\tau, s, z_s + \Phi_s) ds) d\tau, t \in J_i, \\ 0, t \notin J_i. \end{cases}$$

The proof is split into several steps.

Step 1: $\tilde{\mathscr{F}}^2$ maps bounded set into bounded set in $\tilde{\mathscr{P}}_T$.

Note that for $z \in B_\eta$,

$$\|z_t + \Phi_t\|_{\mathscr{P}_0} \le \widetilde{C_1}\eta + \widetilde{C_1} M \|\phi(0)\|_\alpha + \widetilde{C_2} \|\phi\|_{\mathscr{P}_0} =: r^*.$$

Let $u \in B_\eta$. For $i \ge 1$ and $t \in J_i, i = 0, 1, \dots N$, we get

$$\|\tilde{\mathscr{T}} u(t)\|_\alpha = \int_{s_i}^t \mathscr{S}(t-\tau) f(\tau, z_\tau + \Phi_\tau, \int_0^\tau G(\tau, s, z_s + \Phi_s) ds) d\tau \|_\alpha$$

$$\leq \frac{M}{\Gamma(\beta)} \int_{s_i}^{t} (t-\tau)^{\beta-1} m(\tau) W_f(\|z_\tau + \Phi_\tau\|_{\mathscr{P}_0}) ds$$
$$\leq \frac{M}{\Gamma(\beta)} W_f(r^*) \sup_{t\in J_i} \int_{s_i}^{t} (t-\tau)^{\beta-1} m(\tau) d\tau$$
$$\leq \frac{M}{\Gamma(\beta)} W_f(r^*) \sup_{t\in J_i} \int_{s_i}^{t} (t-\tau)^{\beta-1} m(\tau) d\tau$$
$$=: l,$$

where
$$l = \max\{\frac{M}{\Gamma(\beta)} W_f(r^*) \sup_{t\in J_i} \int_{s_i}^{t} (t-\tau)^{\beta-1} m(\tau) d\tau, 0 \leq i \leq N\}.$$

Then for each $z \in B_\eta$, we have $\|\tilde{\mathscr{F}}^2(u)\|_{\hat{\mathscr{P}}} \leq l$.

For convenience, from next step onwards, we take $G(z+\Phi)(\tau) = \int_0^\tau G(\tau, s, z_s+\Phi_s) ds$.

Step 2: $[\mathscr{F}_i^2 y : y \in B_\eta]_i, i = 0, 1, \ldots, N$, is an equicontinuous family of functions on $C([t_i, t_{i+1}], \mathbb{X}_\alpha)$.

Let $l_1, l_2 \in (s_i, t_{i+1}]$, $s_i<l_1<l_2$ and $y \in B_\eta$, we have

$$\|(\widetilde{\mathscr{F}_i^2} y)(l_2) - (\widetilde{\mathscr{F}_i^2} y)(l_1)\|_\alpha$$
$$= \|\int_{s_i}^{l_2} (l_2-\tau)^{\beta-1} \mathscr{S}(l_2-\tau) f(\tau, z_\tau+\Phi_\tau, G(z+\Phi)(\tau)) d\tau$$
$$- \int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} \mathscr{S}(l_2-\tau) f(\tau, z_\tau+\Phi_\tau, G(z+\Phi)(\tau)) d\tau\|$$
$$= \|\int_{l_1}^{l_2} (l_2-\tau)^{\beta-1} \mathscr{S}(l_2-\tau) f(\tau, z_\tau+\Phi_\tau, G(z+\Phi)(\tau)) ds\|$$
$$+ \|\int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} \mathscr{S}(l_2-\tau) f(\tau, z_\tau+\Phi_\tau, G(z+\Phi)(\tau)) d\tau$$
$$- \int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} \mathscr{S}(l_2-\tau) f(\tau, z_\tau+\Phi_\tau, G(z+\Phi)(\tau)) d\tau\|$$
$$+ \|\int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} [(\mathscr{S}(l_2-\tau) - \mathscr{S}(l_2-\tau))] f(\tau, z_\tau+\Phi_\tau, G(z+\Phi)(\tau)) d\tau\|$$
$$\leq \frac{M\beta}{\Gamma(\beta+1)} \int_{l_1}^{l_2} (l_2-\tau)^{\beta-1} m(\tau) W_f(\|z_t+\Phi_t\|_{\mathscr{P}_0}) d\tau$$
$$+ \frac{M\beta}{\Gamma(\beta+1)} \int_{s_i}^{l_1} [(l_2-\tau)^{\beta-1} - (l_2-\tau)^{\beta-1}] m(\tau) W_f(\|z_t+\Phi_t\|_{\mathscr{P}_0}) d\tau$$
$$+ \int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} \|\mathscr{S}(l_2-\tau) - \mathscr{S}(l_2-\tau)\| m(\tau) W_f(\|z_t+\Phi_t\|_{\mathscr{P}_0}) d\tau$$

$$
\begin{aligned}
&\leq \frac{M\beta}{\Gamma(\beta+1)} \int_{l_1}^{l_2} (l_2-\tau)^{\beta-1} m(\tau) W_f(r^*) d\tau \\
&+ \frac{M\beta}{\Gamma(\beta+1)} \int_{s_i}^{l_1} [(l_2-\tau)^{\beta-1} - (l_2-\tau)^{\beta-1}] m(\tau) W_f(r^*) d\tau \\
&+ \int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} \|\mathscr{S}(l_2-\tau) - \mathscr{S}(l_2-\tau)\| m(\tau) W_f(r^*) d\tau \\
&\leq W_f(r^*) \frac{M\beta}{\Gamma(\beta+1)} |\int_{s_i}^{l_2} (l_2-\tau)^{\beta-1} m(\tau) d\tau - \int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} m(\tau) d\tau \| \\
&+ 2W_f(r^*) \frac{M\beta}{\Gamma(\beta+1)} \int_{s_i}^{l_1} [(l_2-\tau)^{\beta-1} - (l_2-\tau)^{\beta-1}] m(\tau) d\tau \\
&+ W_f(r^*) \int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} \|\mathscr{S}(l_2-\tau) - \mathscr{S}(l_2-\tau)\| m(\tau) d\tau \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

Since ${}_{s_i}D_t^{-\beta} m \in C(J_i, \mathbb{R}^+)$, therefore

$$I_1 \to 0 \quad \text{as} \quad l_2 \to l_1.$$

For $l_1 < l_2$,

$$I_2 \leq W_f(r^*) \frac{2M\beta}{\Gamma(\beta+1)} \int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} m(\tau) d\tau.$$

Then by Lemma 1.1.3, we have $I_2 \to 0 \quad \text{as} \quad l_2 \to l_1$.

For $\epsilon > 0$ small enough,

$$
\begin{aligned}
I_3 &= W_f(r^*) \int_{s_i}^{l_1-\epsilon} (l_2-\tau)^{\beta-1} \|\mathscr{S}(l_2-\tau) - \mathscr{S}(l_2-\tau)\| m(\tau) d\tau \\
&+ W_f(r^*) \int_{l_1-\epsilon}^{l_1} (l_2-\tau)^{\beta-1} \|\mathscr{S}(l_2-\tau) - \mathscr{S}(l_2-\tau)\| m(\tau) d\tau \\
&\leq W_f(r^*) \int_{s_i}^{l_1-\epsilon} (l_2-\tau)^{\beta-1} m(\tau) d\tau \sup_{s \in [s_i, l_1-\epsilon]} \|\mathscr{S}(l_2-\tau) - \mathscr{S}(l_2-\tau)\| \\
&+ W_f(r^*) \int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} \|\mathscr{S}(l_2-\tau) - \mathscr{S}(l_1-\tau)\| m(\tau) d\tau \\
&- W_f(r^*) \int_{s_i}^{l_1-\epsilon} (l_2-\tau)^{\beta-1} \|\mathscr{S}(l_2-\tau) - \mathscr{S}(l_1-\tau)\| m(\tau) d\tau \\
&\leq W_f(r^*) \int_{s_i}^{l_1-\epsilon} (l_2-\tau)^{\beta-1} m(\tau) d\tau \sup_{s \in [s_i, l_1-\epsilon]} \|\mathscr{S}(l_2-\tau) - \mathscr{S}(l_2-\tau)\| \\
&+ W_f(r^*) \frac{2M\beta}{\Gamma(\beta+1)} [\int_{s_i}^{l_1} (l_2-\tau)^{\beta-1} m(\tau) d\tau - \int_{s_i}^{l_1-\epsilon} (l_2-\tau)^{\beta-1} m(\tau) d\tau]
\end{aligned}
$$

$$\leq W_f(r^*)\int_{s_i}^{l_1-\epsilon}(l_2-\tau)^{\beta-1}m(\tau)d\tau \sup_{s\in[s_i,l_1-\epsilon]}\|\mathscr{S}(l_2-\tau)-\mathscr{S}(l_2-\tau)\|+$$
$$+W_f(r^*)\frac{2M\beta}{\Gamma(\beta+1)}[\int_{s_i}^{l_1}(l_1-\tau)^{\beta-1}m(\tau)d\tau-\int_{s_i}^{l_1-\epsilon}(l_1-\epsilon-\tau)^{\beta-1}m(\tau)d\tau]$$
$$+W_f(r^*)\frac{2M\beta}{\Gamma(\beta+1)}\int_{s_i}^{l_1-\epsilon}[(l_1-\epsilon-\tau)^{\beta-1}-(l_2-\tau)^{\beta-1}]m(\tau)d\tau$$
$$=: I_{31}+I_{32}+I_{33}.$$

Since $\{Q(t)\}_{t>0}$ is a compact operator, it is an equicontinuous family. Therefore $I_{31}\to 0$ as $l_2\to l_1$. $I_{32}\to 0$, $I_{33}\to 0$ as $\epsilon\to 0$ by virtue of I_2 and I_3. Hence

$$||(\tilde{\mathscr{F}}_i^2 x)(l_2)-(\tilde{\mathscr{F}}_i^2 x)(l_1)||\to 0$$

independent of $y\in B_\eta$ as $l_2\to l_1$. By a similar argument, equicontinuity can be verified in $\tau_1<0<\tau_2\leq T$, whereas it is trivial for $\tau_1<\tau_2\leq 0$.

Thus, $[\tilde{\mathscr{F}}^2 B_\eta]_i$ is equicontinuous and hence $\mathscr{F}_2$ is completely continuous.

Step 3: $\tilde{\mathscr{F}}^2:\acute{\mathscr{P}}\to\acute{\mathscr{P}}$ is continuous.

Let $\{z^n\}_{n=1}^\infty$ be a sequence in $\acute{\mathscr{P}}$ with $z^n\to z\in\acute{\mathscr{P}}$.

By condition $(H3)$, we have

$$\lim_{n\to\infty} f(t,z_t^n+\Phi_t,G(z^n+\Phi)t)\to f(t,z_t+\Phi_t,G(z+\Phi)t).$$

For any $t\in J_i$

$$(t-\tau)^{\beta-1}\|f(\tau,z_\tau^n+\Phi_\tau,G(z^n+\Phi)(\tau))-f(s,z_\tau+\Phi_\tau,G(z+\Phi(\tau))\|$$
$$\leq (t-\tau)^{\beta-1}m(\tau)W_f(r^*).$$

By $(H4)$, the function $\tau\to(t-\tau)^{\beta-1}m(\tau)$ is integrable for $\tau\in[s_i,t_{i+1}]$.

Hence by Lemma 1.1.3, we get

$$\int_{s_i}^t (t-\tau)^{\beta-1}\|f(s,z_\tau^n+\Phi_\tau,G(z^n+\Phi)(\tau))-f(\tau,z_\tau+\Phi_\tau,G(z+\Phi)(\tau))\|d\tau\to 0,$$
$$\text{as } n\to\infty.$$

Thus for $t\in J_i$,

$$\|(\tilde{\mathscr{F}}_i^1 z^n)(t)-(\tilde{\mathscr{F}}_i^1 z)(t)\|$$

$$\leq \tilde{M}_T W_f(r^*) \int_{s_i}^{t} (t-\tau)^{\beta-1} \| f(\tau, z_\tau^n, G(z^n+\Phi)(\tau)) - f(\tau, z_\tau, G(z+\Phi)(\tau)) \| d\tau$$

$$\to 0, \text{ as } n \to \infty.$$

Therefore, $\tilde{\mathscr{F}}_i^1 z^n \to \tilde{\mathscr{F}}_i^1 z$ pointwise on J_i as $n \to \infty$. Thus $\tilde{\mathscr{F}}^2$ is continuous in $\acute{\mathscr{P}}$.

Step 4: Let $t \in (s_i, t_{i+1})$ be fixed and $s(< t) \in (s_i, t_{i+1})$ be such that $\epsilon \in (s_i, s)$. To prove $\bigcup_{l\in[s,t]} \tilde{\mathscr{F}}_i^2 z(l), z \in B_\eta$ is relatively compact in $\mathbb{X}_\alpha$.

For any δ>0, define the set

$$V_\epsilon^\delta(t) = \{((\tilde{\mathscr{F}}_i^2)_\epsilon^\delta z)(t) : z \in B_\eta\},$$

where

$$((\tilde{\mathscr{F}}_i^2)_\epsilon^\delta z)(l) = \beta \int_{s_i}^{l-\epsilon} \int_{\delta}^{\infty} \theta (l-\tau)^{\beta-1} \xi_\beta(\theta) Q((l-\tau)^\beta \theta) f(\tau, z_\tau + \Phi_\tau, G(z+\Phi)(\tau)) d\tau d\theta$$
$$= \beta Q(\epsilon^\delta \theta) \int_{s_i}^{l-\epsilon} \int_{\delta}^{\infty} \theta (l-\tau)^{\beta-1} \xi_\beta(\theta) Q((l-\tau)^\beta \theta - \epsilon^\beta \delta)$$
$$\times f(\tau, z_\tau + \Phi_\tau, G(z+\Phi)(\tau)) d\tau d\theta.$$

Since $Q(\epsilon^\beta \delta), (\epsilon^\beta \delta) > 0$ is compact, the set $V_\epsilon^\delta(\tau)$ is relatively compact in $\mathbb{X}_\alpha$.

On the other hand, for any $z \in B_\eta$,

$$|((\tilde{\mathscr{F}}_i^2 z)(l) - ((\tilde{\mathscr{F}}_i^2)_\epsilon^\delta z)(l)\|_\alpha$$
$$\leq \beta \| \int_{s_i}^{l} \int_{0}^{\psi} \theta (l-\tau)^{\beta-1} \xi_\beta(\theta) Q((l-\tau)^\beta(\theta)) f(\tau, z_\tau + \Phi_\tau, G(z+\Phi)(\tau) d\tau d\theta \|_\alpha$$
$$+ \beta \| \int_{l-\epsilon}^{l} \int_{\delta}^{\infty} \theta (l-\tau)^{\beta-1} \xi_\beta(\theta) Q((l-\tau)^\beta(\theta)) f(\tau, z_\tau + \Phi_\tau, (z+\Phi)(\tau) d\tau d\theta \|_\alpha$$
$$\leq W_f(r^*) \beta M \int_{s_i}^{l} (l-\tau)^{\beta-1} m(\tau) d\tau \int_{0}^{\psi} \theta \xi_\beta(\theta) d\theta$$
$$+ W_f(r^*) \frac{M}{\Gamma(\beta)} \int_{l-\epsilon}^{l} (l-\tau)^{\beta-1} m(\tau) d\tau$$
$$\to 0 \quad as \quad \epsilon \to 0, \delta \to 0.$$

Thus there exist relatively compact sets arbitrary close to the set $\bigcup_{\tau\in[s,t]} \widetilde{\mathscr{F}}_i^2 B_\eta(l)$.

Therefore, the set $\bigcup_{\tau\in[s,t]} \widetilde{\mathscr{F}}_i^2 B_\eta(l)$ is relatively compact in $\mathbb{X}_\alpha$.

Step 5: $\tilde{\mathscr{F}}^1$ is a contraction in B_η.

Let $\mathcal{U}, \mathcal{V} \in \tilde{\mathscr{P}}_T$ and $t \in J_i, 1 \le i \le N$. We have

$$\begin{aligned}\|(\tilde{\mathscr{F}}_i^1\mathcal{U})(t) - (\tilde{\mathscr{F}}_i^1\mathcal{V})(t)\|_\alpha &\le L_{G_i}\|u_t - \mathcal{V}_t\|_{\mathscr{P}_0}\\ &\le \tilde{C}_1 L_{G_i}\|u - \mathcal{V}\|_{\tilde{\mathscr{P}}}.\end{aligned}$$

Also for $\mathcal{U}, \mathcal{V} \in \tilde{\mathscr{P}}$ and $t \in J_i, i = 1, 2, ..., N$, we have

$$\begin{aligned}\|(\tilde{\mathscr{F}}_i^1\mathcal{U})(t) - (\tilde{\mathscr{F}}_i^1\mathcal{V})(t)\|_\alpha &= \|\mathscr{T}(t - s_i)G_i(s_i, \mathcal{U}(s_i) + \Phi(s_i)) - \mathscr{T}(t - s_i)G_i(s_i, \mathcal{V}(s_i) + \Phi_{s_i})\|_\alpha\\ &\le M L_{G_i}\|\mathcal{U}_t - \mathcal{V}_t\|_{\mathscr{P}_0}\\ &\le M\tilde{C}_1 L_{G_i}\|\mathcal{U} - \mathcal{V}\|_{\tilde{\mathscr{P}}}.\end{aligned}$$

Taking the supremum over t, we get

$$\|\tilde{\mathscr{F}}_i^1(\mathcal{U}) - \tilde{\mathscr{F}}_i^1(\mathcal{V})\|_{\tilde{\mathscr{P}}} \le L\|\mathcal{U} - \mathcal{V}\|_{\tilde{\mathscr{P}}}.$$

Hence $\tilde{\mathscr{F}}^1$ is a contraction mapping on $\tilde{\mathscr{P}}$.

Step 6: A priori bounds.

Here we show that the set

$$\mathscr{E} = \{z \in \tilde{\mathscr{P}}_T : z = \delta\tilde{\mathscr{F}}^1 z + \delta\tilde{\mathscr{F}}^2(\frac{z}{\delta}), \text{ for some } 0 < \delta < 1\}$$

is bounded.

For each $t \in J_1$,

$$z(t) = \int_0^t \mathscr{T}(t-\tau)(t-\tau)^{\beta-1} f(\tau, z_\tau + \Phi_\tau, G(z + \Phi)(\tau))d\tau.$$

Hence

$$\|z(t)\|_\alpha \le \frac{M}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} m(\tau) W_f(\|z_t + \Phi_t\|_{\mathscr{P}})d\tau. \tag{5.7}$$

Now

$$\begin{aligned}&\|z_t + \Phi_t\|_{\mathscr{P}_0}\\ &\le \|z_t\|_{\mathscr{P}_0} + \|\Phi_t\|_{\mathscr{P}_0}\\ &\le \tilde{C}_1 \sup_{s\in[0,t]}\|z(s)\|_\alpha + M\tilde{C}_1\|\phi(0)\|_\alpha + \tilde{C}_2\|\phi\|_{\mathscr{P}_0}\end{aligned}$$

$$=: \mu(t).$$

Hence (5.7) becomes

$$\|z(t)\|_\alpha \leq \frac{M}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} m(\tau) W_f(\mu(\tau)) d\tau. \tag{5.8}$$

Using (5.8) in the definition of μ, we have

$$\mu(t) \leq \frac{M}{\Gamma(\beta)} \tilde{C}_1 \int_0^t (t-\tau)^{\beta-1} m(\tau) W_f(\mu(\tau)) d\tau + M\tilde{C}_1 \|\phi(0)\|_\alpha + \tilde{C}_2 \|\phi\|_{\mathscr{P}_0}. \tag{5.9}$$

Thus

$$\mu(t) \leq K_0 + K_1 \int_0^t (t-\tau)^{\beta-1} m(\tau) W_f(\mu(\tau)) d\tau, \tag{5.10}$$

where,

$$K_0 = M\tilde{C}_1 \|\phi(0)\|_\alpha + \tilde{C}_2 \|\phi\|_{\mathscr{P}_0}, K_1 = \frac{M}{\Gamma(\beta)} \tilde{C}_1.$$

If $\mathcal{W}(t) = K_0 + K_1 \int_0^t (t-\tau)^{\beta-1} m(\tau) W_f(\mu(\tau)) d\tau$, then

$$\mu(t) \leq \mathcal{W}(t), \mathcal{W}(0) = K_0 \text{ and } \mathcal{W}'(t) \leq K_1 (s-t)^{\beta-1} m(t) W_f(\mathcal{W}(t)).$$

Thus for $t \in [0, t_1]$, we have

$$\int_{\mathcal{W}(0)}^{v(t)} \frac{d\tau}{W_f(\tau)} \leq K_1 \int_0^t (t-\tau)^{\beta-1} m(\tau) d\tau < \int_{K_0}^{\infty} \frac{d\tau}{W_f(\tau)}.$$

By Lemma 5.2.3, we have

$$\mathcal{W}(t) \leq H^{-1}(K_1) \int_0^t (t-\tau)^{\beta-1} m(\tau) d\tau), t \in [0, t_1],$$

where

$$H(y) = \int_{K_0}^{y} \frac{d\tau}{W_f(\tau)}.$$

Hence

$$\|z_t + \Phi_t\|_{\mathscr{P}_0} < M_{t_0}.$$

From (5.8), we get

$$\|z(t)\|_\alpha \leq \frac{M}{\Gamma(\beta)} \int_0^{t_1} (t-\tau)^{\beta-1} m(\tau) W_f(M_{t_0}) ds.$$

Thus there exists $L_{t_0} > 0$ such that

$$\|z\|_{\acute{\mathscr{P}}} \leq L_{t_0}.$$

For $t \in (t_i, s_i], i = 1, 2, \ldots N$, we have

$$z(t) = G_i(t, z_t + \phi_t).$$

Hence for each $t \in J_i$,

$$\|z(t)\|_\alpha \leq L_{G_i}\|z_t + \Phi_t\|_{\mathscr{P}_0} + \|G_i(t, 0)\|_\alpha. \tag{5.11}$$

If $\|z_t + \Phi_t\|_{\mathscr{P}_0} \leq \mu(t)$, then (5.11) becomes

$$\|z(t)\|_\alpha \leq L_{G_i}\mu(t) + \|G_i(t, 0)\|_\alpha. \tag{5.12}$$

From the definition of $\mu(t)$, we have

$$\mu(t) \leq \tilde{C}_1(L_{G_i}\mu(t) + \|G_i(t, 0)\|_\alpha + M\tilde{C}_1\|\phi(0)\|_\alpha + \tilde{C}_2\|\phi\|_{\mathscr{P}_0}. \tag{5.13}$$

Hence (5.13) gives

$$\mu(t) \leq \frac{\tilde{C}_1(\|G_i(t, 0)\|_\alpha + M\|\phi(0)\|_\alpha) + \tilde{C}_2\|\phi\|_{\mathscr{P}_0}}{1 - \tilde{C}_1 L_{G_i}} =: M_{t_i}.$$

This shows that there is a constant $M_{t_i} > 0$ such that

$$\mu(t) \leq M_{t_i}, t \in (t_i, s_i].$$

Therefore,

$$\|z(t)\|_\alpha \leq L_{G_i}M_{t_i} + \|G_i(t, 0)\|_\alpha.$$

Thus,

$$\|z\|_{\acute{\mathscr{P}}} \leq L_{t_i}.$$

Finally for each $t \in J_i$,

$$z(t) = \mathscr{T}(t - s_i)G_i(s_i, z_{s_i} + \Phi_{s_i}) + \int_{s_i}^t (t - \tau)^{\beta - 1}\mathscr{S}f(\tau, z_\tau + \Phi_\tau + G(z + \Phi)(\tau))d\tau).$$

Now,

$$\|z(t)\|_\alpha \leq ML_{G_i}\|z_{s_i}+\Phi_{s_i}\|_{\mathscr{P}_0}+M\|G_i(t,0)\|_\alpha+\frac{M}{\Gamma(\beta)}\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)W_f(\|z_t+\Phi_t\|_{\mathscr{P}_0})ds.$$

Using $\mu(t)$, we get

$$\|z(t)\|_\alpha \leq ML_{G_i}\mu(t) + M\|G_i(t,0)\|_\alpha + \frac{M}{\Gamma(\beta)}\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)W_f(\mu(\tau))d\tau. \quad (5.14)$$

From the definition of μ, we have

$$\mu(t) \leq \tilde{C}_1(ML_{G_i}\mu(t) + M\|G_i(t,0)\|_\alpha + \frac{M}{\Gamma(\beta)}\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)W_f(\mu(\tau))d\tau)$$
$$+ \tilde{C}_1\|\phi(0)\|_\alpha + \tilde{C}_2\|\phi\|_{\mathscr{P}_0},$$

where,

$$\tilde{K}_0 = \frac{\tilde{C}_1M\|G_i(t,0)\|_\alpha + \tilde{C}_1\|\phi(0)\|_\alpha + \tilde{C}_2\|\phi\|_{\mathscr{P}_0}}{1-\tilde{C}_1ML_{G_i}} \text{ and } \tilde{K}_1 = \frac{\tilde{C}_1M}{\Gamma(\beta)(1-\tilde{C}_1ML_{G_i})}.$$

Thus, we get

$$\mu(t) \leq \tilde{K}_0 + \tilde{K}_1\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)W_f(\mu(\tau))d\tau.$$

If we denote the RHS of the above inequality by $\mathcal{W}(t)$, we have

$\mathcal{W}(0) = \tilde{K}_0, \mu(t) \leq \mathcal{W}(t), t \in [s_i, t_{i+1}], i = 1, 2, \ldots N$ and $\mathcal{W}'(t) \leq \tilde{K}_1(s-t)^{\beta-1}m(t))W_f(\mathcal{W}(t))$.

Thus for $t \in [s_i, t_{i+1}]$,

$$\int_{v(i)}^{\mathcal{W}(t)}\frac{d\tau}{W_f(\tau)} \leq \tilde{K}_1\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)d\tau < \int_{\tilde{K}_0}^{\infty}\frac{d\tau}{W_f(\tau)}.$$

Gy lemma 5.2.3, we have

$$\mathcal{W}(t) \leq H^{-1}(\tilde{K}_1\int_{s_i}^{t_{i+1}}(t-\tau)^{\beta-1}m(\tau)d\tau), s \in [s_i, t_{i+1}],$$

where

$$H(y) = \int_{\tilde{K}_0}^{y}\frac{d\tau}{W_f(\tau)}.$$

Therefore,

$$\|z_t + \Phi_t\|_{\mathscr{P}_0} < M_{t_{i+1}}.$$

From (5.13), we get

$$\|z(t)\|_\alpha \le ML_{G_i}M_{t_{i+1}} + M\|G_i(t,0)\|_\alpha + \frac{M}{\Gamma(\beta)}\int_{s_i}^{t}(t-\tau)^{\beta-1}m(\tau)W_f(M_{t_{i+1}})d\tau =: L_{t_{i+1}}.$$

Thus there exists $L_{t_{i+1}} > 0$ such that

$$\|z\|_{\tilde{\mathscr{B}}} \le L_{t_{i+1}}.$$

This implies that the set $\mathscr{E}$ is bounded. Hence by Burton-Kirk's fixed point theorem, the operator $\tilde{\mathscr{F}}$ has a fixed point since $y(t) = z(t) + \Phi(t), t \in (-\infty, T]$. Then y is a fixed point of the operator $\mathscr{T}$ which is a mild solution of the problem. $\square$

5.5 Example:

Consider the space $\mathbb{X} = L^2([0,\pi],\mathbb{R})$ and the following fractional partial differential equation with infinite delay:

$$\begin{aligned} {}^C D_t^\beta y(t,w) &= \frac{\partial^2 y(t,w)}{\partial z^2} + \sigma(t, y_t(.,z), \int_0^t \sigma_1(t,\tau,y_\tau(.,z))d\tau), \\ & \quad t \in [s_i, t_{i+1}], z \in [0,\pi], & (5.15) \\ y(t,w) &= G_i(t, y_t(.,w)), i = 1,2,\dots N, & (5.16) \\ y(t,0) &= y(t,\pi) = 0, t \in [0,T], & (5.17) \\ y(t,w) &= \phi(t,w), -\infty \le t \le 0, 0 \le w \le \pi, & (5.18) \end{aligned}$$

where $s_i \in (t_i, t_{i+1}], i = 1,2,\dots,N$, in the partition $0 = t_0 < t_1 < ... < t_{N+1} = T$ of the interval $[0,T]$ with $s_0 = 0$, y_t indicates the portion of the solution $y(.,.) : (-\infty, T] \times [0,\pi] \to \mathbb{X}$, that is, for any $t \ge 0$, $y_t(.,.) : (-\infty, 0] \times [0,\pi] \to \mathbb{X}$ is given by

$$y_t(\theta, w) = y(t+\theta, w), \text{ for } \theta \in (-\infty, 0].$$

Let $\mathbb{X} = L^2[0,\pi]$ and define $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ by $Ay = y''$, on
$D(A) = \left\{ y \in \mathbb{X} : \frac{\partial y}{\partial w}, \frac{\partial^2 y}{\partial w^2} \in \mathbb{X} \text{ and } y(0) = y(\pi) = 0 \right\}$.
Then A generates a compact analytic semigroup $\{Q(t)\}_{t\ge 0}$ on $\mathbb{X}$ and there exists a constant $M \ge 1$ such that $\|Q(t)\| \le M$.
Consider the functions

$$y(t)w = y(t,w), t \in J, z \in [0,\pi], \quad (5.19)$$

$$G_i(t,\phi)w = G_i(t,\phi(\theta,w)), \theta \in (-\infty,0], w \in [0,\pi], \quad (5.20)$$

$$f(t,\phi,\int_0^t G(t,\tau,\phi)d\tau)w = \sigma(t,\phi(\theta,w)), \int_0^t \sigma_1(t,\tau,\phi(\theta,w)d\tau),$$

$$\theta \in (-\infty,0], w \in [0,\pi], \quad (5.21)$$

$$\phi(\theta)(w) = \phi(\theta,w), \theta \in (-\infty,0], w \in [0,\pi], \quad (5.22)$$

with the following assumptions:

(i) For each $i = 0,1,\ldots N$, the function $f : [s_i, t_{i+1}] \times \mathscr{P}_0 \times \mathbb{X} \to \mathbb{X}$ defined by (5.21) is continuous and we impose suitable condition on F to satisfy the hypotheses $(H1)$-$(H2)$.

(ii) For each $i = 1,\ldots N$, the function $G_i : (t_i, s_i] \times \mathscr{P}_0 \to \mathbb{X}$ defined by (5.20) is continuous and we impose suitable condition on G_i to satisfy the hypothesis $(H3)$.

With the above setting, the system of equations (5.15)-(5.18) reduces to the system of equations (5.1)-(5.3) satisfying the hypotheses of Theorem 5.4.1 and hence ensuring a mild solution on $(-\infty,T]$.

5.6 Conclusion

Using Burton-Kirk's fixed point theorem and analytical semigroup theory, we prove the existence of mild solution of a class of non-instantaneous impulsive fractional functional equation with infinite delay. An example is provided to illustrate our theory.

Chapter 6

Nonlocal not instantaneous impulsive FDE

In this chapter, we establish a set of sufficient conditions for the existence of mild solution of a class of FDEs with not instantaneous impulses. The results are obtained by using Banach fixed point theorem and Krasnoselskii's fixed point theorem. An example is presented for validation of result.

6.1 Introduction

Differential equations with nonlocal conditions are used to model some physical problems in a more realistic way than the corresponding initial value problem [49]. Some recent works on FDE with nonlocal conditions can be found in [78, 88, 111, 126]. N'Guérékata [88] studied the existence. uniqueness of solution of the following abstract FDE:

$$^{C}D_t^{\beta}y(t) = f(t, y(t)), t \in J = [0, T], \tag{6.1}$$

$$y(0) + g(y) = x_0. \tag{6.2}$$

Wang and Li [113] investigated the periodic BVP of fractional order linear differential equation with Caputo fractional derivative of the form

$$^{C}D_t^{\beta}y(t) = f(t, y(t)), t \in (s_i, t_{i+1}], i = 0, 1, ..., N,$$

$$y(t) = G_i(t, y(t)), t \in (t_i, s_i], i = 1, 2, ..., N,$$
$$y(0) = y(t).$$

Mild solution for not instantaneous impulsive FDE with non local conditions is mostly an untreated subject in literature. Motivated by the work of Wang and Li [113], we discuss the existence and uniqueness of the following problem:

$$^{C}D_t^{\beta} y(t) = f(t, y(t), k(y(t))), t \in (s_i, t_{i+1}], i = 0, 1, ..., N, \tag{6.3}$$
$$y(t) = G_i(t, y(t)), t \in (t_i, s_i], i = 1, 2, ..., N, \tag{6.4}$$
$$y(0) + h(y) = x_0, \tag{6.5}$$

where $J = [0, T], G_i \in C((t_i, s_i] \times \mathbb{R}, \mathbb{R})$, for $i = 1, \ldots, N, f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $k : \mathbb{R} \to \mathbb{R}, h : PC(J, \mathbb{R}) \to \mathbb{R}$ are given functions.

6.2 Preliminaries

Lemma 6.2.1. *Let $f : J \to \mathbb{R}$ be a continuous function. A function $y \in C(J, \mathbb{R})$ is a solution of the fractional integral equation*

$$y(t) = x_c - \frac{1}{\Gamma(\beta)} \int_0^c (c - s)^{(\beta - 1)} f(s) ds + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{(\beta - 1)} f(s) ds$$

if and only if y is a solution of the following fractional Cauchy problem:

$$^{C}D_t^{\beta} y(t) = f(t), t \in J,$$
$$x(c) = x_c, 0 < c < T.$$

Definition 6.2.1. *In view of Lemma* 6.2.1*, we define mild solution as follows:*
A function $y \in PC(J, \mathbb{R})$ is a mild solution of the problem (6.3)-(6.5) *if*
$y(0) + h(y) = x_0$, $y(t) = G_i(t, y(t)), t \in (t_i, s_i], i = 1, 2, ..., N$ *and*

$$y(t) = x_0 - h(y) + \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta - 1} f(\tau, y(\tau), k(y(\tau)), t \in [0, t_1] \text{ and}$$
$$y(t) = G_i(s_i, y(s_i)) - \frac{1}{\Gamma(\beta)} \int_0^{s_i} (s_i - \tau)^{\beta - 1} f(\tau, y(\tau), k(y(\tau)) d\tau$$
$$+ \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta - 1} f(\tau, y(\tau), k(y(\tau)) d\tau, \ t \in [s_i, t_{i+1}], i = 1, 2, ..., N.$$

6.3 Main Results

To establish our results on the existence of solutions, we consider following hypotheses:
(H1) There are constants $L_{G_i} > 0$ such that
$|G_i(t, u_1) - G_i(t, v_1)| \leq L_{G_i}|u_1 - v_1|$ for all $u_1, v_1 \in \mathbb{R}, t \in (t_i, s_i]$ for each $i = 1, 2, ..., N$.
(H2) $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Let W be an open subset of $J \times \mathbb{R} \times \mathbb{R}$ and for each $(t, u_1, v_1) \in W$, there exists a neighborhood $V \subset W$ such that $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following condition:

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq L_f[|u_1 - u_2|| + |v_1 - v_2|],$$

for all $(t, u_1, v_1), (t, u_2, v_2) \in V$ and $L_f > 0$ is a constant.
(H3) $h, k : PC(J, \mathbb{R}) \to \mathbb{R}$ are such that

$$|k(y) - k(z)| \leq L_k||y - z||_{PC}, |h(y) - h(z)| \leq L_h||y - z||_{PC},$$

where $L_k > 0$ and $L_h > 0$ are constants.

Theorem 6.3.1. *Assume that the hypotheses* $(H1)$-$(H3)$ *hold and*

$$L = \max\left\{\max_{1\leq i\leq N}\left\{L_{G_i} + ({s_i}^{\beta} + t_{i+1}^{\beta})\frac{L_f(1 + L_k)}{\Gamma(\beta + 1)}\right\}, L_h + \frac{{t_1}^{\beta}}{\Gamma(\beta + 1)}L_f(1 + L_k)\right\} < 1.$$

Then there exists a unique mild solution $y \in PC(J, \mathbb{R})$ *for the problem* (6.3)-(6.5).

Proof. Define the operator $\mathcal{T} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$\begin{aligned}
\mathcal{T}y(t) &= x_0 - h(y) + \frac{1}{\Gamma(\beta)}\int_0^t (t - \tau)^{\beta-1} f(\tau, y(\tau), ky(\tau))d\tau, t \in [0, t_1],\\
\mathcal{T}y(t) &= G_i(t, y(t)), t \in (t_i, s_i], i = 1, 2, ..., N,\\
\mathcal{T}y(t) &= G_i(s_i, y(s_i)) - \frac{1}{\Gamma(\beta)}\int_0^{s_i} (s_i - t)^{\beta-1} f(\tau, y(\tau), ky(\tau))d\tau\\
&\quad + \frac{1}{\Gamma(\beta)}\int_0^t (t - \tau)^{\beta-1} f(\tau, y(\tau), ky(\tau))d\tau, t \in [s_i, t_{i+1}], i = 1, 2, ..., N.
\end{aligned}$$

From the hypotheses, this definition is well-defined.
We prove that $\mathcal{T}$ is a contraction map on $PC(J, \mathbb{R})$. Let $y, z \in PC(J, \mathbb{R})$.
For $t \in [0, t_1]$,

$$|\mathcal{T}y(t) - \mathcal{T}z(t)|$$

$$= |h(y) - h(z)| + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \{|f(\tau, y(\tau), k(y(\tau))) - f(\tau, z(\tau), k(z(\tau)))|\} d\tau$$
$$\leq [L_h + \frac{{t_1}^\beta}{\Gamma(\beta+1)} L_f(1+L_k)] ||y-z||_{PC},$$

and therefore

$$||\mathcal{T}y - \mathcal{T}z||_{C([0,t_1],R)} \leq \left[L_h + \frac{t_1^\beta}{\Gamma(\beta+1)} L_f(1+L_k) \right] ||y-z||_{PC}.$$

For $t \in [s_i, t_{i+1}], i = 1, 2, ..., N,$

$$|\mathcal{T}y(t) - \mathcal{T}z(t)|$$
$$= |G_i(s_i, y(s_i)) - G_i(s_i, z(s_i))| + \frac{1}{\Gamma(\beta)} \int_0^{s_i} (s_i - \tau)^{\beta-1} |f(\tau, y(\tau), k(y(\tau)))$$
$$- f(\tau, z(\tau), k(z(\tau)))| d\tau + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \{|f(\tau, y(\tau), k(y(\tau))) - f(\tau, z(\tau), k(z(\tau)))|\} d\tau$$
$$\leq [L_{G_i} + ({s_i}^\beta + t_{i+1}^\beta) \frac{1}{\Gamma(\beta+1)} L_f(1+L_k)] ||y-z||_{PC}.$$

Hence,

$$||\mathcal{T}y - \mathcal{T}z||_{C([s_i,t_{i+1}],R)} \leq [L_{G_i} + ({s_i}^\beta + t_{i+1}^\beta) \frac{1}{\Gamma(\beta+1)} L_f(1+L_k)] ||y-z||_{PC}.$$

For $t \in (t_i, s_i], i = 1, 2, ..., N,$

$$\begin{aligned} |\mathcal{T}y(t) - \mathcal{T}z(t)| &= |G_i(t, y(t)) - G_i(t, z(t))| \\ &\leq L_{G_i} ||y-z||_{PC}. \end{aligned}$$

In this case also

$$||\mathcal{T}y - \mathcal{T}z||_{C((t_i,s_i],\mathbb{R})} \leq L_{G_i} ||y-z||_{PC}.$$

Thus we observe that

$$||\mathcal{T}y - \mathcal{T}z||_{PC} \leq L ||y-z||_{PC},$$

which implies that $\mathcal{T}$ is a contraction and hence there exists a mild solution to the problem (6.3)-(6.5). □

To prove the next result, we consider the following condition:

(H4) Assume that the function f Carathéodory Also there exist $m_f \in L^1(J, \mathbb{R}^+)$ and a nondecreasing function $\Omega \in C(J, \mathbb{R}^+)$ such that

$$|f(t, y(t), k(y(t)))| \leq m_f(t)\Omega(||y||), \quad \text{for all} \quad y \in PC(J, \mathbb{R}), \text{ a.e. } t \in J.$$

Theorem 6.3.2. *Assume that the hypotheses* $(H1), (H3)$ *and* $(H4)$ *are satisfied, the functions* $G_i(., 0), k(0)$ *are bounded and* $\lim_{r\to\infty} \sup \frac{\Omega(r)}{r} < \infty$. *Then there exists a mild solution* $y \in PC(J, \mathbb{R})$ *for the problem* (6.3)-(6.5) *provided*

$$L = \max\{L_h, \max_{1\leq i\leq N} L_{G_i}\} < 1.$$

Proof. Let $\eta > 1$ and $0 < \delta < 1$ be such that

$$|x_0| + \max_{1\leq i\leq N} ||G_i(., 0)||_{C((t_i,s_i],\mathbb{R})),\mathbb{R})} < (1-\delta)\eta,$$

$$\max_{1\leq i\leq N}\{L_{G_i} + \frac{s_i^\beta + t_{i+1}^\beta}{\Gamma(\beta+1)}||m_f||_{L^1([s_i,t_{i+1}],\mathbb{R})}\frac{\Omega(s)}{s}\} < \delta, \quad s \geq \eta,$$

$$L_h + \frac{h(0)}{s} + \frac{a^\beta}{\Gamma(\beta+1)}||m_f||_{L^1([0,t_1],\mathbb{R})}\frac{\Omega(s)}{s} < \delta, \quad s \geq \eta.$$

Let us decompose the operator discussed in the previous result as

$$\mathcal{T} = \sum_{i=0}^{N} \mathcal{T}_i^1 + \sum_{i=0}^{N} \mathcal{T}_i^2 = \mathcal{T}^1 + \mathcal{T}^2,$$

where $\mathcal{T}_i^j : PC(J, \mathbb{R}) \to PC(J, \mathbb{R}), i = 1, 2, ..., N, j = 1, 2$ are given by

$$\mathcal{T}_i^1 y(t) = \begin{cases} x_0 - h(y), & t \in [0, t_1], \\ G_i(t, y(t)), & t \in (t_i, s_i], \quad i \geq 1, \\ G_i(s_i, y(s_i)), & t \in (s_i, t_{i+1}] \\ 0, & t \notin (t_i, t_{i+1}], \end{cases}$$

$$\mathcal{T}_i^2 y(t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau, y(\tau), ky(\tau))d\tau - \\ \frac{1}{\Gamma(\beta)} \int_0^{s_i} (s_i - s)^{\beta-1} f(\tau, y(\tau), k(y(\tau)))d\tau, & t \in (s_i, t_{i+1}], \\ 0, & t \notin (s_i, t_{i+1}]. \end{cases}$$

We prove that $\mathcal{T}_i^1$ is a contraction and $\mathcal{T}_i^2$ is completely continuous on

$$B_\eta = \{y \in PC(J, \mathbb{R}) : ||y||_{PC} \leq \eta\}.$$

We complete the proof in several parts.

Step 1: First we show that $\mathcal{T}B_\eta \subset B_\eta$.

Let $y \in B_\eta$. For $i \geq 1$ and $t \in (t_i, t_{i+1}]$, we get

$$|\mathcal{T}y(t)| = |G_i(s_i, y(s_i) - \frac{1}{\Gamma(\beta)}\int_0^{s_i}(s_i - t)^{\beta-1}f(\tau, y(\tau), ky(\tau))d\tau +$$
$$\frac{1}{\Gamma(\beta)}\int_{s_i}^{t}(t-\tau)^{\beta-1}f(\tau, y(\tau), ky(\tau))d\tau|$$
$$\leq L_{G_i}|y(t)| + |G_i(t,0)| + \frac{(s_i{}^\beta + t_{i+1}{}^\beta)}{\Gamma(\beta+1)}\Omega(||y||)||m_f||_{L^1[s_i,t_{i+1}]}$$
$$\leq L_{G_i}\eta + |G_i(t,0)| + \frac{(s_i{}^\beta + t_{i+1}{}^\beta)}{\Gamma(\beta+1)}\Omega(\eta)||m_f||_{L^1[s_i,t_{i+1}]}$$
$$\leq L_{G_i}\eta + \frac{(s_i{}^\beta + t_{i+1}{}^\beta)}{\Gamma(\beta+1)}\Omega(\eta)||m_f||_{L^1[s_i,t_{i+1}]} + |G_i(t,0)|$$
$$\leq \delta\eta + (1-\delta)\eta.$$

For $t \in [0, t_1]$,

$$|\mathcal{T}y(t)| = ||x_0 - h(y)||$$
$$\leq |x_0| + L_h||y|| + |h(0)|$$
$$\leq |x_0| + L_h\eta + |h(0)|$$
$$\leq (1-\delta)\eta + \delta\eta.$$

$$\text{Hence, } ||\mathcal{T}y||_{C([0,t_1],R)} \leq \eta.$$

Similarly, for $t \in (t_i, s_i]$,

$$|\mathcal{T}y(t)| \leq L_{G_i}\eta + |G_i(t,0)| \leq \eta.$$

From these, we can conclude that $||\mathcal{T}y||_{PC} \leq \eta$ and hence $\mathcal{T}$ takes value in B_η. Arguing in a similar fashion, we can also show that for any $y, z \in B_\eta$, $\mathcal{T}^1y + \mathcal{T}^2z \in B_\eta$.

Step 2: $\mathcal{T}^1$ is a contraction mapping in B_η.

For $y, z \in B_\eta$ and $t \in (t_i, t_{i+1}]$, we observe that

$$|\mathcal{T}_i^1y(t) - \mathcal{T}_i{}^1z(t)| \leq L_{G_i}||y-z||_{C((t_i,t_{i+1}],\mathbb{R})}.$$

$$\text{Therefore, } \|\sum_{i=0}^{N}\mathcal{T}_i^1y - \sum_{i=0}^{N}\mathcal{T}_i^1z\|_{PC} \leq L||y-z||_{PC}.$$

By the choice of L, we observe that $\mathcal{T}^1$ is a contraction on B_η.

Step 3: $\mathcal{T}^2$ is a continuous mapping in B_η.

Let $\{y^n\}$ be a sequence in $PC(J,\mathbb{R})$ such that $y^n \to y$.

For $t \in (s_i, t_i + 1]$,

$$
\begin{aligned}
|\mathcal{T}^2 y^n(t) - \mathcal{T}^2 y(t)| \quad &= |\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\{f(\tau, y^n(\tau), ky^n(\tau)) - f(\tau, y(\tau), ky(\tau)\}d\tau \\
&-\frac{1}{\Gamma(\beta)}\int_0^{s_i} (s_i-\tau)^{\beta-1}\{f(\tau, y^n(\tau), ky^n(\tau)) - f(\tau, y(\tau), ky(\tau)\}d\tau| \\
&\le \frac{t_{i+1}^{\beta} + 2s_i{}^{\beta}}{\Gamma(\beta+1)}|f(., y^n(.), ky^n(.)) - f(., y(.), ky(.))| \\
&\longrightarrow 0 \text{ as } n \to \infty \quad, f \text{ and } k \text{ being continuous functions.}
\end{aligned}
$$

Step 4: $\mathcal{T}^2$ is compact in B_η.

From the hypothesis, it is clear that $||\mathcal{T}^2 y|| \le \eta$ in B_η. It is enough to show that $\mathcal{T}^2$ maps bounded sets into equicontinuous sets in B_η.

Let $s_i \le \xi_1 < \xi_2 \le t_{i+1}$ in $(s_i, t_{i+1}], i = 1, 2, ..., N$ and $x \in B_\eta$.

$$
\begin{aligned}
|(\mathcal{T}^2 y)(\xi_1) - (\mathcal{T}^2 y)(\xi_2)| &\le \frac{1}{\Gamma(\beta)}\int_{\xi_1}^{\xi_2} (\xi_2-\tau)^{\beta-1} f(\tau, y(\tau), ky(\tau))ds \\
&+ \frac{1}{\Gamma(\beta)}[(\xi_2-\tau)^{\beta-1} - (\xi_1-\tau)^{\beta-1}]f(\tau, y(\tau), ky(\tau))d\tau \\
&\le \frac{\Omega(\eta)||m_f||_{L^1([s_i,t_{i+1}],\mathbb{R})}}{\Gamma(\beta+1)}[\xi_2{}^{\beta} - \xi_1{}^{\beta} + 2(\xi_2-\xi_1)^{\beta}] \\
&\le \frac{\Omega(\eta)||m_f||_{L^1([s_i,t_{i+1}],\mathbb{R})}}{\Gamma(\beta+1)} 3(\xi_2-\xi_1)^{\beta} \\
&\longrightarrow 0 \text{ as } \xi_2 \to \xi_1.
\end{aligned}
$$

As a result, we can conclude that $\mathcal{T} : B_\eta \to B_\eta$ is completely continuous. By using Krasnoselskii's fixed point theorem, $\mathcal{T} = \mathcal{T}^1 + \mathcal{T}^2$ has a fixed point which is a solution of the problem. □

6.4 Application

In this Section we present an application of the result obtained in section 6.3.

Example 6.4.1. *Consider the following FDE with impulsive condition of the form*

$$
\begin{aligned}
{}^{C}D^{\frac{1}{2}}y(t) &= \frac{|y(t)|}{(9+e^t)(1+|y(t)|)} + \frac{1}{9}\int_0^t e^{-\frac{1}{4}y(\sin\tau)}d\tau, t \in (0,1]\cup(2,3], && (6.6)\\
y(t) &= \frac{|y(t)|}{(9t+1)(1+|y(t)|)}, t\in(1,2], && (6.7)\\
y(0) &= x_0 + \sum_{i=1}^{10} c_i y(p_i), && (6.8)
\end{aligned}
$$

where, c_i *are constants with* $\max\limits_{1\le i\le 10} c_i < 0.01$ *and*
$0 < p_1 < p_2 < ... < p_{10} < 3$, $\beta = \frac{1}{2}$, $J = [0,3], 0 = s_0 < t_1 = 1 < s_1 = 2 < t_2 = 3$.
Take

$$
\begin{aligned}
f(t, y(t), ky(t)) &= \frac{|y(t)|}{(9+e^t)(1+|y(t)|)} + \frac{1}{9}\int_0^t e^{-\frac{1}{4}y(\tau)}d\tau, t\in(0,1]\cup(2,3],\\
k(y(t)) &= \int_0^t e^{-\frac{1}{4}y(\sin\tau)}d\tau, t\in(0,1]\cup(2,3],\\
G_1(t, y(t)) &= \frac{|y(t)|}{(9t+1)(1+|y(t)|)}, t\in(1,2],\\
h(y) &= \sum_{i=1}^{10} c_i y(p_i).
\end{aligned}
$$

For $y, z \in PC(J,\mathbb{R})$ *and* $t \in (1,2]$,

$$|G_1(t,y) - G_1(t,z)| \le \frac{1}{10}|y(t)-z(t)| \le \frac{1}{10}||y-z||_{PC}.$$

For $y, z \in PC(J,\mathbb{R})$, *and* $t \in (0,1]\cup(2,3]$,

$$
\begin{aligned}
&|f(t, y(t), k(y(t))) - f(t, z(t), k(z(t))\\
&|\le \tfrac{1}{10}|y(t)-z(t)| + \tfrac{1}{9}||k(y)-k(z)||\\
&\le \tfrac{1}{9}[||y-z||_{PC} + ||k(y)-k(z)||_{PC}].
\end{aligned}
$$

For $y, z \in PC(J,\mathbb{R})$ *and* $t \in (0,1]\cup(2,3]$,

$$|(ky)(t) - (kz)(t)| \le \frac{1}{4}||y(\sin t) - z(\sin t)|| \le \frac{1}{4}||y-z||_{PC}.$$

$$\|ky - kz\|_{PC} \le \frac{1}{4}||y(\sin t) - z(\sin t)|| \le \frac{1}{4}||y-z||_{PC}.$$

For $y, z \in PC(J,\mathbb{R})$,

$$||h(y) - h(z)|| < \frac{1}{10}||y-z||_{PC}.$$

Put $L_{G_1} = L_h = L_f = \frac{1}{10}, L_k = \frac{1}{4}$,

$$L = \max\left\{\frac{1}{10} + (\sqrt{2}+\sqrt{3})\frac{(0.1\times 1.25)}{\Gamma(\frac{3}{2})}, \frac{1}{10} + \frac{1}{\Gamma(\frac{3}{2})}0.1\times(1+.25)\right\} = 0.543 < 1.$$

Thus, since all the assumptions in Theorem 6.3.1 are fulfilled, our results can be applied to the problem given by equations (6.6)-(6.8).

6.5 Conclusion

This work studies the existence of mild solution of a class of nonlocal FDE with not instantaneous impulses. The proof of the main result is based on the Banach contraction theorem and Krasnoselskii's fixed point theorem. An example is presented to illustrate the applicability of the obtained result.

Chapter 7

FEE with an almost sectorial operator

In the previous chapters, we investigated the existence of mild solutions of different classes of FDEs with non-instantaneous impulses. This chapter is concerned with the existence of mild solution of a class of abstract FEE with an almost sectorial operator.

7.1 Introduction

It is observed that FEEs with linear part being infinitesimal generator of a C_0 semigroup, or a compact semigroup, or an analytic semigroup, or a Hille-Yosida operator have been extensively studied [24, 87, 91]. But the number of works carried out on the FEE with an almost sectorial operator is very scanty. Arrieta et al. [13–15] defined a domain of a dumb-bell with a thin handle in a series of articles. The domain is of the form

$$\Omega_\epsilon = D_1 \cup S_\epsilon \cup D_2, \quad (\epsilon \ll 1),$$

where D_1 and D_2 are mutually disjoint bounded domains in $\mathbb{R}^n (n \geq 2)$ with smooth boundaries, joined by a thin channel S_ϵ, which degenerates to a 1-dimensional line segment S_0 as ϵ approaches zero.

Carvalho et al. [38] showed that in the limiting case ($\epsilon \to 0$), the parabolic evolution equation of the form

$$U_t - \Delta U + U = f(U), \quad x \in \Omega_\epsilon,\ t > 0,$$
$$\frac{\partial U}{\partial \eta} = 0, \quad \mathscr{X} \in \partial\Omega_\epsilon,$$

gets transformed to an abstract evolution equation of $U(\mathscr{X}, t)$ in which the linear part generates an analytic semigroup A_0 in some special domain for which the spectrum is contained in a sector but the resolvent estimate is different from the case of a sectorial operator. For details, one can refer to [93].

In recent past, Wang et al. [116] established the existence of classical and mild solution of linear and semilinear abstract fractional Cauchy problem using an almost sectorial operator. Fang Li [77] investigated the existence of mild solution to delay FDEs with an almost sectorial operator. The existence theorems of mild solutions for Riemann Liouville fractional Cauchy problem with an almost sectorial have been studied by Zhang and Zhou [121].

In this chapter, we prove the existence and uniqueness of mild solution of a class of semilinear impulsive Cauchy problem with an almost sectorial operator in the following form:

$$^{C}D_t^{\beta} y(t) = Ay(t) + f(t, y(t)), \quad t \in (0, T], \tag{7.1}$$
$$y(0) = x_0, \tag{7.2}$$

with $0 < \beta < 1, A : D(A) \subset \mathbb{X} \to \mathbb{X}$ as an almost sectorial operator on $\mathbb{X}$ and $f : J \times \mathbb{X} \to \mathbb{X}$ as a given function and $J = [0, T]$.

7.2 Preliminaries

Taking clue from [121], we present the following definition and some properties of an almost sectorial operator:

Let S_q^0 with $0 < q < \pi$ be the open sector

$$\{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < q\}$$

and S_q be its closure.

Definition 7.2.1. *Let $-1 < p < 0$ and $0 < \omega < \frac{\pi}{2}$. By Θ^p_ω, we denote the family of all closed operators $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ which satisfy*

(i) $\sigma(A) \subset S_\omega = \{\lambda \in \mathbb{C} \setminus 0 : |\arg \lambda| \leq \omega\} \cup \{0\}$, *and*

(ii) for every $\omega < q < \pi$, there exists a constant $c_q > 0$ such that

$$\|R(\lambda; A)\|_{B(\mathbb{X})} \leq c_q |\lambda|^p, \quad \text{for all } \lambda \in \mathbb{C} \setminus S_q,$$

A linear operator A is called an almost sectorial operator on $\mathbb{X}$ if $A \in \Theta^p_\omega$.

If $A \in \Theta^p_\omega$, then A generates a semigroup $Q(t)$ with singular behaviour at $t = 0$ and $Q(t)$ is analytic in an open sector of the complex plane $\mathbb{C}$. In fact, for $t \in S^0_{\frac{\pi}{2}-\omega}$,

$$Q(t) = e^{-t\lambda}(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{-t\lambda} R(\lambda; A) d\lambda,$$

where the integral contour $\Gamma_\theta = \{\mathbb{R}^+ e^{i\theta}\} \cup \{\mathbb{R}^+ e^{-i\theta}\}$ is oriented counter-clockwise and $\omega < \theta < q < \frac{\pi}{2} - |\arg t|$, forms an analytic semigroup of growth order $1 + p$.

Proposition 7.2.1. *[121] Let $A \in \Theta^p_\omega(\mathbb{X})$, with $-1 < p < 0$ and $0 < \omega < \frac{\pi}{2}$. Then the following properties remain true:*

(i) $Q(t)$ is analytic in $S^0_{\frac{\pi}{2}-\omega}$ and $\frac{d^n}{dt^n} Q(t) = (-A)^n Q(t),\ t \in S^0_{\frac{\pi}{2}-\omega}$,

(ii) the functional equation $Q(s + t) = Q(s)Q(t)$ for all $s, t \in S^0_{\frac{\pi}{2}-\omega}$ holds,

(iii) there is a constant $c_0 = c_0(p) > 0$ such that $\|Q(t)\|_{B(\mathbb{X})} \leq c_0 t^{-p-1} (t > 0)$,

(iv) if $\beta > 1 + p$, then $D(A^\alpha) \subset \Sigma_Q = \{u \in \mathbb{X} : \lim_{t \to 0^+} Q(t)u = u\}$,

(v) $R(\lambda; A) = \int_0^\infty e^{-\lambda t} Q(t) dt$ for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$.

For each $t \in S^0_{\frac{\pi}{2}-\omega}$, define operators $\mathscr{S}_\beta(t)$ and $\mathscr{P}_\beta(t)$ by

$$\mathscr{S}_\beta(t) = \int_0^\infty M_\beta(\theta) Q(t^\beta \theta) d\theta,$$
$$\mathscr{P}_\beta(t) = \int_0^\infty \beta\theta M_\beta(\theta) Q(t^\beta \theta) d\theta,$$

where $M_\beta(\theta)$ is a Wright-type function given by

$$M_\beta(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\,\Gamma(1 - \beta n)}, \quad \theta \in \mathbb{C},\ 0 < \beta < 1.$$

Lemma 7.2.1. *[119] For each* $t \in S^0_{\frac{\pi}{2}-\omega}$, $\mathscr{S}_\beta(t)$ *and* $\mathscr{P}_\beta(t)$ *are linear and bounded operators on* $\mathbb{X}$*. Moreover, for all* $t > 0$,

$$\|\mathscr{S}_\beta(t)\| \leq c_1 t^{-\beta(1+p)} \quad \text{and} \quad \|\mathscr{P}_\beta(t)\| \leq c_2 t^{-\beta(1+p)},$$

where $c_1 = c_0 \frac{\Gamma(-p)}{\Gamma(1-\beta(1+p))}$ *and* $c_2 = \beta c_0 \frac{\Gamma(1-p)}{\Gamma(1-\beta p)}$.

Lemma 7.2.2. *[119] For* $t > 0$, $\mathscr{S}_\beta(t)$ *and* $\mathscr{P}_\beta(t)$ *are strongly continuous on* $\mathbb{X}$.

Lemma 7.2.3. *[119] Let* $\alpha > 1 + p$*. For all* $u \in D(A^\alpha)$, *we have*

$$\lim_{t\to 0^+} \mathscr{S}_\beta(t)u = u \quad \text{and} \quad \lim_{t\to 0^+} \mathscr{P}_\beta(t)u = \frac{u}{\Gamma(\beta)}.$$

7.3 Existence and uniqueness of mild solutions

In this section, we state the existence of mild solutions for the system (7.1)-(7.2) by using fixed point theorems and Hausdorff measure of noncompactness for the cases when the associated semigroup $Q(t)$ is compact or noncompact.

For $\eta > 0$, define $B_\eta = \{y \in C(J, \mathbb{X}) : \|y\| \leq \eta\}$. Then B_η is a bounded, closed and convex subset of $C(J, \mathbb{X})$.

The mild solution of our problem is defined as follows:

Definition 7.3.1. *[121] By a mild solution of the Cauchy problem (7.1)-(7.2), we mean a function* $y \in C(J, \mathbb{X})$ *which satisfies*

$$y(t) = \mathscr{S}_\beta(t)x_0 + \int_0^t (t-\tau)^{\beta-1}\mathscr{P}_\beta(t-\tau)f(\tau, y(\tau))d\tau, \quad t \in (0, T]. \tag{7.3}$$

Remark 7.3.1. *[119] For* $x_0 \in D(A^\alpha), \alpha > 1 + p$, *the mild solution is continuous at* $t = 0$.

Now, we introduce the following hypotheses:

(H1) For each $t > 0$, $Q(t)$ is equicontinuous.

(H2) For each $t \in J$, the function $f(t, .) : \mathbb{X} \to \mathbb{X}$ is continuous and for each $u \in \mathbb{X}$, the function $f(., u) : J \to \mathbb{X}$ is strongly measurable.

(H3) There exists a constant $L_f \geq 0$ such that for any $u_1, v_1 \in \mathbb{X}$ satisfying $\|u_1\|, \|v_1\| \leq \eta$, and $t \in J$, we have

$$\|f(t, u_1) - f(t, v_1)\| \leq L_f \|u_1 - v_1\|.$$

(H4) There exists a function $m \in L(J, \mathbb{R}^+)$ such that

$${}_0D_t{}^{\beta p} m(t) \in C(J, \mathbb{R}^+), \quad \lim_{t \to 0^+} {}_0D_t{}^{\beta p} m(t) = 0 \text{ and}$$

$$\|f(t, y)\| \leq m(t) \quad \text{for all } y \in B_\eta \text{ and a.e. } t \in J.$$

(H5) There exists a constant $\eta > 0$ such that

$$c_1\|x_0\| + c_2 \sup_{t\in[0,T]} \int_0^t (t-\tau)^{-(1+\beta p)} m(\tau) d\tau \leq \eta.$$

We define the operator $F : B_\eta \to B_\eta$ by

$$(\mathscr{F}y)(t) = \mathscr{S}_\beta(t)x_0 + \int_0^t (t-\tau)^{\beta-1} \mathscr{P}_\beta(t-\tau) f(\tau, y(\tau)) d\tau, \ t \in [0, T].$$

The following lemmas can be easily proved.

Lemma 7.3.1. *Assume that (H1),(H2) and (H4) hold. Then the set of functions* $\{\mathscr{F}y : y \in B_\eta\}$ *is an equicontinuous subset of* B_η.

Lemma 7.3.2. *Assume that (H2),(H4) and (H5) hold. Then* $F(B_\eta) \subset B_\eta$ *and* F *is continuous in* B_η .

For compact semigroup case, we may obtain the following result:

Theorem 7.3.1. *Assume that the hypotheses (H2),(H4) and (H5) are satisfied and the semigroup* $\{Q(t)\}_{t>0}$ *is compact. Then there exists a mild solution* $y \in C(J, \mathbb{X})$ *of the problem (7.1)-(7.2) provided* $x_0 \in D(A^\alpha)$ *with* $\alpha > 1 + p$.

Noncompact semigroup case:

If $Q(t)$ is not compact, we consider the following assumption:

(H6) there exists $l > 0$ such that for any bounded set $D \subset \mathbb{X}$,

$$\chi(f(t, D)) \leq l\chi(D).$$

Theorem 7.3.2. *Assume that (H1), (H2), (H4), (H5) and (H6) hold. Then for each* $x_0 \in D(A^\alpha)$ *with* $\alpha > 1 + p$*, the problem (7.1)-(7.2) has atleast one mild solution in* B_η.

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The existence of unique mild solution may be obtained using hypothesis $(H3)$ and $(H5)$.

Theorem 7.3.3. *Assume that the hypotheses (H3) and (H5) hold. Then for every* $x_0 \in D(A^\alpha)$ *with* $\alpha > 1 + p$*, the problem (7.1)-(7.2) has a unique mild solution* $y \in B_\eta$ *provided* $-\frac{c_2 L_f T^{-\beta p}}{\beta p} < 1$.

The above three theorems may be attempted by in-calculating the idea of measure of non-compactness and using some fixed point theorems, viz., Schauder's fixed point theorem and Banach fixed point theorems.

Elaborate procedure of obtaining the result of existence of mild solution is not being described here. However, it is expected that it will not be a difficult task to obtain the desired results.

It may be noted that similar idea can be found in [125].

Chapter 8

Conclusion and future plan

In this thesis we study various types of fractional differential equations. First four works deal with some classes of fractional evolution equations subject to non instantaneous impulses when the linear operator is the generator of a compact semigroup or an integrated semigroup or an analytic semigroup. Using some classical fixed point theorems such as Banach, Krasnoselskii's and Burton-Kirk's, we prove the existence of mild solution with the techniques of fractional calculus and semigroup theory. In the fifth problem, we prove the existence of mild solution of a class of fractional differential equation subject to non instantaneous impulse and nonlocal condition. Furthermore, existence of mild solution of a class of fractional evolution equation with an almost sectorial operator is discussed as the sixth problem. At the end of each chapter, appropriate examples are presented to demonstrate the results obtained.

Nonlinear fractional evolution equation with time dependent linear part is in its infant state. As the partial differential operator of some parabolic evolution equation depends on the time t, as observed in [109], so it would be interesting and significant to extend our result for impulsive fractional evolution equation to time dependent as well as time and state dependent linear part.

Secondly, in comparison to a deterministic model, a stochastic model is seen as more realistic and challenging. We would like to investigate the existence of mild solution for non-instantaneous impulsive stochastic FEE with an almost sectorial operator. In our considered problem, we assumed $0 < \beta < 1$. We plan to extend our work for $\beta > 1$.

There are very few works in fractional differential inclusions subject to non-instantaneous impulses. We wish to investigate the existence of fractional evolution inclusions with finite and infinite delay and varying or non-varying base point of the fractional differential operator.

There exists no literature on non-instantantaneous impulsive FDE with Riemann-Liouville fractional derivative. In our future plan, we want to investigate the existence, stability and dependence of initial data of the mild solution of Riemann-Liouville FDE subject to non-instantaneous impulses.

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Published and Communicated Papers

Based on our works carried out in this thesis, the following articles are published/accepted/communicated:

1. Jayanta Borah and Swaroop Nandan Bora. Existence of mild solution for mixed Volterra-Fredholm integro fractional differential equations with non-instantaneous impulses, *Differential Equations and Dynamical Systems* (Published online: 12 February 2018) Doi.org/10.1007/s12591-018-0410-1.

2. Jayanta Borah and Swaroop Nandan Bora. Existence of mild solution of a class of nonlocal fractional order differential equation with not instantaneous impulses, *Fractional Calculus and Applied Analysis* (Acccepted).

3. Jayanta Borah and Swaroop Nandan Bora. Non-instantaneous impulsive fractional semilinear evolution equation with finite delay (Communicated).

4. Jayanta Borah and Swaroop Nandan Bora. Sufficient conditions for existence of integr'al solution for non-instantaneous impulsive evolution equations (Communicated).

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